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MASTER'S THESIS

A short thesis on Shelah's Singular Compactness Theorem for modules.

Mitos Achilleas

Committee:

Pr. Emmanouil Ioannis (supervisor)

Associate Pr. Michail Sykiotis

Assistant Pr. Ntokas Ioannis

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Abstract

Shelah's Singular (cardinal) Compactness Theorem has been a cornerstone in the marriage of Homological Algebra and Set Theory for 50 years now. This essay pays homage to this much-celebrated theorem of mathematics, by attempting to provide the basics for someone with a graduate understanding of mathematics to get involved. We introduce and present the necessary definitions and propositions from Homological Algebra and Set Theory. We, then, give a historical overview of the most prevalent breakthroughs regarding the Singular Compactness Theorem. Finally, we conclude the essay with an analysis of a recent (2020) paper by Saroch and Stovicek that shone a light upon a new, set-theoretic and general proof of the Theorem.

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Introduction

Shelah's Singular Compactness Theorem is a well-known result in the field of Homological Algebra, which in broad terms states that if a mathematical object (structure) contains "enough" smaller mathematical objects (sub-structures) with a certain property P, then the original large object also has that same property P. The main purpose of this essay is to gather bibliography on the matter, explain the methods and tools used to prove some more general results based on Shelah's original theorem, and provide an easy and streamlined way for a reader of a graduate level to understand the concepts and proofs that lead to this fascinating cross-over between Homological Algebra and Set Theory.

The dissertation is constructed as follows: The first chapter is dedicated to a short introduction on Homological Algebra; the reader is eased into the world of Category Theory (with the very basic definitions of *category, morphism* and *functor*, together with some propositions covered in a 3-month period taught in a class of a relevant subject), and then is immediately thrust upon the study of Homological Algebra (with definitions such as that of a *module* and the *Hom* functor, as well as propositions such as the *Snake Lemma*). Chapter 2 discusses some elementary Set Theory (in the *ZFC* axiomatic system) with the goal to introduce the notions of *cardinals* and *ordinals* and basic arithmetic of them. Since these two chapters are introductory and not the main material of this paper, we stick to the bare minimum material we need in Chapter 3, skipping most of the (nice) proofs we could present (though references to other works for all of them are provided throughout the paper). Most of the work is being done in the last chapter, where we begin with a historic account of relevant results and finally present the main theorem of the Šaroch and Šťovíček [18] paper, which we state below:

Theorem. *Let R be a ring with enough idempotents. Let κ be a singular cardinal, M be a κ-presented module and $\mathscr{C}$ a filter-closed class of modules. Assume that there is an infinite cardinal ν such that, for all successor cardinals $\nu < \lambda < \kappa$, there is a system S_λ witnessing that M is almost $(\mathscr{C}, \lambda)$-projective. Then $M \in^\perp \mathscr{C}$*

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Chapter 1

Homological Algebra: a short introduction

Homological algebra is the branch of mathematics that concerns itself with the study of *homology* and *functors*. It stems from algebraic topology and abstract algebra (although, for our purposes, we will barely touch on these two subjects). A relatively young mathematical branch, homological algebra has proven its worth in a wide range of other areas, such as commutative algebra, algebraic geometry and number theory, complex analysis and many more. As Rotman says in his great book [12] (which will be our reference point for many of the omitted proofs and ideas in this chapter), Homological Algebra has known great growth due to the appearance of category theory, hence the first pages are given to a basic study of categories. After we have also given some elementary module-theory background, we will then move on to Homological Algebra, up to the point where we introduce the *Tor* and *Ext* functors, which appear in Chapter 3.

1.1 Some basics of Category Theory

Category theory introduces the notion of *categories*, a general mathematical structure that provides a template for already-existing structures (such as sets, groups etc), and the notion of *functors* (a generalized notion of a function), in order to study them from an abstract point of view (often with the goal of applying these more abstract results to more concrete examples as mentioned). Let us begin by introducing these two stars:

Definition 1.1.1. A *category* $\mathscr{C}$ is comprised of three parts:

- the class of the objects in $\mathscr{C}$, denoted by $\text{obj}(\mathscr{C})$
- a set of morphisms, denoted by $\text{Hom}(A,B)$, for every two objects A,B in $\text{obj}(\mathscr{C})$ [1]
- a composition operation $\text{Hom}(A,B) \times \text{Hom}(B,C) \longrightarrow \text{Hom}(A,C)$, $(f,g) \mapsto gf$ for all $A,B,C \in \text{obj}(\mathscr{C})$

with the following rules:

- if $A,A',B,B' \in \text{obj}(\mathscr{C})$ with $(A,B) \neq (A',B')$ then their morphism sets are disjoint
- for every $A \in \text{obj}(\mathscr{C})$ there exists the identity morphism $1_A \in \text{Hom}(A,A)$ such that $f1_A = f$ and $1_B f = f$ for all $f \in \text{Hom}(A,B)$
- for all morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ [2] we have $h(gf) = (hg)f$ (composition of morphisms is associative)

Remark 1.1.2. Some common examples of categories, which also display the power of such an abstract definition for categories, are the following:

- (Category of) Sets: the objects of this category are the sets, the morphisms are the functions between sets, and composition is the normal composition of functions
- (Category of) Top: the objects are topological spaces, the morphisms are the continuous functions, and composition is the normal composition of functions
- (Category of) Groups: the objects are algebraic groups, the morphisms are the group homomorphisms, and composition is the normal composition of functions
- (Category of) Ab: the objects are abelian groups
- (Category of) Rings: the objects are algebraic rings
- (Category of) ${}_R$Mod: the objects are left R-modules

[1] Although $\text{obj}(\mathscr{C})$ is generally a class, not a set, we will often use the notation $A \in \text{obj}(\mathscr{C})$ to denote that some element A is an object of a category.

[2] By $A \xrightarrow{f} B$ we mean a morphism $f \in \text{Hom}(A,B)$.

Definition 1.1.3. A category $\mathscr{S}$ is called a *subcategory* of $\mathscr{C}$ if:

- every object of $\mathscr{S}$ is also an object of $\mathscr{C}$
- $\mathrm{Hom}_{\mathscr{S}}(A,B) \subseteq \mathrm{Hom}_{\mathscr{C}}(A,B)$ for all $A,B \in \mathrm{obj}(\mathscr{S})$
- for all $f \in \mathrm{Hom}_{\mathscr{S}}(A,B)$, $g \in \mathrm{Hom}_{\mathscr{S}}(B,C)$ the result $gf \in \mathrm{Hom}_{\mathscr{S}}(A,C)$ of their composition in $\mathscr{S}$ is the same as their result in $\mathscr{C}$
- for every object A in $\mathscr{S}$ its identity morphisms in $\mathscr{S}$ and $\mathscr{C}$ are the same

We say that $\mathscr{S}$ is *complete* if for every $A,B \in \mathrm{obj}(\mathscr{S})$ we have $\mathrm{Hom}_{\mathscr{S}}(A,B) = \mathrm{Hom}_{\mathscr{C}}(A,B)$.

Definition 1.1.4. Let $\mathscr{C},\mathscr{D}$ be two categories. Then the map $T : \mathscr{C} \longrightarrow \mathscr{D}$ is called a *(covariant) functor* when:

- for all $A \in \mathrm{obj}(\mathscr{C})$ we have $T(A) \in \mathrm{obj}(\mathscr{D})$
- for all $f \in \mathrm{Hom}_{\mathscr{C}}(A,A')$ we have $T(f) \in \mathrm{Hom}_{\mathscr{D}}(T(A),T(A'))$
- for all morhpisms $A \xrightarrow{f} A' \xrightarrow{g} A''$ we have $T(A) \xrightarrow{T(f)} T(A') \xrightarrow{T(g)} T(A'')$ in $\mathscr{D}$, and $T(gf) = T(g)T(f)$
- for all $A \in \mathrm{obj}(\mathscr{C})$ we have $T(1_A) = 1_{T(A)}$

Let us now look at a functor which lies at the heart of Category Theory:

Definition 1.1.5. Let $\mathscr{C}$ be a category, and $A \in \mathrm{obj}(\mathscr{C})$. We define the Hom functor to be the functor T_A from $\mathscr{C}$ to the category of Sets for which:

- $T_A(B) = \mathrm{Hom}(A,B)$ for all $B \in \mathrm{obj}(\mathscr{C})$
- for all $f \in \mathrm{Hom}(B,B')$ and all $B,B' \in \mathrm{obj}(\mathscr{C})$ the morphism $T_A(f) : \mathrm{Hom}(A,B) \longrightarrow \mathrm{Hom}(A,B')$ is given by: $T_A(f) : h \mapsto fh$

One can easily see that this is indeed a functor. We will denote it by $\mathrm{Hom}(A,\Box)$. We will also write f_* for $T_A(f)$ and call this the induced (by f) map.

There is also a dual notion to the functor as defined previously, that of a *contravariant functor*:

Definition 1.1.6. Let $\mathscr{C},\mathscr{D}$ be two categories. Then the map $T : \mathscr{C} \longrightarrow \mathscr{D}$ is called a *contravariant functor* when:

- for all $A \in \mathrm{obj}(\mathscr{C})$ we have $T(A) \in \mathrm{obj}(\mathscr{D})$
- for all $f \in \mathrm{Hom}_{\mathscr{C}}(A,A')$ we have $T(f) \in \mathrm{Hom}_{\mathscr{D}}(T(A'),T(A))$
- for all morphisms $A \xrightarrow{f} A' \xrightarrow{g} A''$ we have $T(A'') \xrightarrow{T(g)} T(A') \xrightarrow{T(f)} T(A)$ in $\mathscr{D}$, and $T(gf) = T(f)T(g)$
- for all $A \in \mathrm{obj}(\mathscr{C})$ we have $T(1_A) = 1_{T(A)}$

So a contravariant functor is a functor that "reverses the arrows".

The contravariant dual to the Hom functor is denoted by $\mathrm{Hom}(\square,B)$ for an object B of a category $\mathscr{C}$. The induced morphism $\mathrm{Hom}(f,B) : h \mapsto hf$ is denoted by f^*.

Definition 1.1.7. A morphism $f : A \longrightarrow B$ in a category $\mathscr{C}$ is an *isomorphism* if there exists a morphism $g : B \longrightarrow A$ in $\mathscr{C}$ such that $gf = 1_A$ and $fg = 1_B$. The morphism g is then called the *inverse* of f.

Proposition 1.1.8. *Functors preserve isomorphisms. That is, if $T : \mathscr{C} \longrightarrow \mathscr{D}$ is a functor and f is an isomorphism in $\mathscr{C}$, then $T(f)$ is an isomorphism in $\mathscr{D}$.*

Proof. Let g be the inverse morphism of f in $\mathscr{C}$. Then $T(g)$ is a morphism in $\mathscr{D}$. Furthermore, $T(f)T(g) = T(fg) = T(1) = 1 = T(gf) = T(g)T(f)$ if T is covariant, or $T(f)T(g) = T(gf) = T(1) = 1 = T(fg) = T(g)T(f)$ if T is contravariant. This means that $T(g)$ is the inverse morphism of $T(f)$. □

Definition 1.1.9. Let $S,T : \mathscr{A} \longrightarrow \mathscr{B}$ be two functors. A *natural transformation* $\tau : S \longrightarrow T$ is a family $(\tau_A : S(A) \longrightarrow T(A))_{A\in\mathrm{obj}(\mathscr{C})}$ of morphisms in $\mathscr{B}$ such that for every morphism $f : A \longrightarrow A'$ in $\mathscr{A}$ we have: $T(f)\tau_A = \tau_A S(f)$.
A natural transformation is called a *natural isomorphism* if every τ_A in the above definition is an isomorphism. Two functors S,T are *naturally isomorphic* when there exists a natural isomorphism between them. We will then write $S \cong T$.

Definition 1.1.10. Let $\tau : S \longrightarrow T$, $\sigma : T \longrightarrow U$ be two natural transformations between the functors $S,T,U : \mathscr{A} \longrightarrow \mathscr{B}$ that are of the same type (either all covariant, or all contravariant). We define their composition $\sigma\tau : S \longrightarrow U$ to be the natural transformation such that $(\sigma\tau)_A = \sigma_A\tau_A$ for all $A \in \text{obj}(\mathscr{A})$.

Proposition 1.1.11. *Let $S : \mathscr{A} \longrightarrow \mathscr{B}$ be a functor, and $\omega_S : S \longrightarrow S$ be its identity natural transformation (that is, $(\omega_S)_A = 1_{S(A)} : S(A) \longrightarrow S(A)$ for all objects A in $\mathscr{A}$). A natural transformation $\tau : S \longrightarrow T$ is an isomorphism if-f there exists a natural transformation $\sigma : T \longrightarrow S$ such that $\sigma\tau = \omega_S$ and $\tau\sigma = \omega_T$.*

Proof. Suppose that $\tau : S \longrightarrow T$ is an isomorphism. By definition, for every object A in $\mathscr{A}$ the functor τ_A is an isomorphism, that is there exists a functor σ_A such that $\tau_A\sigma_A = 1_{T(A)}$ and $\sigma_A\tau_A = 1_{S(A)}$. Define $\sigma : T \longrightarrow S$ to be the family $(\sigma_A : T(A) \longrightarrow S(A))_{A\in\text{obj}(\mathscr{A})}$. It's easy to see that this family is a functor (check Definition 1.1.9 directly). Hence, $\sigma\tau = \omega_S$ and $\tau\sigma = \omega_T$.

The converse is easily seen to also hold true. □

Whenever $F, G : \mathscr{A} \longrightarrow \mathscr{B}$ are two functors of the same type (either both covariant or both contravariant), then we will denote by $\text{Nat}(F,G)$ the class of all the natural transformations from $\tau : F \longrightarrow G$.

Lemma 1.1.12 (Yoneda's Lemma)**.** *Let $\mathscr{C}$ be a category, $A \in obj(\mathscr{C})$ and $G : \mathscr{C} \longrightarrow Sets$ be a covariant functor. Then there exists a 1-1 and onto function $y : Nat\big(Hom_{\mathscr{C}}(A,\Box), G\big) \longrightarrow G(A)$ such that $y(\tau) = \tau_A(1_A)$.*

Proof. Rotman p.37(25) □

Definition 1.1.13. A covariant functor $F : \mathscr{C} \longrightarrow$ Sets is called *representable* if there exists an object A in $\mathscr{C}$ such that $F \cong \text{Hom}_{\mathscr{C}}(A,\Box)$.

Corollary 1.1.14. *If $\mathscr{C}$ is a category, $A, B \in obj(\mathscr{C})$ and $Hom_{\mathscr{C}}(A,\Box) \cong Hom_{\mathscr{C}}(B,\Box)$, then there exists an isomorphism $f : A \longrightarrow B$. The converse also holds.*

Proof. Rotman p.39(27) □

Definition 1.1.15. We define the *Hilbert space* $\mathscr{H}$ to be the set $\{(x_i)_{i=0}^{\infty} \mid \forall i\, (x_i \in \mathbb{R}) \wedge \sum_{i=o}^{\infty} x_i^2 < \infty\}$. The euclidean space $\mathbb{R}^n$ is then the subspace of $\mathscr{H}$ such that $x_i = 0$ for all $i \geq n$.

Definition 1.1.16. We define the *normal n-complex* to be the set of all convex combinations

$$\Delta^n = [e_0, \dots, e_n] = \{ \sum_{i=0}^{\infty} t_i e_i \mid t_i \geq 0 \land \sum_{i=0}^{n} t_i = 1 \}$$

where $e_i \in \mathscr{H}$ with the i-th coordinate equal to 1 and every other coordinate equal to 0.

We call e_i the *i-th vertex* of Δ^n. A *j-th edge* of Δ^n is the set of all the convex combinations between $j+1$ vertices.

Definition 1.1.17. Let X be a topological space. A *singular n-complex* in X is a continuous map $\sigma : \Delta^n \longrightarrow X$.

We define the sequence of *singular n-chains* in X recursively:

- $S_{-1}(X) = \{0\}$
- for all $n \geq 0$, $S_n(X)$ is the free abelian group on the set of all singular n-complexes in X

Definition 1.1.18. We define the *i-th face map* $e_i^n : \Delta^{n-1} \longrightarrow \Delta^n$ as follows:
If $(t_0, t_1, \dots, t_{n-1}) = t_0 e_0 + t_1 e_1 + \cdots + t_{n-1} e_{n-1}$, then $e_i^n(t_0, t_1, \dots, t_{n-1}) = (t_0, t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$

Lemma 1.1.19. *Let* $0 \leq j < i \leq n-1$. *Then* $e_i^n e_j^{n-1} = e_j^n e_{i-1}^{n-1} : \Delta^{n-2} \longrightarrow \Delta^n$.

Proof. Verify directly from the above definitions and the composition of functions. □

Definition 1.1.20. Let X be a topological space, and σ be a singular 0-complex in X. We define $\partial_0(\sigma) = 0$. Now, let σ be a singular n-complex in X. We define $\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma e_i^n$. We define the *singular border map* $\partial_n : S_n(X) \longrightarrow S_{n-1}(X)$ to be the linear extension of $\partial_n(\sigma)$.

Proposition 1.1.21. *For all* $n \geq 1$ *we have* $\partial_{n-1}\partial_n = 0$.

Proof. Since ∂_m is the linear extension of a singular m-complex in X, it suffices to show that $\partial_{n-1}\partial_n(\sigma) = 0$ for any singular n-complex in X.
By direct calculations we have:

$$\partial_{n-1}\partial_n(\sigma) = \partial_{n-1}\Big(\sum_{i=0}^{n} (-1)^i \sigma e_i^n\Big) = \sum_{i=0}^{n} (-1)^i \partial_{n-1}(\sigma e_i^n) = \sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \sigma e_i^n e_j^{n-1}$$

and by the above lemma we get:

$$\sum_{i=0}^{n} (-1)^i \sum_{j=0}^{n-1} (-1)^j \sigma e_j^n e_{i-1}^{n-1}$$

□

Definition 1.1.22. For all $n \geq 0$ we define:

- the *group of singular n-cycles* to be $Z_n(X) := \ker(\partial_n)$
- the *group of singular n-borders* to be $B_n(X) := \operatorname{im}(\partial_{n+1})$

Corollary 1.1.23. *For all $n \geq 0$ we have $B_n(X) \subseteq Z_n(X)$.*

Proof. It follows immediately from Proposition 1.1.21. □

This corollary allows us the following definition:

Definition 1.1.24. We define the *n-th singular homology group* of the topological space X to be the group quotient $H_n(X) = Z_n(X)/B_n(X)$.

Remark 1.1.25. If $f : X \longrightarrow Y$ is a continuous map, and $\sigma : \Delta^n \longrightarrow X$ is a singular n-complex in X, then their composition $f\sigma : \Delta^n \longrightarrow Y$ is a singular n-complex in Y.

Definition 1.1.26. Let X, Y be topological spaces, and $f : X \longrightarrow Y$ a continuous map. We define the *chain map* $f_\# : S_n(X) \longrightarrow S_n(Y)$ to be the map $\sum_\sigma m_\sigma \sigma \mapsto \sum_\sigma m_\sigma f\sigma$.

Lemma 1.1.27. *If $f : X \longrightarrow Y$ is a continuous function, then $\partial_n f_\# = f_\# \partial_n$ for all $n \in \mathbb{N}$.*

Proof. Simply apply Definitions 1.1.18, 1.1.20 and 1.1.26. □

Theorem 1.1.28. *For all $n \geq 0$ we have that the singular homology $H_n : Top \longrightarrow Ab$ is a functor with actions on:*

- *the objects $X \in obj(Top)$ as $X \mapsto H_n(X) = Z_n(X)/B_n(X)$*
- *the morphisms $f \in Hom(X,Y)$ as $f \mapsto H_n(f) : H_n(X) \longrightarrow H_n(Y)$ with $H_n(f)(z_n + B_n(X)) = f_\# z_n + B_n(Y)$ for all $z_n \in Z_n(X)$*

Proof. One only needs to apply Definitions 1.1.4, 1.1.24 and 1.1.26 to prove the theorem. □

Corollary 1.1.29. *If X, Y are two topological spaces with the same homotopy type, then $H_n(X) \cong H_n(Y)$ for all $n \geq 0$.*

Definition 1.1.30. A functor (covariant or contravariant) $T : {}_R\text{Mod} \longrightarrow \text{Ab}$ is called *additive* if for all R-linear functions $f, g : A \longrightarrow B$ we have $T(f+g) = T(f) + T(g)$.

Proposition 1.1.31. *Let R be a ring, and A, B be left R-modules.*

- *The $Hom_R(A, \Box)$ functor is additive*
- *Let $Z(R)$ be the center of R, and define $rf : a \mapsto f(ra)$ for $r \in Z(R)$ and $f \in Hom_R(A,B)$. Then $Hom_R(A,B)$ becomes a $Z(R)$-module.*

Proposition 1.1.32. *Let $T : {}_R Mod \longrightarrow Ab$ be an additive functor. Then:*

- *If $0 : A \longrightarrow B$ is the zero map, then $T(0) = 0$*
- *$T(\{0\}) = \{0\}$*

Definition 1.1.33. If $f : M \longrightarrow N$ is an R-linear map between two left R-modules, we define $\text{coker}(f) = N/\text{im}(f)$.

Definition 1.1.34. A (finite or infinite) sequence of R-maps and left R-modules

$$\cdots \longrightarrow M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \longrightarrow \ldots$$

is called *exact* if for all $n \in \mathbb{N}$ we have $\text{im}(f_{n+1}) = \ker(f_n)$.

A *short exact* sequence is an exact sequence of the form $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$. It is also called an *extension* of A by C.

Proposition 1.1.35. *Let $T : {}_R Mod \longrightarrow Ab$ be an additive functor. Then $T(A \oplus B) \cong T(A) \oplus T(B)$.*

Definition 1.1.36. A submodule S of a left R-module M is called a *retract* of M if there is an R-map $p : M \longrightarrow S$ such that $p(s) = s$ for all $s \in S$.

Corollary 1.1.37. *Let S be a submodule of M. Then there exists a submodule T such that $M = S \oplus T$ if-f there exists a retraction $p : M \longrightarrow S$.*

Proof. If the module T exists, put $p : M \longrightarrow S$ to be $s + t \mapsto s$ for all $s \in S$ and $t \in T$. Verify that p is an R-map and a retraction.
If a retraction p exists, then by the First Isomorphism Theorem for modules we have $\text{im}(p) \cong M/\ker(p)$, where $\text{im}(p) = S$. Put $T = \ker(p)$. Verify that each $m \in M$ can be written as one and only one sum $s + t$ where $s \in S$ and $t \in T$. $\Box$

Corollary 1.1.38. *Let $M = S \oplus T$ be a left R-module.*

- *If $S \subseteq N \subseteq M$ and N is a submodule, then $N = S \oplus (N \cap T)$*
- *If $S \subseteq S'$ and S' is a submodule, then $M/S' = S/S' \oplus (T + S')/S'$*

Definition 1.1.39. A short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ is said to *split* if there exists a map $j : C \longrightarrow B$ such that $pj = 1_C$.

Proposition 1.1.40. *If a short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ splits, then $B \cong A \oplus C$.*

Definition 1.1.41. A left R-module F is called *free* if there exists a set B such that $F = \bigoplus_{b \in B} R_b$ with $R_b = \langle b \rangle \cong R$. In that case, B is called the *base* of F.

Remark 1.1.42. A free $\mathbb{Z}$-module is a free abelian group.

Proposition 1.1.43. *For every ring R and every set B, there exists a free left R-module F with base B.*

Proposition 1.1.44. *Let R be a ring, $X \subseteq R$ and F be a free left R-module with base X. Let M be also a left R-module and $f : X \longrightarrow M$ be a function. Then there exists a unique R-function $\tilde{f} : F \longrightarrow M$ such that $\tilde{f}(x) = f(x)$ for all $x \in X$.*

Theorem 1.1.45. *Every left R-module M is a quotient of a free left R-module F. Furthermore, M is finitely generated if-f F is finitely generated.*

Theorem 1.1.46 (Left exactness)**.** *If $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C$ is an exact sequence of left R-modules, and if X is a left R-module, then there exists an exact sequence of $Z(R)$-modules $0 \longrightarrow Hom_R(X,A) \xrightarrow{i_*} Hom_R(X,B) \xrightarrow{p_*} Hom_R(X,C)$.*

Definition 1.1.47. A covariant functor $T : {}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$ is called *left exact* if exactness of $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C$ implies exactness of $0 \longrightarrow T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C)$.

Definition 1.1.48. A covariant functor $T : {}_R\mathrm{Mod} \longrightarrow \mathrm{Ab}$ is called *exact* if exactness of $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ implies exactness of $0 \longrightarrow T(A) \xrightarrow{T(i)} T(B) \xrightarrow{T(p)} T(C) \longrightarrow 0$.
An analogous definition applies to contravariant functors.

Theorem 1.1.49. *Let F, A, A'' be left R-modules with F=free. If $p : A \longrightarrow A''$ is onto, then for every $h : F \longrightarrow A''$ there exists an R-homomorphism g such that $p \cdot g = h$.*

Definition 1.1.50. Let C, A, A'' be left R-modules and $p : A \longrightarrow A''$ be an onto map. A *lift* of the map $h : C \longrightarrow A''$ is a map $g : C \longrightarrow A$ such that $p \cdot g = h$.

Definition 1.1.51. A left R-module P is called *projective* if for every onto map $p : A \longrightarrow A''$ and every $h : P \longrightarrow A''$ there exists a lift of h.

Remark 1.1.52. Obviously, every free left R-module is projective. The converse is not always true. We will now give an equivalent definition for projectiveness.

Proposition 1.1.53. *A left R-module P is projective if-f the functor $Hom_R(P, \square)$ is exact.*

Proposition 1.1.54. *A left R-module P is projective if-f every short exact sequence $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} P \longrightarrow 0$ splits.*

Corollary 1.1.55. *If $A \subseteq B$ are two left R-modules and B/A is projective, then A has a compliment in B; that is, there exists a submodule $C \subseteq B$ such that $C \cong B/A$ and $B = A \oplus C$.*

Theorem 1.1.56. *A left R-module P is projective if-f there exist left R-modules F, A with F=free such that $F = A \oplus P$.*

Corollary 1.1.57.

- *Every term in a direct sum of a projective module is itself projective.*
- *Every direct sum of projective modules is itself projective.*

Remark 1.1.58. Let P be a left R-module and $(P_n)_{n\in\mathbb{N}}$ be submodules such that:

- $P = \bigcup_{n\in\mathbb{N}} P_n$
- $\{0\} = P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n \subseteq P_{n+1} \subseteq \dots$
- $\forall n \in \mathbb{N}\ \exists X_n \subseteq P\ (P_{n+1} = P_n \oplus X_n)$

Proposition 1.1.59 (Kaplansky)**.** *If R is a ring, $P \oplus Q = \bigoplus_{i\in I} M_i$ for any set I and all M_i are countably generated left R-modules, then P is a direct sum of countably generated left R-modules.*

Corollary 1.1.60.

- *Every projective left R-module P is a direct sum of countably generated projective left R-modules.*
- *If every countably generated projective left R-module is free, then every projective left R-module is free.*

Proposition 1.1.61. *A left R-module A is projective if-f there exist $(a_i)_{i\in I} \subseteq A$ and R-maps $(\phi_i : A \longrightarrow R)_{i\in I}$ such that:*

- *for every $x \in A$ almost every $\phi_i(x)$ is 0*

- $\forall x \in A \left(x = \sum_{i \in I} (\phi_i(x)) a_i\right)$

Moreover, A is generated by the set $\{a_i, i \in I\}$.

Definition 1.1.62. The families $(a_i)_{i \in I}, (\phi_i)_{i \in I}$ in the above proposition are called the *projective base* of A.

Definition 1.1.63. Let $X = \{x_i, i \in I\}$ be a base of a free left R-module F, and $Y = \{\sum_{i \in I} r_{ij} x_i, j \in J\} \subseteq F$. If $K = \langle Y \rangle$, then we say that a module $M \cong F/K$ has *generators* X and *relations* Y. We also say that $(X|Y)$ is a *representation* of M.

Definition 1.1.64. A left R-module M is *finitely presented* if there exists an exact sequence $R^m \longrightarrow R^n \longrightarrow M \longrightarrow 0$ for some $m, n \in \mathbb{N}$.

Remark 1.1.65. A left R-module M is finitely presented if-f it has a representation $(X|Y)$ where both X and Y are finite sets.

Proposition 1.1.66. *Every finitely generated projective left R-module is finitely presented.*

Proposition 1.1.67 (Schanuel's Lemma)**.** *Let $0 \longrightarrow K \xrightarrow{i} P \xrightarrow{\pi} M \longrightarrow 0$ and $0 \longrightarrow K' \xrightarrow{i'} P' \xrightarrow{\pi'} M \longrightarrow 0$ be two short exact sequences where P, P' are projective modules. Then $K \oplus P' = K' \oplus P$.*

Corollary 1.1.68. *If M is finitely presented and $0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$ is exact, F being a finitely generated free module, then K is finitely generated.*

Definition 1.1.69. A left R-module E is called *injective* if for all left R-modules $A \subseteq B$ and all $f \in \mathrm{Hom}_R(A, E)$ there exists $g \in \mathrm{Hom}_R(B, E)$ such that $g \cdot i = f$, where $i : A \longrightarrow B$ is the normal embedding map.

Proposition 1.1.70. *A left R-module E is injective if-f the functor $Hom_R(\square, E)$ is exact.*

Proposition 1.1.71. *If E is injective, then every short exact sequence $0 \longrightarrow E \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$ splits.*

Proposition 1.1.72.

- *The direct sum of injective modules is itself injective.*
- *Every term in a direct sum of an injective module is itself injective.*

Theorem 1.1.73 (Baer's Criterion)**.** *A left R-module E is injective if-f every R-map $f : I \longrightarrow E$ with I being an ideal of R can be extended to an R-map $g : R \longrightarrow E$.*

Theorem 1.1.74. *Let R be a ring. Then every left R-module is embeddable in an injective left R-module.*

Proposition 1.1.75. *A left R-module E is injective if-f every short exact sequence $0 \longrightarrow E \longrightarrow B \longrightarrow C \longrightarrow 0$ splits.*

Definition 1.1.76. Let M, E be left R-modules. Then E is called an *essential extension* of M if there exists a one-to-one R-map $\alpha : M \longrightarrow E$ such that $S \cap \alpha(M) \neq \{0\}$ for every non-zero submodule $S \subseteq E$.

Proposition 1.1.77. *E is injective if-f it does not have any proper essential extensions.*

Definition 1.1.78. Let M, E be left R-modules. Then E is called an *injective envelope* of M if E is injective and an essential extension of M and $M \subseteq E$. We denote E by $\mathrm{Env}(M)$.

Theorem 1.1.79 (Eckman-Schöpf)**.** *Let M be a left R-module. Then:*

- *There exists an injective envelope of M*
- *If E, E' are two injective envelopes of M, then there exists an R-isomorphism $\phi : E \longrightarrow E'$ such that $\phi(x) = x$ for all $x \in M$.*

1.2 Additive categories and functors

Definition 1.2.1. A category $\mathscr{C}$ is called *additive* if all of the below hold:

- $\mathrm{Hom}_{\mathscr{C}}(A, B)$ is an abelian group for all objects A, B
- for all morphisms $X \xrightarrow{a} A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{b} Y$ we have $b(f+g) = bf + bg$ and $(f+g)a = fa + ga$
- $\mathscr{C}$ has a zero object
- for all objects A, B we have $A \sqcup B, A \sqcap B \in \mathrm{obj}(\mathscr{C})$

Definition 1.2.2. If A, B are objects in $\mathscr{C}$, then their *coproduct* is the triplet $(A \sqcup B, \alpha, \beta)$ where $A \sqcup B \in \mathrm{obj}(\mathscr{C})$ and $\alpha : A \longrightarrow A \sqcup B$, $\beta : B \longrightarrow A \sqcup B$ are morphisms (called *embeddings*) such that for all $X \in \mathrm{obj}(\mathscr{C})$ and morphisms $f : A \longrightarrow X$, $g : B \longrightarrow X$ there exists a unique morphism $\theta : A \sqcup B \longrightarrow X$ such that $\theta\alpha = f$ and $\theta\beta = g$.

Definition 1.2.3. A $A \in \text{obj}(\mathscr{C})$ is called an *initial object* if for every object X in $\mathscr{C}$ there exists a unique morphism $\alpha : A \longrightarrow X$.

Definition 1.2.4. If A, B are objects in $\mathscr{C}$, then their *product* is the triplet $(A \sqcap B, p, q)$ where $A \sqcap B \in \text{obj}(\mathscr{C})$ and $p : A \sqcap B \longrightarrow A$, $q : A \sqcap B \longrightarrow B$ are morphisms (called *projections*) such that for all $X \in \text{obj}(\mathscr{C})$ and morphisms $f : X \longrightarrow A$, $g : X \longrightarrow B$ there exists a unique morphism $\theta : X \longrightarrow A \sqcap B$ such that $p\theta = f$ and $q\theta = g$.

Definition 1.2.5. A $A \in \text{obj}(\mathscr{C})$ is called a *terminal object* if for every object X in $\mathscr{C}$ there exists a unique morphism $\alpha : X \longrightarrow A$.

Definition 1.2.6. If an object is both initial and terminal, then it is called a *zero object* in its category.

Definition 1.2.7. Let $\mathscr{C}, \mathscr{D}$ be two additive categories. A functor $T : \mathscr{C} \longrightarrow \mathscr{D}$ is called *additive* if for every morphism f, g in $\mathscr{C}$ we have $T(f+g) = Tf + Tg$.

Lemma 1.2.8. *Let $\mathscr{C}$ be an additive category, and $M, A, B \in obj(\mathscr{C})$. Then $M \cong A \sqcap B$ if-f there exist morphisms $i : A \longrightarrow M$, $j : B \longrightarrow M$, $p : M \longrightarrow A$ and $q : M \longrightarrow B$ such that $pi = 1_A$, $qj = 1_B$, $pj = 0$, $qi = 0$ and $ip + jq = 1_M$. Furthermore, $A \sqcap B$ is a corpoduct with embeddings i, j, and so $A \sqcap B \cong A \sqcup B$.*

Definition 1.2.9. A morphism $u : B \longrightarrow C$ in a category $\mathscr{C}$ is called *monomorphism* (resp. *epimorphism*) if for all morphisms a, b we have $(au = ab) \longrightarrow (u = b)$ (resp. $(ua = ba) \longrightarrow (u = b)$).

Definition 1.2.10. If $u \in \text{Hom}_{\mathscr{C}}(A, B)$ with $\mathscr{C}$=additive, then the *kernel* of u is a morphism $\ker(u) : K \longrightarrow A$ such that $u\ker(u) = 0$ and for all morphisms $g : X \longrightarrow A$ with $ug = 0$ there exists a unique morphism $\theta \in \text{Hom}_{\mathscr{C}}(X, K)$ such that $\ker(u)\theta = g$.
We also define the *cokernel* of u to be a morphism $\text{coker}(u) : B \longrightarrow C$ such that $\text{coker}(u)u = 0$ and for all morphisms $h : B \longrightarrow Y$ with $hu = 0$ there exists a unique morphism $\theta \in \text{Hom}_{\mathscr{C}}(C, Y)$ such that $\theta\text{coker}(u) = h$.

Proposition 1.2.11. *Let $u \in Hom_{\mathscr{A}}(A, B)$ where $\mathscr{A}$ is additive. Then:*

- *If $ker(u)$ exists, then u is a monomorphism if-f $ker(u) = 0$*
- *If $coker(u)$ exists, then u is an epimorphism if-f $coker(u) = 0$*

Definition 1.2.12. An additive category $\mathscr{C}$ is called *abelian* if:

- every morphism in it has a kernel and a cokernel
- every monomorphism (resp. epimorphism) is a kernel (resp. cokernel)

Definition 1.2.13. Let $f : A \longrightarrow B$ be a morphism in an abelian category. We define its *image* to be: $\mathrm{im}(f) = \ker(\mathrm{coker}(f))$.

Remark 1.2.14. A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\ker(g) = \mathrm{im}(f)$.

Remark 1.2.15. We write $S.(X) = \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$ the singular complex of a topological space X.

Definition 1.2.16. A *chain complex* in an abelian category $\mathscr{A}$ is a sequence of objects and morphisms in $\mathscr{A}$ (called *differentials*)

$$(C.,d.) = \cdots \longrightarrow A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1} \longrightarrow \ldots$$

such that $d_n d_{n+1} = 0$ for all $n \in \mathbb{Z}$.
If $(C.,d.),(C.',d.')$ are chain complexes, then a *chain map* $f = f. : (C.,d.) \longrightarrow (C.',d.')$ is a sequence of morphisms $f_n : C_n \longrightarrow C_n'$ such that $d_{n+1}' f_{n+1} = f_n d_{n+1}$ for all $n \in \mathbb{Z}$.

Definition 1.2.17. If $\mathscr{A}$ is an abelian category, then the category of all of its complexes is denoted by $\mathrm{Comp}(\mathscr{A})$. In particular, if R is a ring, then we write ${}_R\mathrm{Comp} := \mathrm{Comp}({}_R\mathrm{Mod})$.

Definition 1.2.18. A complex $(A.,\delta.)$ is called a *sub-complex* of $(C.,d.)$ if there exists a chain map $i : A. \longrightarrow C.$ where every i_n is an isomorphism.

Proposition 1.2.19. *If $\mathscr{A}$ is an abelian category, then Comp$(\mathscr{A})$ is also an abelian category.*

Definition 1.2.20. A *projective resolution* of $A \in \mathrm{obj}(\mathscr{A})$ where $\mathscr{A}$ is abelian is an exact sequence $P = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$ where P_n are projective modules.
If $\mathscr{A}$ is the category of (left or right) R-modules and every P_n is free, then the resolution is also called free.

Remark 1.2.21. Consider the projective resolution of A above. Then $A \cong \mathrm{coker}(d_1)$.

Proposition 1.2.22. *Every left R-module has a free projective resolution.*

As a dual to the projective resolution, the *injective resolution* of A is an exact sequence $E = 0 \longrightarrow A \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \ldots$ where every E^n is injective. We also have the respective definition for a *free injective resolution* as well as the above proposition for a free injective resolution.

Definition 1.2.23. Given a projective resolution P of $A \in \mathrm{obj}(\mathscr{A})$ as in Definition 1.2.20 we define $K_0 = \ker(\varepsilon)$ and $K_n = \ker(d_n)$ for all $n \geq 1$. The K_n is called the *n-th syzygy* of P.
Dually, for an injective resolution E of A we define $V^0 = \mathrm{coker}(\eta)$ and $V^n = \mathrm{coker}(d^{n-1})$ for all $n \geq 1$. The V^n is called the *n-cosyzygy* of E.

Definition 1.2.24. If (C,d) is a complex in $\mathrm{Comp}(\mathscr{A})$, we define:

- the *n-th chain* to be the object C_n
- the *n-th cycle* to be $Z_n(C) := \ker(d_n)$
- the *n-th border* to be $B_n(C) := \mathrm{im}(d_{n+1})$
- the *n-th homology* to be $H_n(C) := Z_n(C)/B_n(C)$

Proposition 1.2.25. *If $\mathscr{A}$ is an abelian category, then the functor $H_n : Comp(\mathscr{A}) \longrightarrow \mathscr{A}$ is additive for all $n \in \mathbb{Z}$.*

Proposition 1.2.26. *Let $\mathscr{A}$ be an abelian category. If $0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0$ is an exact sequence in $Comp(\mathscr{A})$ then for every $n \in \mathbb{Z}$ there exists a morphism $\partial_n \in Hom_{\mathscr{A}}(H_n(C''), H_{n-1}(C'))$ such that $\partial_n(cls(z''_n) = cls(i_{n-1}^{-1} d_n p_n^{-1} z''_n)$.*

Definition 1.2.27. The morphisms ∂_n of the above proposition are called *connecting homomorphisms*.

Theorem 1.2.28. *Let $\mathscr{A}$ be an abelian category. If $0 \longrightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \longrightarrow 0$ is an exact sequence in $Comp(\mathscr{A})$, then there exists an exact sequence in $\mathscr{A}$:*

$$\cdots \longrightarrow H_{n+1}(C'') \xrightarrow{\partial_{n+1}} H_n(C') \xrightarrow{i_*} H_n(C) \xrightarrow{p_*} H_n(C'') \xrightarrow{\partial_n} H_{n-1}(C') \longrightarrow \dots$$

Corollary 1.2.29 (Snake's Lemma)**.** *Let $\mathscr{A}$ be an abelian category. If the following diagram with exact rows is commutative:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}$$

then there exists an exact sequence in $\mathscr{A}$:

$$0 \longrightarrow ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \longrightarrow coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$$

Definition 1.2.30. Let C,D be complexes, $p \in \mathbb{Z}$. A *map* $s : C \longrightarrow D$ *of degree* p is a sequence $s_n : C_n \longrightarrow D_{n+p}$.

Definition 1.2.31. Two chain maps $f,g : (C,d) \longrightarrow (C',d')$ are *homotopic*, we write $f \cong g$, if for all $n \in \mathbb{Z}$ there exists a map $s = (s_n)$ of degree $+1$ such that $f_n - g_n = d'_{n+1}s_n + s_N n - 1d_n$.

Theorem 1.2.32 (Comparison theorem)**.** *Let $\mathscr{A}$ be an abelian category, and $f \in Hom_{\mathscr{A}}(A,A')$. Consider the following diagram:*

$$\begin{array}{ccccccccccc} \longrightarrow & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\varepsilon} & A & \longrightarrow & 0 \\ & \downarrow{\check{f}_2} & & \downarrow{\check{f}_1} & & \downarrow{\check{f}_0} & & \downarrow{f} & & \\ \longrightarrow & P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\varepsilon'} & A' & \longrightarrow & 0 \end{array}$$

where the rows are complexes. If $\forall n \in \mathbb{Z}$ $(P_n = projective)$ and if the below row is exact, then there exists a chain map $\check{f} : P_A \longrightarrow P'_A$ such that the complete diagram is commutative. Furthermore, every two such chain maps are homotopic.

Definition 1.2.33. If $f : A \longrightarrow A'$ is a morphism, and P_A, P'_A are deleted projective resolutions of A, A' respectively, then a chain map $\check{f} : P_A \longrightarrow P'_A$ is called *over* f if $f\varepsilon = \varepsilon' \check{f}_0$.

1.3 Left derived functors

Let $T : \mathscr{A} \longrightarrow \mathscr{C}$ be an additive covariant functor between two abelian categories, where $\mathscr{A}$ has 'enough projectives'. We fix a projective resolution

$$P = \cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} A \longrightarrow 0$$

for every object A in $\mathscr{A}$. From the deleted resolution P_A we form the complex

$$TP_A = \cdots \longrightarrow T(P_2) \xrightarrow{T(d_2)} T(P_1) \xrightarrow{T(d_1)} T(P_0) \longrightarrow 0$$

For all $n \in \mathbb{Z}$ we define $(L_nT)A = H_n(TP_A)$.

Let $f : A \longrightarrow A'$ be a morphism. From the Comparison Theorem there exists a chain map $\check{f} : P_A \longrightarrow P'_A$ over f. Then $T\check{f} : TP_A \longrightarrow TP'_A$ is also a chain map. We define $(L_nT)f : (L_nT)A \longrightarrow (L_nT)A'$ such that $(L_nT)f = H_n(T\check{f}) = (T\check{f})_{n*}$.

Definition 1.3.1. The functors $L_nT : \mathscr{A} \longrightarrow \mathscr{C}$ in the above construction are called *left derived*.

Theorem 1.3.2. *The left derived functors of the above construction are additive covariant functors for all $n \in \mathbb{Z}$.*

Proposition 1.3.3. *We have $(L_nT)A = 0$ for all $n < 0$ and all $A \in obj(\mathscr{A})$.*

Definition 1.3.4. Let B be a left R-module, and $T = \Box \otimes_R B$. We define $Tor_n^R(\Box, B) = L_nT$. Similarly, if $T' = A \otimes_R \Box$, we define $tor_n^R(A, \Box) = L_nT'$.

Proposition 1.3.5. *Given two choices for the projective resolution of an object A, the respective left derived functors are naturally isomorphic.*

Corollary 1.3.6. *Let $A \in obj(\mathscr{A})$ and P as above. Put $K_0 = ker(\varepsilon), K_n = ker(d_n)$ for all $n \geq 1$. Then $(L_{n+1}T)A \cong (L_nT)K_0 \cong (L_{n-1}T)K_1 \cong \ldots \cong (L_1T)K_{n-1}$.*

Proposition 1.3.7 (Horshoe Lemma)**.** *Let*

$$
\begin{array}{ccccccccc}
 & & \downarrow & & & & \downarrow & & \\
 & & P_1' & & & & P_1'' & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & P_0' & & & & P_0'' & & \\
 & & \downarrow{\scriptstyle \varepsilon'} & & & & \downarrow{\scriptstyle \varepsilon''} & & \\
0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{q} & A'' & \longrightarrow & 0
\end{array}
$$

be a diagram in an abelian category $\mathscr{A}$ with 'enough projectives', whose columns are projective resolutions, and the last row is exact. Then there exists a projective resolution of A: $P = \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0$ and chain maps such that, when input into the diagram above, the three columns form an exact sequence of complexes.

Corollary 1.3.8. *Let $0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$ be a short exact sequence in $_RMod$. If A', A'' are finitely presented, then so is A.*

Corollary 1.3.9. *If $T : {_RMod} \longrightarrow_S Mod$ is an additive functor, then L_0T is right-exact.*

Theorem 1.3.10. *The following hold:*

- *If the additive functor $T : \mathscr{A} \longrightarrow \mathscr{B}$ is right-exact, and $\mathscr{A}, \mathscr{B}$ are additive categories with $\mathscr{A}$ having 'enough projectives', then T is naturally isomorphic to L_0T.*
- *For every $A_R, {_RB}$ modules, we have: $Tor_0^R(A, B) \cong A \otimes_R B \cong tor_0^R(A, B)$.*

Theorem 1.3.11. *Let $A_R, {}_R B$ be modules. Then for all $n \geq 0$ we have $Tor_n^R(A,B) \cong tor_n^R(A,B)$.*

Theorem 1.3.12 (Axioms for Tor)**.** *Let $(T_n : Mod_R \longrightarrow Ab)_{n \geq 0}$ be a sequence of additive covariant functors. If:*

- *for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ there exists an exact sequence with natural connecting homomorphisms:*

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\Delta_{n+1}} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\Delta_n} T_{n-1}(A) \longrightarrow \dots$$

- *T_0 is naturally isomorphic to some $\Box \otimes_R M$*

- *$T_n P = \{0\}$ for every projective right R-module P and $n \geq 1$*

then for all $n \geq 0$ the functor T_n is naturally isomorphic to $Tor_n^R(\Box, M)$.

1.4 Right derived functors

In a dual way to how we constructed the left derived functors, we can also construct the right derived functors. This time, we fix an injective resolution

$$E = 0 \longrightarrow B \xrightarrow{\eta} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} \cdots$$

of every object B in the abelian category $\mathscr{A}$ (which has enough injectives). From its deleted resolution E^B we can form the complex

$$TE^B = \cdots \longrightarrow T(E^2) \xrightarrow{T(d^2)} T(E^1) \xrightarrow{T(d^1)} T(E^0) \longrightarrow 0$$

for an additive covariant functor $T : \mathscr{A} \longrightarrow \mathscr{C}$ between abelian categories. Then, for all $n \in \mathbb{Z}$ we define $(R^n T)B = H^n(TE^B)$ and similarly for any homomorphism $f : B \longrightarrow B'$ we can define how these functors act on it: $(R^n T)f = (T\check{f})_{n_*}$.

The duals of the theorems presented in the last section also hold for right derived functors, that is the functor $R^n T : \mathscr{A} \longrightarrow \mathscr{C}$ is an additive covariant functor for all $n \in \mathbb{Z}$, we have $(R^n T)B = 0$ for all negative integers n and objects B, and the constructed right derived functors are unique in a sense that they are naturally isomorphic to any such derived functors beginning with a different injective resolution.

Next is the definition that will make the statement of Shelah's Singular Compactness Theorem more compact:

Definition 1.4.1. For $T = \mathrm{Hom}_R(A, \square)$, we define $Ext^n_R(A, \square) = R^n T$ for all integers n.

Continuing in the dual nature of our previous statements we have:

Corollary 1.4.2. *If $T : {}_R Mod \longrightarrow_S Mod$ is an additive functor, then $R^0 0T$ is left-exact.*

Theorem 1.4.3. *The following hold:*

- *If the additive functor $T : \mathscr{A} \longrightarrow \mathscr{C}$ is left-exact, and $\mathscr{A}, \mathscr{C}$ are additive categories with $\mathscr{A}$ having 'enough injectives', then T is naturally isomorphic to $R^0 T$.*
- *For all left R-modules A, B we have: $Hom_R(A, B) \cong Ext^0_R(A, B)$.*

And finally:

Theorem 1.4.4 (Axioms for Ext)**.** *Let $(F^n : {}_R Mod \longrightarrow Ab)_{n \geq 0}$ be a sequence of additive covariant functors. If:*

- *for every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ there exists an exact sequence with natural connecting homomorphisms:*

$$\cdots \longrightarrow F^{n-1}(C) \xrightarrow{\Delta_{n-1}} F^n(A) \longrightarrow F^n(B) \longrightarrow F^n(C) \xrightarrow{\Delta_n} F^{n+1}(A) \longrightarrow \dots$$

- *F^0 is naturally isomorphic to some $Hom_R(M, \square)$*
- *$F^n(E) = \{0\}$ for every injective left R-module E and $n \geq 1$*

then for all $n \geq 0$ the functor F^n is naturally isomorphic to $Ext^n_R(M, \square)$.

Chapter 2

Set Theory: Cardinals and Ordinals

Set Theory is the branch of mathematics that concerns itself with developing a framework upon which the rest of mathematics can build. It involves the basic notions of set and element, along with some axioms on how these interact with each other. Although there are a lot of different axiomatic systems, each with its pros and cons, the most common system and the one we will be working on is the so-called Zermelo-Fraenkel system with the Axiom of Choice (or ZFC for short). We will state the axioms here. However, our main objective is to present the tools necessary to Chapter 3. These are mainly the properties of ordered sets and of cardinals and ordinals. The interested reader can find a more detailed start to axiomatic Set Theory in [11].

2.1 Axioms of ZFC

It is in the author's belief that every mathematical text revolving around some Set Theory (and not only) should present the axiomatic system in which the work is being done. Here is the list of axioms of ZFC which we will follow:

1. **Axiom of Extensionality:** Two sets are equal if they have the same elements.
$$\forall x \forall y \big(\forall z (z \in x \iff z \in y) \implies x = y\big)$$
2. **Axiom of Regularity:** Every non-empty set x contains a member y such that x and y are disjoint sets.
$$\forall x \big(\exists a (a \in x) \implies \exists y (y \in x \land \neg \exists z (z \in y \land z \in x))\big)$$

3. **Axiom schema of Specification:** Let φ be any formula in the language of ZFC with all free variables among $x, z, w_1, \ldots, w_n$. Then the set $\{x \in z \mid \varphi(x)\}$ always exists.

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x \big(x \in y \iff ((x \in z) \wedge \varphi)\big)$$

4. **Axiom of Pairing:** If x and y are sets, then there exists a set which contains x and y as elements.

$$\forall x \forall y \exists z (x \in z \wedge y \in z)$$

5. **Axiom of Union:** For any set of sets F there is a set A containing every element that is a member of some member of F.

$$\forall F \exists A \forall Y \forall x \big((x \in Y \wedge Y \in F) \implies x \in A\big)$$

6. **Axiom schema of Replacement:** Let φ be any formula in the language of ZFC with all free variables among $x, y, A, w_1, \ldots, w_n$. If φ is a function of x, A is its domain and $\varphi(x)$ is always a set, then the image $\varphi(A)$ is contained in a set B.

$$\forall A \forall w_1, \ldots, \forall w_n \big(\forall x (x \in A \implies \exists ! y \varphi) \implies \exists B \forall x (x \in A \implies \exists y (y \in B \wedge \varphi))\big)$$

7. **Axiom of Infinity:** There exists an inductive [1] set X.

$$\exists X (X = \text{ inductive})$$

8. **Axiom of Power set:** For any set x there exists a set y which contains every subset of x.

$$\forall x \exists y \forall z \big(z \subseteq x \implies z \in y\big)$$

9. **Axiom of Choice:** Let A, B be non-empty sets, and $P \subseteq A \times B$ be a relation such that $\forall x \in A \big(\exists y \in B (xPy)\big)$. Then there exists a function $f : A \longrightarrow B$ such that $\forall x \in A (xPf(x))$.

[1] A set A is called inductive if:

- $\emptyset \in A$
- $\forall Y \in A (Y \cup \{Y\} \in A)$

2.2 Partial Orders and Linear Orders

By direct use of the axioms above, one can immediately prove some elementary statements (such as the proposition: *There is no set that contains every set*) and give basic definitions (such as that of the union, intersection, empty set and power set, some of which we have already used in the formulation of the axioms). We now move on to the concept of partial orders.

First, let us remind ourselves of the definition of the ordered pair (due to Kuratowski):

Definition 2.2.1. For every two objects x, y we define the *ordered pair* (x, y) as the set $\{\{x\}, \{x, y\}\}$.

For every sets A, B it is easy to prove that there exists a unique set Z such that

$$z \in Z \iff \exists x \in A \exists y \in B (z = (x, y))$$

This set is called the (cartesian) product of A, B and is denoted by $A \times B$.

Definition 2.2.2. Let A, B be non-empty sets. A *binary relation* R from A to B is a subset of their cartesian product $A \times B$.
If $(x, y) \in R$ we will simply write xRy.

Definition 2.2.3. A binary relation $\leq$ from a set A to itself is called a *partial order* if:

- $\forall x \in A (x \leq x)$ (reflexivity)
- $\forall x, y \in A \big((x \leq y \land y \leq x) \implies x = y\big)$ (antisymmetry)
- $\forall x, y, z \in A \big((x \leq y \land y \leq z) \implies x \leq z\big)$ (transitivity)

A set on which we have defined a partial order is called a *partially ordered set*. We will often write $(P, \leq)$ and talk about a *partially ordered space* (poset from now on), where $P \neq \emptyset$ is the underlying set, and $\leq$ is the partial order on this set.

Furthermore, if $x \leq y$ and $x \neq y$ then we simply write $x < y$.

Definition 2.2.4. A partial order $\leq$ on a set A is called *linear* if $\forall x, y \in A \big((x \leq y) \lor (y \leq x)\big)$, that is: every two elements of A are $\leq$-comparable.

Definition 2.2.5. Let $(P,\leq)$ be a poset, $S\subseteq P$ and $M\in P$. The element M is called:

- an *upper bound* of S if $\forall x\in S(x\leq M)$
- a *maximum* element of S if it is an upper bound of S and $M\in S$
- a *supremum* of S if it is the smallest of all upper bounds of S (so that M is an upper bound of S and if M' is an upper bound of S, then $M\leq M'$). We will write $M=\sup S$ (notice that the supremum of a set is unique).

We may give similar definitions for when all of the above $\leq$-inequalities are reversed, by turning "upper" into "lower", and "supremum" into "infimum".

Definition 2.2.6. Let $(P,\leq)$ be a poset and $S\subseteq P$. The set S is called a *chain* of P if every two elements in S are $\leq$-comparable (so that the restriction of $\leq$ on S is a linear order).

Definition 2.2.7. A poset $(P,\leq)$ is called *inductive* if every chain of P has a supremum.

The following definition lies in the core of the part of Set Theory that interests us in this essay. It will be referenced to time and time again (in silence, for the most part), and is essential to cardinals and ordinals.

Definition 2.2.8. Let $\leq$ be a partial order on a (non-empty) set P. Then $\leq$ is called a *well-order* if it is linear and every non-empty subset of P has a minimum element. Furthermore, $(P,\leq)$ will then be called a *well-ordered space*.
We will write 0_P to denote the space's minimum element (or simply 0 when there is no risk of confusion).

One can think of well-ordered spaces as consecutive line segments, each with a starting point (and possibly not an end point). This image can help with the visualization and understanding of the proofs, but as with all helping tools in mathematics, it should be taken with a grain of salt, and not be solely relied upon to "prove" something.

Definition 2.2.9. Let $(U,\leq)$ be a well-ordered space and $x\in U$ be an element of U that is not its maximum. We define the *successor* of x to be the element $S_U(x)=\min\{y\in U\mid x<y\}$.
The images of the function S_U are called successor elements of U, or simply successors.

Definition 2.2.10. Let $(U, \leq)$ be a well-ordered space and $x \in U$. If $x \neq 0$ and $x \notin S_U[U]$, then x is called a *limit point* of U. We denote this by $\text{Limit}(x)$.
We also write $\omega_U = \min\{x \in U \mid \text{Limit}(x)\}$.

Remark 2.2.11. The minimum element 0 of a well-ordered space is neither a successor nor a limit point.

Definition 2.2.12. Let $(U, \leq)$ be a well-ordered space and $I \subseteq U$. The set I is called an *initial segment* of U if $\forall x \in I(y \leq x \implies y \in I)$ (we also say that I is "downwards closed"). We write $I \sqsubseteq U$.
If $x \in U$, then the initial segment defined by x is the set $\text{seg}_U(x) = \{y \in U \mid y < x\}$.

Remark 2.2.13. Notice that $\text{seg}_U(S_U(x)) = \text{seg}_U(x) \cup \{x\}$.

Proposition 2.2.14. *Let $(U, \leq)$ be a well-ordered space, and $I \subseteq U$. Then $I \sqsubseteq U$ if-f $I = U$ or $\exists x \in U(I = seg_U(x))$.*

Proof. Obviously, if $I = U$ or $\exists x \in U(I = \text{seg}_U(x))$ then I is an initial segment of U. So let us suppose that I is an initial segment of U. If $I = U$, we have nothing to show. If $I \subset U$ then there exists a point $x_0 \in U$ with $x_0 \notin I$. Consider the set $P = \{z \in U \mid z \notin I\}$. This set is non-empty ($x_0 \in P$). Since the space $(U, \leq)$ is well-ordered, the set P has a least element, say x. We will show that $I = \text{seg}_U(x)$.

First we have that $x \notin I$, since $x = \min P \in P$. Now pick a $y < x$ in U. Then $y \notin P$, and so $y \in I$. We have just shown that $\text{seg}_U(x) \subseteq I$. Furthermore, I is an initial segment and $x \notin I$, therefore $\forall y \in U(y > x \implies y \notin I)$. This shows that $I = \text{seg}_U(x)$. □

Definition 2.2.15. Let $(P, \leq_P), (Q, \leq_Q)$ be two posets, and $\pi : P \longrightarrow Q$ be a function. We say that π is an *order-embedding* if $\forall x, y \in P\big(x \leq_P y \iff \pi(x) \leq_Q \pi(y)\big)$.

Remark 2.2.16. Obviously, an order-embedding is one-to-one.

Definition 2.2.17. Let $(P, \leq_P), (Q, \leq_Q)$ be two posets, and $\pi : P \longrightarrow Q$ be a function. We say that π is an *order-isomorphism* if it is one-to-one, onto and order-embedding.
Two posets $(P, \leq_P), (Q, \leq_Q)$ are called *order-isomorphic* if there exists an order-isomorphism $\pi : P \longrightarrow Q$. We write $P =_o Q$.

The following proposition allows us to think of the order-isomorphic "relation" as some sort of an equivalence relation between posets.

Proposition 2.2.18. *The following hold:*

1. *For every poset $(P, \leq)$ we have $P =_o P$*
2. *For every two posets $(P, \leq_P), (Q, \leq_Q)$ we have $P =_o Q \implies Q =_o P$*
3. *For every three posets $(P, \leq_P), (Q, \leq_Q), (R, \leq_R)$ we have $(P =_o Q \wedge Q =_o R) \implies P =_o R$.*

Proof. We will prove the 3 points in order.
1) Obviously, the identity function is always an order-isomorphism, so $P =_o P$.
2) Let $\pi : P \longrightarrow Q$ be an order-isomorphism between the two posets. Then its inverse function $\pi^{-1} : Q \longrightarrow P$ is easily seen to be an order-isomorphism as well.
3) Let $\pi : P \longrightarrow Q$, $\rho : Q \longrightarrow R$ be two order-isomorphisms. Then their composition $\rho \circ \pi : P \longrightarrow R$ is easily seen to be an order-isomorphism as well. □

Proposition 2.2.19. *Let $(P, \leq_P), (Q, \leq_Q)$ be two order-isomorphic posets, $P =_o Q$. If $(Q, \leq_Q)$ is a well-ordered space, then so is $(P, \leq_P)$.*

Proof. Pick any two elements $x, y \in P$. Then their images under the order-isomorphism $\pi : P \longrightarrow Q$ are $\leq_Q$-comparable, say $\pi(x) \leq_Q \pi(y)$. By definition of the order-isomorphism, we also have $x \leq_P y$, so x, y are $\leq_P$-comparable. This makes the ordering on P a linear order.

Now consider a non-empty set $S \subseteq P$. Then its image $\pi[S] \subseteq Q$ is also non-empty, and since $(Q, \leq_Q)$ is well-ordered, the set $\pi[S]$ has a minimum element. Put $y = \min \pi[S]$. Then there exists $x \in S$ such that $\pi(x) = y$. This x is also the minimum element of S; indeed, pick a $z \in S$. Then $\pi(z) \in \pi[S]$ and so $\pi(z) \geq_Q y = \pi(x)$, and by definition of the order-isomorphism we get $x \leq_P z$. This proves that the linear order on P is a well-order. □

Definition 2.2.20. Let $(P, \leq)$ be a poset, and $\pi : P \longrightarrow P$ be a function. Then π is called an *extension* if $\forall x \in P (x \leq \pi(x))$.

Theorem 2.2.21. *Let $(U, \leq)$ be a well-ordered space and $\pi : U \longrightarrow U$ be an order-embedding. Then π is also an extension.*

Proof. Suppose that there exists a point $x \in U$ such that $\pi(x) < x$. Put $A = \{y \in U \mid \pi(y) < y\}$, and notice that $x \in A$. Since $(U, \leq)$ is well-ordered, A must have a minimum element, say x_0. Obviously, $x_0 \in A$ and so $\pi(x_0) < x_0$. But π is an order-embedding, and so $\pi(\pi(x_0)) < \pi(x_0)$. This dictates that $\pi(x_0) \in A$, which is a contradiction (since $\pi(x_0) < x_0 = \min A$). □

This theorem has the following important consequence:

Corollary 2.2.22. *No well-ordered space $(U, \leq)$ is order-isomorphic with a proper initial segment $I \sqsubset U$.*

Proof. Suppose $(U, \leq)$ is a well-ordered space which has an order-isomorphic proper initial segment $I \sqsubset U$, and consider an order-isomorphism $\pi : U \longrightarrow I$. Put $A = \{y \in U \mid y \notin I\}$. This set is non-empty (since I is a proper subset of U), therefore it must have a minimum element, say $x = \min A$. By Theorem 2.2.21 we have $x \leq \pi(x)$. But $\pi(x) \in \pi[U] = I$ and I is an initial segment, so $x \in I$, a contradiction. □

Theorem 2.2.23 (Transfinite Induction)**.** *Let $(U, \leq)$ be a well-ordered space and P be a formula with one free variable. Suppose that $\forall y \in U\big(\forall x < y(P(x)) \implies P(y)\big)$. Then it is true that $\forall y \in U\big(P(y)\big)$.*

Proof. Suppose to the contrary that there is a $x_0 \in U$ such that $P(x_0)$ does not hold. Then the set $A = \{y \in U \mid \neg P(y)\}$ is a non-empty subset of U, therefore it has a minimum element $y^* = \min A$. Now $\forall x \in U(x < y^* \implies x \notin A)$, hence $\forall x \in U(x < y^* \implies P(x))$. By the assumption of the theorem we get that $P(y^*)$ holds, which is a contradiction (since $y^* \in A$). □

Definition 2.2.24. Let $(U, \leq)$ be a well-ordered space. The *successor space* of U is the well-ordered space $\mathrm{Succ}(U) = U \cup \{t_U\}$ where t_U is an element that does not belong to U [2], and where we define its ordering $\leq$ by the following:

$$x \leq y \text{ if-f } (x, y \in U \wedge x \leq y) \vee (x \in U \wedge y = t_U) \vee (x = y = t_U)$$

Remark 2.2.25. The successor space of a space U is unique up to order-isomorphism.

Remark 2.2.26. For every well-ordered space U we have $U = \mathrm{seg}_{\mathrm{Succ}(U)}(t_U) \sqsubset \mathrm{Succ}(U)$.

Closely related to induction is recursion. Similarly from what we ordinarily know from the natural numbers, we can state the transfinite recursion theorem (the proof of which we will omit for brevity reasons).

Theorem 2.2.27 (Transfinite Recursion)**.** *Let U be a well-ordered space, $A \subseteq U$, $E \neq \emptyset$ a set and $h : (A \longrightarrow E) \longrightarrow E$ a function [3]. Then there exists a unique function $f : U \longrightarrow E$ such that $f(x) = h(f|_{seg(x)})$ for all $x \in U$.*

[2]Such an element always exists; consider for example the set $r(U) = \{x \in U \mid x \notin x\}$

[3]By $(A \longrightarrow E)$ we denote the set of all functions from A to E

Definition 2.2.28. Let U,V be well-ordered spaces and $\pi : U \longrightarrow V$ be a function. Then π is called an *initial order-isomorphism* if it is an order-embedding and $\pi[U] \sqsubseteq V$. We will write $U \leq_o V$ whenever an initial order-isomorphism from U to V exists.

Lemma 2.2.29. *The composite of two initial order-isomorphisms is again an initial order-isomorphism.*

Proof. We already know that the composite of two order-embeddings is an order embedding. It is also easy to see that the image of the composite is an initial segment of the target space. □

Proposition 2.2.30. *For all well-ordered spaces U,V,W we have:*

1. $U \leq_o U$
2. $(U \leq_o V) \wedge (V \leq_o W) \implies U \leq_o W$
3. $(U \leq_o V) \wedge (V \leq_o U) \implies U =_o V$

Proof. For (1), consider the identity function on U. For (2), use Lemma 2.2.29. For (3), again use the previous lemma and remember Corollary 2.2.22. □

The next theorem is quite intuitive, however its proof requires some work.

Theorem 2.2.31. *Let U,V be two well-ordered spaces and $\pi : U \longrightarrow V$ be a function. Then π is an initial order-isomorphism if-f* $\forall x \in U \left[\pi(x) = \min\{y \in V \mid \forall u \in U(u < x \implies \pi(u) < y)\}\right]$

Proof. $\underline{\implies}$) For every $x \in U$ define $A = \{y \in V \mid \forall u \in U(u < x \implies \pi(u) < y)\}$ (which is non-empty, since $\pi(x) \in A$) and put $z = \min A$. Obviously $z \leq \pi(x)$, but suppose that $z < \pi(x)$. Since $\pi[U] \sqsubseteq V$ we get $z \in \pi[U]$. This means that $\exists u \in U(z = \pi(u))$. Moreover, since π is an order-embedding, we have $\pi(u) < \pi(x) \iff u < x$. However $z \in A$ implies that $\pi(u) < z$, a contradiction.
$\underline{\impliedby}$) Pick an $x \in U$ and consider a $u \leq x$. If $u = x$ then obviously $\pi(u) \leq \pi(x)$. If $u < x$, then $\pi(u) < \pi(x) = \min\{y \in V \mid \forall z \in U(z < x \implies \pi(z) < y)\}$. So $u \leq x \implies \pi(u) \leq \pi(x)$.
On the other hand, consider a $u \in U$ such that $\pi(u) \leq \pi(x)$. Then we cannot have $x < u$, because $\pi(x) < \pi(u) = \min\{y \in V \mid \forall z \in U(z < u \implies \pi(z) < y)\}$. So $\pi(u) \leq \pi(x) \implies u \leq x$. We have proven that π is an order-embedding. It remains to show that $\pi[U] \sqsubseteq V$.

Pick any $x \in U$ and consider any $z \in V$ with $z < \pi(x)$. Suppose that $z \notin \pi[U]$. Then $z \notin \{y \in V \mid \forall u \in U(u < x \implies \pi(u) < y)\}$, and so there exists a $u < x$ such that $\pi(u) \geq z$. Consider the least such $u \in U$. By $z \notin \pi[U]$ we get $z \neq \pi(u)$, which means that $z < \pi(u)$. However $\pi(u) = \min\{y \in V \mid \forall v \in U(v < u \implies \pi(v) < y)\}$, and so there exists $v < u$ with $\pi(v) \geq z$. This contradicts the minimality of u. □

The important connection between well-ordered spaces and order relations becomes evident in the following theorem:

Theorem 2.2.32. *For every well-ordered space U and V we either have $U \leq_o V$ or $V \leq_o U$.*

Proof. If $V = \emptyset$ then obviously $V \leq_o U$. Let V be a non-empty set and 0_V be its minimum element. Also, put $A_x = \{y \in V \mid \forall u \in U(u < x \implies \pi(u) < y)\}$ for every $x \in U$. By transfinite recursion (Theorem 2.2.27) we may define a function $\pi : U \longrightarrow V$ such that

$$\pi(x) = \begin{cases} \min A_x & \text{, if } A_x \neq \emptyset \\ 0_v & \text{, otherwise} \end{cases}$$

Consider the following two cases:
1) $\forall x \in U(x \neq 0_U \implies \pi(x) \neq 0_V)$: Then $\forall x \in U(\pi(x) = \min A_x)$, which implies that π is an initial order-isomorphism (due to the last theorem), giving $U \leq_o V$.
2) $\exists a \in U(a \neq 0_U \wedge \pi(a) = 0_v)$: Consider the minimum such $a \in U$ with this property, and put $\rho = \pi|_{\text{seg}_U(a)}$. Then $\forall x < a(\rho(x) = \pi(x) = \min A_x)$, meaning that ρ is an initial order-isomorphism (again due to the last theorem). So $\rho[\text{seg}_U(a)] \sqsubseteq V$.
Suppose that $\rho[\text{seg}_U(a)] \neq V$. Then there exists $z \in V$ such that $\rho[\text{seg}_U(a)] = \text{seg}_V(z)$. Since $a \neq 0_U$ we get $\text{seg}_U(a) \neq \emptyset$ and so $\text{seg}_V(z) \neq \emptyset$, giving $z \neq 0_V$. So, by $\forall x < a(\pi(x) \in \pi[\text{seg}_U(a)] = \text{seg}_V(z))$ we have $\forall x < a(\pi(x) < z)$. Moreover, z is the minimum element with this property, and so $z = \pi(a) \neq 0_V$. This is a contradiction of the hypothesis of case (2). This proves that $\rho[\text{seg}_U(a)] = V$, meaning $V =_o \text{seg}_U(a) \leq_o U$. □

Corollary 2.2.33. *Let $\mathscr{E}$ be a non-empty class of well-ordered spaces. Then there exists a $\leq_o$-minimum element in $\mathscr{E}$, meaning a space U_0 such that $U_0 \leq_o U$ for all $U \in \mathscr{E}$.*

Proof. Pick any $W \in \mathscr{E}$. If W satisfies $W \leq_o U$ for every $U \in \mathscr{E}$ then we are done. Otherwise, there exists some space which is order-isomorphic to some proper initial segment of W. Consider the minimum element $a \in W$ for which there exists a space $U \in \mathscr{E}$ such that $U =_o \text{seg}_W(a)$, and put $U_0 = \text{seg}_W(a)$. Due to Theorem 2.2.32 for every $U \in \mathscr{E}$ we either have $U <_o U_0$ or $U_0 \leq_o U$. However we cannot have $U <_o U_0$, for that would mean that there exists a $a' < a$ in W such that $U =_o \text{seg}_W(a')$, contradicting the minimality of a. □

2.3 Ordinals

We have done most of the work required to make an introduction to what ordinals are and how we can do arithmetic on them. We are only missing a theorem about well-defined operators (or *class functions*) analogous to transfinite recursion, which we will state (but not prove) here.

Theorem 2.3.1. *Let H be a well-defined operator with one free variable, and U be any well-ordered space. Then there exists a unique set B and a unique function $f : U \longrightarrow f[U] = B$ such that $f(x) = H(f|_{seg_U(x)})$ for all $x \in U$.*

Consider the operator H defined by: $H(w) = \begin{cases} \text{Image}(w) & , w = \text{ function} \\ \emptyset & , \text{otherwise} \end{cases}$. Let U be a well-ordered space and apply Theorem 2.3.1. There exists a unique function $v_U : U \longrightarrow v_U[U]$ such that $v_U(x) = H(v_U|_{\text{seg}_U(x)})$ for all $x \in U$.

Definition 2.3.2. We say that the unique v_U defined above is the *von Neumann surjection* of the well-ordered space U.
We define the *ordinal number* of U to be the set $\text{ord}(U) = v_U[U] = \{v_U(x) \mid x \in U\}$. Also, any set α for which there exists a well-ordered space U such that $\alpha = \text{ord}(U)$ will be called an ordinal number. We write ON to denote the class of all ordinal numbers.

Remark 2.3.3. This rather strange definition becomes more understood with the following remarks:

- Since $\emptyset$ is a well-ordered space, the set $0 := \emptyset$ is an ordinal number with $\text{ord}(0) = 0$ (it is in fact the smallest ordinal, as we will later see).
- The set $1 := 0 \cup \{0\} = \{\emptyset\}$ is an ordinal number with $\text{ord}(1) = 1$.
- The set $2 := 1 \cup \{1\} = \{\emptyset, \{\emptyset\}\}$ is an ordinal number with $\text{ord}(2) = 2$.
- More generally, we can define the set $n^+ := n \cup \{n\}$ for every set n. If n is an ordinal, then so is n^+ with $\text{ord}(n^+) = n^+$. This way, we have constructed the *von Neumann natural numbers*, the set of which we denote by $\omega = \{0, 1, 2, 3, \dots\}$.
- The set ω of all natural numbers is an ordinal with $\omega = \text{ord}(\omega)$.
- $\text{ord}(\omega \cup \{\omega\}) = \omega \cup \{\omega\}$

Proposition 2.3.4. *Let U, V be well-ordered spaces and $\pi : U \longrightarrow V$ be an initial order-isomorphism. Then $v_V(\pi(x)) = v_U(x)$ for all $x \in U$.*

Proof. Suppose that there exists some point in U for which the conclusion does not hold. Let x be the least such point, so that $v_V(\pi(x)) \neq v_U(x)$. By definition of the von Neumann surjection, the fact that π is an initial order-isomorphism and the choice of x we have:

$$\begin{aligned} v_V(\pi(x)) &= \{v_V(y) \mid y <_V \pi(x)\} \\ &= \{v_V(\pi(t)) \mid t <_U x\} \\ &= \{v_U(t) \mid t <_U x\} \\ &= v_U(x) \end{aligned}$$

which is a contradiction. □

Proposition 2.3.5. *Let U be a well-ordered space, $x \in U$ and put $W = seg_U(x)$. Then W is a well-ordered space with $ord(W) = v_U(x)$.*

Proof. Apply Proposition 2.3.4 to W and U with π being the identity function. For every $y <_U x$ we get that $v_U(y) = v_U(\pi(y)) = v_W(y)$, and so:

$$v_U(x) = \{v_U(y) \mid y <_U x\} = \{v_W(y) \mid y <_U x\} = \text{ord}(W)$$

□

Corollary 2.3.6. *For every ordinal number α:*

- *Every element of α is an ordinal number*
- *There exists an ordinal β such that $\alpha \in \beta$*

Proof. Pick an element $\gamma \in \alpha$. Then there exists $x \in \alpha$ such that $\gamma = v_\alpha(x)$, which is an ordinal number as Proposition 2.3.5 shows.

Now put $\beta = \alpha^+ = \alpha \cup \{\alpha\}$. As in Remark 2.3.3 we get that β is an ordinal. □

Lemma 2.3.7. *Let U be a well-ordered space, $\alpha = ord(U)$ and $v : U \longrightarrow \alpha$ be the von Neumann surjection of U. Then $\forall x, y \in U (x < y \iff v(x) \in v(y))$.*

Proof. Obviously, if $x < y$ then $v(y) = \{v(z) \mid z < y\}$ by definition and so $v(x) \in v(y)$. Now consider two elements $x, y \in U$ such that $v(x) \in v(y)$. By definition, there exists $z < y$ such that $v(x) = v(z)$. If $x < z$, then $v(x) \in v(z) = v(x)$ by the above argument, a contradiction. If $x > z$, then we arrive at the same contradiction. All that remains is $x = z < y$, concluding the proof. □

Proposition 2.3.8. *Let U be a well-ordered space, $\alpha = ord(U)$ and $v : U \longrightarrow \alpha$ be the von Neumann surjection of U. Then α is well-ordered by the relation $u \leq_\alpha v$ if-f $u = v$ or $u \in v$.*

Proof. Using Lemma 2.3.7 we can easily see that this is indeed a linear ordering on α. It is also a well-order. Indeed, for any non-empty set $S \subseteq \alpha$ consider its pre-image under the von Neumann surjection $v^{-1}[S]$, which is a non-empty subset of the well-ordered set U. Pick x to be the minimum element of $v^{-1}[S]$. Then $v(x)$ is the minimum element of S. □

Corollary 2.3.9. *For every well-ordered space U it holds: $U = ord(U)$.*

Proof. Put $\alpha = \mathrm{ord}(U)$ and $v : U \longrightarrow \alpha$ to be the von Neumann surjection of U. Lemma 2.3.7 and Proposition 2.3.8 give us $x \leq_U y \iff v(x) \leq_\alpha v(y)$ (so v is an order-embedding) and one-to-one. Furthermore, it is also onto by definition, and so the von Neumann surjection is an order-isomorphism. □

The proof of the next theorem is now immediate:

Theorem 2.3.10. *Let U, V be well-ordered spaces with $U \leq_o V$. Then $ord(U) \sqsubseteq ord(V)$.*

Also immediate is the following:

Corollary 2.3.11. *Let U, V be well-ordered spaces. Then $U =_o V$ if-f $ord(U) = ord(V)$.*

We are ready to prove the most useful theorem in visualizing ordinals so far.

Theorem 2.3.12. *For every ordinal α we have: $\alpha = \{\beta \in ON \mid \beta <_o \alpha\}$*

Proof. Let U be a well-ordered space such that $\alpha = \mathrm{ord}(U)$. By Proposition 2.3.5 we get that $\alpha = \{v_U(x) \mid x \in U\} = \{\mathrm{ord}(\mathrm{seg}_U(x)) \mid x \in U\}$. Since every well-ordered space that is $<_o U$ is order-isomorphic to an initial segment of U, this equation gives us the desired result. □

Remark 2.3.13. It is easy to see that for any two ordinals α, β we have: $\alpha =_o \beta \iff \alpha = \beta$.

We can extend the previous remark to a lemma, which will then give us the much-desired definition of a well-ordering in the ON class.

Lemma 2.3.14. *For every two ordinals α, β the following statements are equivalent:*

- $\alpha \leq_o \beta$
- $\alpha = \beta$ *or* $\alpha \in \beta$
- $\alpha \sqsubseteq \beta$
- $\alpha \subseteq \beta$

Proof. Combine the previous theorem and remark. □

Definition 2.3.15. We define the (well) *ordering in the class ON* as follows: $\alpha \leq \beta \iff \alpha \leq_o \beta$.

Proposition 2.3.16. *For every two ordinals α, β we have exactly one of the following:*

- $\alpha < \beta$
- $\alpha = \beta$
- $\beta < \alpha$

Proof. Since ordinals are themselves well-ordered spaces, this result is immediate from Theorem 2.2.32. □

Proposition 2.3.17. *Let $\mathscr{E}$ be a non-empty class of ordinals. Then $\mathscr{E}$ has a $\leq$-least element.*

Proof. Remember Corollary 2.2.33. □

This fact allows us to define the "next" ordinal of any ordinal number.

Definition 2.3.18. Let $\alpha \in \mathrm{ON}$. We define $\alpha^+ := s(\alpha) := \min\{\beta \in \mathrm{ON} \mid \alpha < \beta\}$ to be *the successor ordinal of α*.

This definition falls in-line with the successor of a natural number as this was defined in Remark 2.3.3, since one easily proves that $s(\alpha) = \alpha \cup \{\alpha\}$.

Definition 2.3.19. Let A be a set of ordinals. We define the *least upper bound of* A to be the ordinal number $\sup A := \min\{\beta \in \mathrm{ON} \mid \forall \alpha \in A(\alpha \leq \beta)\}$.

Proposition 2.3.20. *For every set of ordinals A we have:* $\sup A = \bigcup A$.

Proof. Obviously any $\alpha \in A$ is a subset of $\bigcup A$. Additionally, $\bigcup A$ is an ordinal number, which gives us $\alpha \leq \bigcup A$ for every member of A. Therefore, $\bigcup A$ is an upper bound of A, and $\sup A \leq \bigcup A$. But now consider any $\beta \in \bigcup A$. Then there exists an $\alpha \in A$ such that $\beta \in \alpha$, or equivalently: $\beta < \alpha$, giving us $\beta < \sup A \iff \beta \in \sup A$, and so $\bigcup A \subseteq \sup A \iff \bigcup A \leq \sup A$. □

Next, a classification of the ordinals:

Definition 2.3.21. Let α be an ordinal number. Then:

- α is called *a successor ordinal* if there exists a $\beta \in \mathrm{ON}$ such that α is the successor of β (as in Definition 2.3.18).
- α is called *a limit ordinal* if it is not zero nor a successor ordinal.

This classification of the ordinals into three categories (one containing only the zero ordinal, one containing the successor ordinals, and one containing the limit ordinals) is essential for inductive proofs on the class ON. The following proposition gives us a straight-forward way to handle limit ordinals:

Proposition 2.3.22. *Let* $\lambda \in ON$*. Then* λ *is a limit ordinal if-f it is not zero and* $\lambda = \sup\{\alpha \in ON \mid \alpha < \lambda\}$.

Proof. If λ is a successor ordinal, say $\lambda = s(\beta)$, then $\sup\{\alpha \in \mathrm{ON} \mid \alpha < \lambda\} = \beta < \lambda$. On the other hand, if λ is a limit ordinal, then for every $\alpha < \lambda$ there exists a β such that $\alpha < \beta < \lambda$. This means that $\bigcup \lambda = \lambda$, and by Proposition 2.3.20 and Theorem 2.3.12 we get the desired result. □

We can now give an example of a limit ordinal: define ω_1 to be the smallest uncountable ordinal number. Then ω_1 is a limit ordinal.

Let us now present the theorems regarding induction and recursion on the class of ordinal numbers. However, we shall skip over their proofs for brevity reasons.

Theorem 2.3.23 (Induction on the Ordinals)**.** *Let P be a formula with one free variable. If* $\forall \alpha \in ON\big(\forall \xi < \alpha(P(\xi)) \implies P(\alpha)\big)$*, then* $\forall \alpha \in ON(P(\alpha))$.

Theorem 2.3.24 (Recursion on the Ordinals)**.** *For every well-defined operator H with one free variable there exists a well-defined operator F with one free variable such that $F(\alpha) = H(F|_\alpha)$ for all ordinal numbers α.*

2.3.1 Ordinal Arithmetic

Finally, we have all the tools necessary to establish arithmetic on ordinals! In this section we shall define what addition and multiplication of ordinals is, and discuss their most useful properties, which we will need in Chapter 3 of this essay. Let us begin with the first operation: addition.

Definition 2.3.25. Let α, β be ordinals, and E, F be well-ordered spaces such that $\mathrm{ord}(E) = \alpha$, $\mathrm{ord}(F) = \beta$ and $E \cap F = \emptyset$. Equip the union $E \cup F$ with the ordering [4]:

$$x \leq y \quad \text{if-f} \quad (x \in E \wedge y \in F) \text{ or } (x \leq_E y) \text{ or } (x \leq_F y)$$

We write $E +_o F$ for the well-ordered space $(E \cup F, \leq)$ and call it *the (ordinal) sum of* E, F. We then define:

$$\alpha + \beta = \mathrm{ord}(E +_o F)$$

A straightforward application of this definition should suffice to prove the following:

Proposition 2.3.26. *For all ordinals α, β, γ we have:*

- $\alpha + 0 = 0 + \alpha = \alpha$
- $\alpha + 1 = s(\alpha)$
- $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

Remark 2.3.27. Instead of the previous definition for the addition of ordinals, we could have equivalently defined the ordinal sum recursively:
For all ordinals α, β, λ we define $\alpha + 0 = \alpha$, $\alpha + s(\beta) = s(\alpha + \beta)$ and $\alpha + \lambda = \sup\{\alpha + \gamma \mid \gamma < \lambda\}$ whenever λ is a limit ordinal.

[4] Essentially, we are placing the elements of E before the elements of F, preserving the ordering of each space respectively. This is easily shown to be a well-order in $E \cup F$.

Let us now present ordinal multiplication in a similar manner, before moving on to more advanced properties of these operations.

Definition 2.3.28. Let α, β be ordinals, and A, B be well-ordered spaces such that $\text{ord}(A) = \alpha$ and $\text{ord}(B) = \beta$. Equip the product $A \times B$ with the ordering [5]:

$$(x_1, y_1) \leq (x_2, y_2) \quad \text{if-f} \quad (y_1 <_B y_2) \text{ or } (y_1 = y_2 \text{ and } x_1 \leq_A x_2)$$

We write $A \cdot_o B$ for the well-ordered space $(A \times B, \leq)$ and call it *the (ordinal) product of* A, B. We then define:

$$\alpha \cdot \beta = \text{ord}(A \cdot_o B)$$

Again, apply the above definition to proove:

Proposition 2.3.29. *For all ordinals* α, β, γ *we have:*

- $\alpha \cdot 0 = 0 \cdot \alpha = 0$
- $\alpha \cdot 1 = 1 \cdot \alpha = \alpha$
- $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$

Remark 2.3.30. Equivalently, we could have defined multiplication of ordinals recursively:
For all ordinals α, β, λ we define: $\alpha \cdot 0 = 0$, $\alpha \cdot s(\beta) = \alpha \cdot \beta + \alpha$ and $\alpha \cdot \lambda = \sup\{\alpha \cdot \gamma \mid \gamma < \lambda\}$ whenever λ is a limit ordinal.

Addition and multiplication have certain properties regarding preservation of ordinal inequalities. One should treat them with care however, as the order of the ordinals is important!

Lemma 2.3.31. *For all ordinals* α, β, γ *the following hold:*

- $\alpha \leq \beta \implies \alpha + \gamma \leq \beta + \gamma$
- $\alpha < \beta \implies \gamma + \alpha < \gamma + \beta$
- $\alpha + \beta = \alpha + \gamma \implies \beta = \gamma$

[5] Essentially, we are creating a latice with α rows and β columns, considering the lowest element to be the one that's most left, or the one that's most up in case the two elements are on the same column. This is easily shown to be a well-order in $A \cdot_o B$.

Proof. The proof will be an inductive one. We shall do the first point, the rest are left as an exercise. Keep in mind what we already know for addition from Proposition 2.3.26 and Remark 2.3.27.
Let $\alpha \leq \beta, \gamma$ be ordinals.
$\circ\gamma = 0$: Then $\alpha + \gamma = \alpha + 0 = \alpha \leq \beta = \beta + 0 = \beta + \gamma$.
$\circ\gamma = s(\delta)$: Then $\alpha + \gamma = \alpha + s(\delta) = s(\alpha + \delta) \leq s(\beta + \delta) = \beta + s(\delta) = \beta + \gamma$, where the inequality stems from the inductive hypothesis that $\alpha + \delta \leq \beta + \delta$.
$\circ\gamma$ is a limit ordinal: Then $\alpha + \gamma = \sup\{\alpha + \delta \mid \delta < \gamma\} \leq \sup\{\beta + \delta \mid \delta < \gamma\} = \beta + \gamma$, where the inequality again stems from our inductive hypothesis. □

Lemma 2.3.32. *For all ordinals α, β, γ the following hold:*

- $\alpha \leq \beta \implies \alpha \cdot \gamma \leq \beta \cdot \gamma$
- $(1 \leq \alpha) \wedge (\beta < \gamma) \implies \alpha \cdot \beta < \alpha \cdot \gamma$
- $(\alpha \cdot \beta = \alpha \cdot \gamma) \wedge (\alpha \neq 0) \implies \beta = \gamma$

Proof. Similar to that of Lemma 2.3.31. □

The following two lemmas are stated without their proofs. They will prove useful later on.

Lemma 2.3.33. *Let S be a non-empty set of ordinal numbers, and $\alpha \in ON$. Then $\alpha + supS = sup\{\alpha + \beta \mid \beta \in S\}$.*

Lemma 2.3.34. *Let S be a non-empty set of ordinal numbers, and $\alpha \in ON$. Then $\alpha \cdot supS = sup\{\alpha \cdot \beta \mid \beta \in S\}$.*

The next proposition is a left distribution law. Notice that the right distribution law does not hold for ordinal arithmetic.

Proposition 2.3.35. *For all ordinals α, β, γ we have: $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$.*

Proof. The proof is by induction on γ. Let α, β be two ordinals.
$\circ\gamma = 0$: Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + \alpha \cdot 0 = \alpha \cdot \beta + \alpha \cdot \gamma$.
$\circ\gamma = s(\delta)$: Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot (\beta + s(\delta)) = \alpha \cdot s(\beta + \delta) = s(\alpha \cdot (\beta + \delta)) = s(\alpha \cdot \beta + \alpha \cdot \delta) = \alpha \cdot \beta + s(\alpha \cdot \delta) = \alpha \cdot \beta + \alpha \cdot s(\delta) = \alpha \cdot \beta + \alpha \cdot \gamma$.
$\circ\gamma$ is a limit ordinal: Then $\alpha \cdot (\beta + \gamma) = \alpha \cdot \sup\{\beta + \delta \mid \delta < \gamma\} = \sup\{\alpha \cdot (\beta + \delta) \mid \delta < \gamma\} = \sup\{\alpha \cdot \beta + \alpha \cdot \delta \mid \delta < \gamma\} = \alpha \cdot \beta + \sup\{\alpha \cdot \delta \mid \delta < \gamma\} = \alpha \cdot \beta + \alpha \cdot \gamma$. □

Theorem 2.3.36 (Subtraction)**.** *For any $\alpha, \gamma \in ON$ with $\alpha \leq \gamma$, there exists a unique ordinal β such that $\alpha + \beta = \gamma$.*

Proof. Consider the class $\Delta = \{\delta \in \text{ON} \mid \alpha + \delta \leq \gamma\}$. Then Δ is a non-empty set, since $0 \in \Delta$ and $\Delta \subseteq s(\gamma)$. Put $\beta = \sup\Delta$. Then $\alpha + \beta = \alpha + \sup\Delta = \sup\{\alpha + \delta \mid \delta \in \Delta\} \leq \gamma$. Suppose now that $\alpha + \beta < \gamma$. Then $\alpha + s(\beta) = s(\alpha + \beta) \leq \gamma$, and so $s(\beta) \in \Delta$, a contradiction. Therefore $\alpha + \beta = \gamma$. If β_0 is another ordinal with the property $\alpha + \beta_0 = \gamma$, then $\alpha + \beta_0 = \alpha + \beta \implies \beta_0 = \beta$ by Lemma 2.3.31. Hence, β is unique. □

Definition 2.3.37. Let α, β be ordinals with $\alpha \neq 0$. We define the power α^β recursively:

- $\alpha^0 = 1$
- $\alpha^{s(\beta)} = \alpha^\beta \cdot \alpha$
- $\alpha^\lambda = \sup\{\alpha^\beta \mid \beta < \lambda\}$, when λ is a limit ordinal.

Analogous to the lemmas we have just seen about addition and multiplication is the following:

Lemma 2.3.38. *For all ordinals α, β, γ with $\alpha > 1$ we have: $\beta < \gamma \implies \alpha^\beta < \alpha^\gamma$.*

Proof. The proof is by induction on γ:
$\circ\gamma = 1$: Then $\beta = 0$, therefore $\alpha^\beta = \alpha^0 = 1 < \alpha = \alpha^1 = \alpha^\gamma$.
$\circ\gamma = s(\delta)$: Then $\beta \leq \delta \implies \alpha^\beta \leq \alpha^\delta < \alpha^\delta \cdot \alpha = \alpha^\gamma$, with the last equality stemming from Remark 2.3.30.
$\circ\gamma$ is a limit ordinal: Then there exists an ordinal δ such that $\beta < \delta < \gamma$. Therefore $\alpha^\beta < \alpha^\delta \leq \alpha^\gamma$. □

Lemma 2.3.39. *Let S be a non-empty set of ordinals, and $\alpha \in ON$. Then: $\alpha^{\sup S} = \sup\{\alpha^\beta \mid \beta \in S\}$.*

The following theorem is a well-known property of powers.

Theorem 2.3.40. *For all ordinals α, β, γ we have: $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$.*

Proof. The proof is analogous to that of Proposition 2.3.35. □

Proposition 2.3.41. *For all ordinals* α, β, γ *we have:* $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}$.

Proof. Again, the proof is by induction on γ. □

Remark 2.3.42. Unfortunately, the property $(\alpha \cdot \beta)^{\gamma} = \alpha^{\gamma} \cdot \beta^{\gamma}$ does not hold in general. For example: $(\omega \cdot 2)^2 = (\omega \cdot 2) \cdot (\omega \cdot 2) = \omega \cdot (2 \cdot \omega) \cdot 2 = \omega \cdot \omega \cdot 2 = \omega^2 \cdot 2 < \omega^2 \cdot 4 = \omega^2 \cdot 2^2$.

With this, we conclude the things we needed to say about ordinals and their arithmetic. Now we can move on to the cardinals.

2.4 Cardinals

Cardinality in a broad sense is a measure of how many elements there are in a set. One would expect sets like $\{1,2,3\}, \{4,5,9\}, \{\text{sun}, \text{shoes}, \text{beach}\}$ to have cardinality 3, whereas a set like the natural numbers, the real numbers and the complex numbers to have cardinality ∞.

In order to define cardinality (and cardinal numbers) mathematically, we need to talk about a closely-related idea: that of equinumerous sets. We want two sets to be equinumerous when they have "the same number" of elements. This can be achieved in the following way:

Definition 2.4.1. Two sets A, B are called *equinumerous* if there exists a one-to-one and onto function $f : A \longrightarrow B$. We will write $A =_c B$.
We will also write $A \leq_c B$ if there simply exists a on-to-one function $f : A \longrightarrow B$. We will write $A <_c B$ if $A \leq_c B$ but not $A =_c B$.

Immediately from the definition, certain desirable properties of equinumerosity jump out.

Proposition 2.4.2. *For all sets* A, B, C *we have:*

1. $A =_c A$
2. $A =_c B \implies B =_c A$
3. $(A =_c B) \wedge (B =_c C) \implies A =_c C$

Properties 1 and 3 hold if we replace the equation $=_c$ *with inequality* $\leq_c$.

Proof. For the first property, consider the identity function $id_A : A \longrightarrow A$. For the second property, remember that the inverse of a one-to-one and onto function is again a one-to-one and onto function. Finally, for the third property, notice that the composition of two one-to-one (and onto) functions is again a one-to-one (and onto) function. □

With this new notion of how to "count" elements in a set, we can rephrase our goal regarding cardinality: for every set A we want to define a set $|A|$ such that $A =_c |A|$, and additionally for any set B such that $A =_c B$ we have $|A| = |B|$. So, without further ado, let's define the cardinals!

Definition 2.4.3. Let A be any set. We define the cardinality of A to be the ordinal

$$|A| = \min\{\alpha \in \mathrm{ON} \mid \alpha =_c A\}$$

Moreover, any set that serves as cardinality of some other set will be called a cardinal number.

Remark 2.4.4. By Proposition 2.3.17 and a consequence of the Axiom of Choice, the above ordinal always exists. Further more, the two requirements we set above the definition are satisfied, due to Proposition 2.4.2.

Definition 2.4.5. Let $\omega = \{0, 1, 2, 3, \dots\}$ be the set of natural numbers, and $\mathscr{P}(\omega)$ be its power-set. Then, we define $\aleph_0$ to be the cardinality of ω, and $\mathfrak{c}$ to be the cardinality of $\mathscr{P}(\omega)$.

Since cardinals are ordinals, it makes sense to have "the next largest" cardinal. More rigorously, the following theorem holds:

Theorem 2.4.6. *Every non-empty class $\mathscr{E}$ of cardinals has a $\leq_c$-least element.*

Proof. $\mathscr{E}$ is a non-empty class of ordinals. Apply Proposition 2.3.17, and show that this least element is also $\leq_c$-least. □

This theorem is the main tool in proving the following corollary:

Corollary 2.4.7. *Let $\mathscr{E}$ be a non-empty class of cardinals. Then there exists a cardinal number κ such that $\forall \alpha \in \mathscr{E}(\alpha \leq_c \kappa)$ and it is the smallest cardinal with this property.*

Proposition 2.4.8. *For every cardinal κ there exists a unique cardinal κ^+ such that $\kappa <_c \kappa^+$, and it is the smallest cardinal with this property.*

Proof. From Cantor's Theorem, the cardinal of $\mathscr{P}(\kappa)$ is strictly larger than κ. Therefore, the class of cardinals which are strictly larger than κ is non-empty. Apply the previous Corollary. □

Definition 2.4.9. The unique cardinal κ^+ defined by the previous Proposition is called the successor (cardinal) of κ.

Definition 2.4.10. Let α be any ordinal. We define the cardinal $\aleph_\alpha$ recursively:

- if $\alpha = s(\beta)$, then $\aleph_\alpha = \left(\aleph_\beta\right)^+$
- if α is a limit ordinal, then $\aleph_\alpha = \sup\{\aleph_\beta \mid \beta <_c \alpha\}$

With all that out of the way, let's lay out the first cardinals. To begin with, all natural numbers $0, 1, 2, 3, \dots$ are both cardinals and ordinals. The set ω of all natural numbers is again both a cardinal and an ordinal (although we typically write $\aleph_0$ when referring to this set as a cardinal). Immediately after $\aleph_0$ we have the cardinals $\aleph_1, \aleph_2, \aleph_3$ and so on, up to $\aleph_\omega$ and counting. We have defined $\mathfrak{c}$ to be the cardinal of $\mathscr{P}(\omega)$ (which can be shown to be the cardinal of $\mathbb{R}$ as well, i.e. $\mathbb{R} =_c \mathscr{P}(\omega)$).

On the other hand, the ordinals $\omega+1, \omega+2, \omega \cdot 2$ are not cardinals (they are equinumerous to ω but strictly larger than ω).

2.4.1 Cardinal Arithmetic

Now that we know some elementary cardinals, it's time for us to talk about how to do arithmetic on them. Fortunately, cardinal arithmetic is easier than ordinal arithmetic.

Definition 2.4.11. Let κ, λ be cardinal numbers, and K, Λ be disjoint sets such that $|K| = \kappa$ and $|\Lambda| = \lambda$.[6] We define *the (cardinal) sum* $\kappa + \lambda := |K \cup \Lambda|$.[7]

Two properties of cardinal addition can readily be proven. Notice how these are some of the well known properties of the addition in the real numbers that we are used to.

[6] Such sets always exist; take for example K to be κ and Λ to be $\lambda \times \{0\}$.
[7] This operation is easily seen to be well-defined, i.e. for other such sets K', Λ' we have $|K \cup \Lambda| = |K' \cup \Lambda'|$

Proposition 2.4.12. *For any cardinals κ, λ, μ we have:*

1. $\kappa + \lambda = \lambda + \kappa$ *(commutativity)*
2. $\kappa + (\lambda + \mu) = (\kappa + \lambda) + \mu$ *(associativity)*

Proof. It is enough to see that the analogous properties hold for the union of sets:

1. $K \cup \Lambda = \Lambda \cup K$
2. $K \cup (\Lambda \cup M) = (K \cup \Lambda) \cup M$

Now pick three suitable sets K, Λ, M that are pairwise disjoint, and write out the definition for each cardinal addition. □

Definition 2.4.13. Let κ, λ be cardinal numbers, and K, Λ be sets such that $|K| = \kappa$ and $|\Lambda| = \lambda$. We define *the (cardinal) product* $\kappa \cdot \lambda := |K \times \Lambda|$.[8]

Two properties of cardinal multiplication can readily be proven. As with addition, these are also well-known properties of multiplication on the real numbers.

Proposition 2.4.14. *For any cardinals κ, λ, μ we have:*

1. $\kappa \cdot \lambda = \lambda \cdot \kappa$ *(commutativity)*
2. $\kappa \cdot (\lambda \cdot \mu) = (\kappa \cdot \lambda) \cdot \mu$ *(associativity)*

Proof. It is enough to see that the following properties hold for the product of sets:

1. $K \times \Lambda =_c \Lambda \times K$
2. $K \times (\Lambda \times M) =_c (K \times \Lambda) \times M$

Now pick three suitable sets K, Λ, M, and write out the definition for each cardinal multiplication. □

[8]This operation is easily seen to be well-defined, i.e. for other such sets K', Λ' we have $|K \times \Lambda| = |K' \times \Lambda'|$

Continuing our work on proving things that we have known to hold for the real numbers since high school, we have the distributive property of addition and multiplication:

Proposition 2.4.15. *For any cardinals* κ, λ, μ *we have:* $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$.

Proof. It is enough to see that for disjoint sets Λ and M, and K any set, we have

$$K \times (\Lambda \cup M) = (K \times \Lambda) \cup (K \times M)$$

and that the sets $K \times \Lambda$, $K \times M$ are again disjoint. □

Definition 2.4.16. Let κ, λ be cardinal numbers, and K, Λ be sets such that $|K| = \kappa$ and $|\Lambda| = \lambda$. We define *the (cardinal) power* $\kappa^\lambda := |(\Lambda \longrightarrow K)|$,[9] where the set $(\Lambda \longrightarrow K)$ is the set containing all functions from Λ to K.

Proposition 2.4.17. *For any cardinals* κ, λ, μ *we have:*

1. $\kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu$
2. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu$
3. $(\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$

Proof. The proof utilizes the definitions of the cardinals operations we have just laid out. We shall prove the first point, leaving the rest as an exercise.
Pick sets K, Λ, M such that $\Lambda \cap M = \emptyset$ and $|K| = \kappa$, $|\Lambda| = \lambda$, $|M| = \mu$. Then:

$$\kappa^{\lambda+\mu} = |(\Lambda \cup M \longrightarrow K)| = |(\Lambda \longrightarrow K) \times (M \longrightarrow K)| = \kappa^\lambda \cdot \kappa^\mu$$

since every function $f : \Lambda \cup M \longrightarrow K$ is uniquely decomposed into two functions $g : \Lambda \longrightarrow K$ and $h : M \longrightarrow K$ (Λ and M are disjoint sets). □

Just as in ordinals, we can define an ordering on the cardinals. However, to prove that it is in fact an ordering relationship, one has to show the Schröder-Bernstein Theorem first.

Theorem 2.4.18 (Schröder-Bernstein)**.** *For all sets* A, B, *if* $A \leq_c B$ *and* $B \leq_c A$, *then* $A =_c B$.

[9]This operation is easily seen to be well-defined, i.e. for other such sets K', Λ' we have $|(\Lambda \longrightarrow K)| = |(\Lambda' \longrightarrow K')|$

We shall omit this proof, but provide the statement of a lemma which is essential in proving the above theorem. The interested reader may fill in the blanks themselves, or look for the proofs in [11].

Lemma 2.4.19. *Let $C \subseteq B$ be sets. If there exists a one-to-one function $f : B \longrightarrow C$, then $B =_c C$.*

Definition 2.4.20. Let κ, λ be cardinals, and K, Λ be sets such that $|K| = \kappa$ and $|\Lambda| = \lambda$. We define the relationship:

$$\kappa \leq \lambda \quad \text{if-f} \quad K \leq_c \Lambda$$

We also write $\kappa < \lambda$ if $\kappa \leq \lambda$ and $\kappa \neq \lambda$.

Combining Theorem 2.4.18 above and Proposition 2.4.2, one sees that this relation satisfies all the properties required for it to be an ordering, as in Definition 2.2.3 (note that this is a binary relation on the *class* of cardinals, not on a set as in the aforementioned definition).

A very important theorem concerning this ordering is Cantor's Theorem about the cardinality of the power-set. Again, the proof of this can be found in [11].

Theorem 2.4.21 (Cantor)**.** *For any set A we have: $A <_c \mathscr{P}(A)$.*

At this point, we cannot omit to mention the cardinality of the power-set:

Proposition 2.4.22. *For any set K we have: $|\mathscr{P}(K)| = 2^{|K|}$.*

Proof. We need to show that $\mathscr{P}(K) =_c (K \longrightarrow \{0,1\})$. For that, define the function $f : \mathscr{P}(K) \longrightarrow (K \longrightarrow \{0,1\})$ that maps any subset A of K to its identity function $id_A : K \longrightarrow \{0,1\}$,

$$id_A(x) = \begin{cases} 1 & , x \in A \\ 0 & , x \notin A \end{cases}$$

The function f is obviously one-to-one. Furthermore, it is onto. Indeed, pick any function $g : K \longrightarrow \{0,1\}$ and put $A = g^{-1}(\{1\})$. Then $f(A) = g$. This concludes the proof. □

Corollary 2.4.23. *For every cardinal κ we have: $\kappa < 2^\kappa$.*

Let us now see that this ordering is preserved by the cardinal arithmetic we have defined:

Theorem 2.4.24. *For all cardinals κ, λ, μ we have:*

1. $\kappa \leq \lambda \implies \kappa + \mu \leq \lambda + \mu$
2. $\kappa \leq \lambda \implies \kappa \cdot \mu \leq \lambda \cdot \mu$
3. $\kappa \leq \lambda \implies \kappa^{\mu} \leq \lambda^{\mu}$
4. $(\kappa \leq \lambda) \wedge (\mu > 0) \implies \mu^{\kappa} \leq \mu^{\lambda}$ [10]

Some remarks about certain properties of cardinal arithmetic and ordering follow. These will serve as a preliminary to proving some more general facts about infinite cardinal arithmetic.

Remark 2.4.25. In the following, κ is any cardinal, $\aleph_0 = |\omega|$, and $\mathfrak{c} = |\mathscr{P}(\omega)|$:

- $\forall n \in \omega\ (n + \aleph_0 = \aleph_0)$
- $\aleph_0 + \aleph_0 = \aleph_0$
- $\aleph_0 \cdot \aleph_0 = \aleph_0$
- $\kappa + 0 = \kappa$
- $\kappa \cdot 0 = 0$
- $\kappa \cdot 1 = \kappa$
- $\kappa^0 = 1$
- $0^{\kappa} = 0$, if $\kappa > 0$
- $\mathfrak{c} = 2^{\aleph_0}$
- $\kappa + \kappa = 2 \cdot \kappa$
- $\mathfrak{c} + \mathfrak{c} = \mathfrak{c}$
- $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$
- $\aleph_0 \cdot \mathfrak{c} = \mathfrak{c}$
- $\aleph_0^{\aleph_0} = \mathfrak{c}$

2.4.2 Infinite Cardinal Arithmetic

This is perhaps the section we are mostly going to refer to in Chapter 3 of this essay. While it is the culmination of everything we have said so far about ordinals and cardinals, it is far from difficult to understand. Contrary to ordinals, infinite cardinals are much more intuitive.

This section requires a small detour into the Axiom of Choice, equivalent statements and some of its consequences. However, this is beyond the scope of our paper, and we shall leave this detour to the reader. Details can be found in [11].

[10] The only problem with $\mu = 0$ is in $\kappa = 0 < \lambda$, since then $\mu^{\kappa} = 1 > 0 = \mu^{\lambda}$.

Theorem 2.4.26. *Every infinite set contains an infinite and countable subset. Consequently, if κ is an infinite cardinal, then $\aleph_0 \leq \kappa$.*

Proof. Fix an infinite set M, and construct the following sequence $(A_n)_{n\in\omega}$ of finite sets recursively:
$\circ A_0$: Since M is infinite, it is non-empty. Pick an element $a_0 \in M$ and put $A_0 = \{a_0\}$
$\circ A_{n+1}$: Suppose you have constructed all sets A_k for $0 \leq k \leq n$. Since A_n is finite and M is infinite, there exists an element $a_{n+1} \in M \backslash A_n$. Put $A_{n+1} = A_n \cup \{a_{n+1}\}$.
Now consider the union of all these sets, $A = \bigcup_{n\in\omega} A_n = \{a_0, a_1, a_2, \dots\}$. This is a countable subset of M, thus proving the first claim.

One important consequence of (and in fact, an equivalent statement to) the Axiom of Choice is that all cardinals are pairwise comparable. Hence, if κ is an infinite cardinal, then the above states that we cannot have $\kappa < \aleph_0$, giving us $\aleph_0 \leq \kappa$. □

Proposition 2.4.27. *Let α, β be cardinal numbers, with α = finite and β = infinite. Then $\alpha + \beta = \beta$.*

Proof. Pick sets A, B such that $|A| = \alpha$ and $|B| = \beta$. Since β is infinite, by the above theorem B contains a countable subset, say C. We will show that $\alpha + \aleph_0 = \aleph_0$. From that, we will have:

$$\alpha + \beta = \alpha + (|C| + |B \backslash C|) = (\alpha + \aleph_0) + |B \backslash C| = \aleph_0 + |B \backslash C| = \beta$$

To that end, suppose without loss of generality that $A \cap C = \emptyset$, and write $C = \{c_0, c_1, c_2, \dots\}$ and $A = \{a_0, a_1, \dots, a_{\alpha-1}\}$. Define the function $f : A \cup C \longrightarrow C$ with

$$f(x) = \begin{cases} c_n & , \quad x = a_n \\ c_{n+\alpha} & , \quad x = c_n \end{cases}$$

This is obviously a one-to-one function (since A, C are disjoint). It is also onto, for if we pick an element $c_n \in C$, then:
1) $f(a_n) = c_n$, if $n < \alpha$; or
2) $f(c_{n-\alpha}) = c_n$, if $n \geq \alpha$.
This concludes the proof; $A \cup C =_c C \implies |A \cup C| = |C| \iff \alpha + \aleph_0 = \aleph_0$. □

The following theorem needs Zorn's Lemma and inductive spaces to be proven, which is why we simply state it here.

Theorem 2.4.28. *For every infinite cardinal α we have: $\alpha + \alpha = \alpha$.*

Corollary 2.4.29. *For all cardinals α, β, if at least one of them is infinite, then $\alpha+\beta = max\{\alpha,\beta\}$.*

Proof. Without loss of generality and since the two cardinals α, β are comparable (by the Axiom of Choice), assume $\alpha \leq \beta$ (so that β is infinite). By Theorem 2.4.24 we have $\beta \leq \alpha+\beta$ and $\alpha+\beta \leq \beta+\beta$. Apply Theorem 2.4.28 to get that $\alpha+\beta \leq \beta$. □

Similarly to the last theorem and corollary we have the following regarding multiplication. Proving them can be done in a likewise manner.

Theorem 2.4.30. *For every infinite cardinal α we have: $\alpha \cdot \alpha = \alpha$.*

Corollary 2.4.31. *For all non-zero cardinals α, β, if at least one of them is infinite, then $\alpha \cdot \beta = max\{\alpha, \beta\}$.*

Corollary 2.4.32. *Let α be an infinite cardinal, and $\gamma > 0$ be a finite cardinal. Then $\alpha^\gamma = \alpha$.*

Proof. The proof is by induction on γ, using the last corollary and Proposition 2.4.17. □

Corollary 2.4.33. *Let $\alpha, \beta \geq 2$ be finite cardinals, and γ be an infinite cardinal. Then $\alpha^\gamma = \beta^\gamma$.*

Proof. Without loss of generality, suppose that $\alpha \leq \beta$. We have $\aleph_0 \leq \alpha^\gamma \leq \beta^\gamma$. There also exists[11] a natural number n for which $\beta \leq \alpha^n$ (since $\alpha \geq 2$). So, $\beta^\gamma \leq (\alpha^n)^\gamma = \alpha^(n \cdot \gamma) = \alpha^\gamma$, using already-proven properties of cardinal arithmetic. This gives us: $\alpha^\gamma \leq \beta^\gamma \leq \alpha^\gamma$, which proves the statement. □

We are now moving on to the final leg of our journey in the land of cardinals. We want to define what an infinite sum/product of cardinals is. Our intuition serves us well.

Definition 2.4.34. Let $(a_i)_{i\in I}$ be a non-empty collection of cardinals. Pick any pairwise disjoint sets $(A_i)_{i\in I}$ such that $\forall i \in I\ (|A_i| = a_i)$. We define the *infinite sum*

$$\sum_{i\in I} a_i = |\bigcup_{i\in I} A_i|$$

Definition 2.4.35. Let $(a_i)_{i\in I}$ be a non-empty collection of cardinals. Pick any collection of sets $(A_i)_{i\in I}$ such that $\forall i \in I\ (|A_i| = a_i)$. We define the *infinite product*

$$\prod_{i\in I} a_i = |\prod_{i\in I} A_i|$$

[11]This remains to be proven. Hint: use finite induction on β.

Remark 2.4.36. Both of the infinite sum and the infinite product of cardinals are well-defined.

Corollary 2.4.37. *Let $(a_i)_{i\in I}$ be a non-empty collection of cardinals, and α be an infinite cardinal such that $\forall i \in I\ (a_i \leq \alpha)$ and $|I| \leq \alpha$. Then $\sum_{i\in I} a_i \leq \alpha$.*

Proof. Pick a collection of pairwise disjoint sets $(A_i)_{i\in I}$ such that $|A_i| = a_i$ for all $i \in I$. Also, pick a set A such that $|A| = \alpha$. Then the collection $(A \times \{\lambda\})_{\lambda\in A}$ consists of pairwise disjoint sets, each with cardinality α. Hence:

$$\sum_{i\in I} a_i = |\bigcup_{i\in I} A_i| \leq |\bigcup_{\lambda\in A} A \times \{\lambda\}| = |A \times A| = \alpha \cdot \alpha = \alpha$$

□

Remark 2.4.38. In the previous proof, we showed that $\sum_{\lambda\in\alpha} \alpha = \alpha \cdot \alpha$, which follows our intuition on what relationship addition and multiplication should have. More generally, it can be proven that $\sum_{\lambda\in\beta} \alpha = \alpha \cdot \beta$ for any cardinals α, β. In a similar manner, one can prove the following about multiplication and powers:

If α, β are cardinals, then $\alpha^\beta = \prod_{\lambda\in\beta} \alpha$.

At this point, we cannot not state the famous König's Theorem, though its proof requires the Axiom of Choice, and is therefore left unexamined.

Theorem 2.4.39 (König)**.** *Let $(a_i)_{i\in I}, (b_i)_{i\in I}$ two non-empty collections of cardinals, such that $\forall i \in I\ (a_i < b_i)$. Then we have: $\sum_{i\in I} a_i < \prod_{i\in I} b_i$.*

Corollary 2.4.40. *There does not exist any sequence $(\alpha_n)_{n\in\omega}$ of cardinals strictly smaller than $\mathfrak{c}$, such that $\mathfrak{c} = \sum_{n\in\omega} \alpha_n$.*

Proof. Apply König's Theorem to the sequence of cardinals, and notice that $\mathfrak{c} = \mathfrak{c}^{\aleph_0} = \prod_{n\in\omega} \mathfrak{c}$. □

Finally, we present a proposition that seems intuitively right, but requires the Axiom of Choice to be proven:

Proposition 2.4.41. *Let A, B be two sets, and $h : A \longrightarrow B$ be an onto function. Then $B \leq_c A$.*

2.4.3 Orders and Cardinals

Now that we have learned all we needed to learn about cardinals and partial orders, it's time to combine the two and give some final definitions that will be used in the next chapter.

Definition 2.4.42. Let $(I, \leq)$ be a poset, and J be a subset of I. Then J is called *bounded* if there exists an upper bound for its elements, that is: $\exists i \in I \left(\forall j \in J \ (j \leq i)\right)$.
Furthermore, if J is not bounded, then J is called *cofinal*.

Definition 2.4.43. Let I be an unbounded partially ordered set (the case where I is a cardinal number is of interest to us). We define its *cofinality* to be the cardinal number given by:

$$cf(I) = \min\{\text{card}(J) \mid J \subseteq I \wedge J = \text{cofinal}\}$$

Notice that, in the case of cardinals, we can identify two types of situations: either the cardinal number is equal to its own cofinality, or the cardinal number is larger than it. This gives rise to the following notions, both of which are essential to the understanding of the Singular Compactness Theorem.

Definition 2.4.44. Let κ be an infinite cardinal. Then κ is called:

- *regular*, if $cf(\kappa) = \kappa$
- *singular*, if $cf(\kappa) < \kappa$

In the presence of the axiom of choice, one can also use the following proposition to prove that the cardinals $\aleph_0, \aleph_1, \ldots, \aleph_n \ldots$ are regular, whereas $\aleph_\omega$ is the first singular cardinal. Remember that each cardinal is an ordinal, hence it is a set $\kappa = \{\alpha \in \text{ON} \mid \alpha < \kappa\}$, and note that the supremum of a set A of ordinals is just the union $\bigcup A$ (Proposition 2.3.20).

Proposition 2.4.45. *Let κ be a cardinal number. All of the following are equivalent:*

- *κ is regular*
- *If $\kappa = \sum_{i \in I} \lambda_i$ and $\lambda_i < \kappa$ for all $i \in I$, then $|I| \geq \kappa$*
- *If $S = \bigcup_{i \in I} S_i$ and $|I| < \kappa$ and $|S_i| < \kappa$ for all $i \in I$, then $|S| < \kappa$*

Proof. $1 \implies 2)$ Suppose to the contrary that there exists a set I with $|I| < \kappa$ and there exist cardinals $\lambda_i < \kappa$ for all $i \in I$ such that $\kappa = \sum_{i \in I} \lambda_i$. Then $\forall i \in I\ (\lambda_i \in \kappa)$ meaning that the set $J = \{\lambda_i \mid i \in I\}$ is a subset of κ. Moreover, it is cofinal in it: indeed, if that weren't the case, then we could find a $\lambda \in \kappa$ such that $\forall i \in I\ (\lambda_i \leq \lambda)$ and $|I| \leq \lambda$; apply Corollary 2.4.37 to see that $\kappa = \sum_{i \in I} \lambda_i \leq \lambda < \kappa$, a contradiction. So, we have found a cofinal subset of κ that has cardinality $|J| = |I| < \kappa$, meaning that κ is a singular cardinal, contradicting our hypothesis.
$2 \implies 3)$ Suppose to the contrary that there exists a set S with $|S| \geq \kappa$, and there exist sets $I, (S_i)_{i \in I}$ with cardinalities strictly smaller than κ, such that $S = \bigcup_{i \in I} S_i$. By substituting S, S_i with appropriate subsets S', S'_i (for each $i \in I$), we may assume that $|S| = \kappa$; furthermore, we may assume that all of the S_i are pairwise disjoint (otherwise, substitute $\tilde{S}_i = S_i \times \{i\}$ for S_i). Then by Definition 2.4.34 we have: $\kappa = |S| = |\bigcup_{i \in I} S_i| = \sum_{i \in I} |S_i|$. By hypothesis, it must be $|I| \geq \kappa$, which is a contradiction.
$3 \implies 1)$ Suppose to the contrary that κ is singular. Then by definition there must exist a cofinal subset $I \subseteq \kappa$ with $|I| < \kappa$. For each $i \in I$ define the set $S_i = \{\alpha \in \kappa \mid \alpha < i\}$, and put $S = \bigcup_{i \in I} S_i$. Notice that $|S_i| = \text{card}(i) \leq i < \kappa$. By applying our hypothesis we get that $|S| < \kappa$. However, I being cofinal in κ means that $S = \kappa$, a contradiction. □

Definition 2.4.46. Let $(I, \leq)$ be a poset. Then:

- it is called *directed*, if $\forall i, i' \in I\ \big(\exists j \in I\ (i \leq j\ \wedge\ i' \leq j)\big)$
- if λ is a regular cardinal, the poset is called λ*-directed* if every subset of I with cardinality less than λ has an upper bound.

Definition 2.4.47. Let $(I, \leq)$ be a poset, and λ be a regular cardinal. Then we say that I is λ*-closed* if the supremum of any of its chains with cardinality less than λ lies in I.

Chapter 3

Shelah's Singular Compactness Theorem

It is finally time to discuss the main theorem of this report: Shelah's Singular Compactness Theorem. We will begin with a short background on the approaches to this result done in the past, and then we are ready to begin developing and presenting the tools necessary to prove this very exciting theorem.

3.1 Historical recount

No study of a subject would be complete without at least some notes on how the main ideas of it came to be, how they took off and were morphed to aid in other areas of interest, and how these ideas stand in modern times. However, therein lies an author's probably most daunting and difficult task: how is one to decide when an idea was born? How can one gather all the important information on a, say, theorem, without risking going off on tangents, the other ends of which might be completely unrelated to and of no interest to the scholar of the original theorem? How can one do justice by the shape of these ideas in modern times?

For a theorem of mathematics, such as our own, that borrows from so many different areas, it is impossible to carry out such a heavy task. Yet, the author of this essay has made as best an effort as possible to present a coherent and streamlined version of the history of Shelah's Singular Compactness Theorem. To this end, only a superficial recount of the most important and relevant to us breakthroughs will be shone a light upon, sacrificing many of the other, equally beautiful, breakthroughs that have been discovered along the way by past contributors.

In many ways, the beginning of an idea is both the hardest and easiest to pinpoint. Shelah's celebrated theorem was written in ink and paper in 1974, but in order to witness its beginnings, we have to go back a few years; more specifically to 1952, when J. C. Whitehead is said (Ehrenfeucht [3]) to have first presented his famous Whitehead problem on abelian groups. The problem states:

(WH): Is every abelian group with $Ext^1(A,\mathbb{Z}) = 0$ a free abelian group?

An abelian group A that satisfies the equation $Ext^1(A,\mathbb{Z}) = 0$ is called a *Whitehead group*. By the recount of Eklof ([4]), the answer to the above question was already proven to be affirmative when dealing with countable groups, i.e. every countable *W-group* is free.

It was this very problem that piqued Shelah's interest, which led to his first paper on the subject in the year 1973 (see [13]). In this paper, Shelah proved that (WH) for abelian groups of cardinality $\aleph_1$ is independent of ZFC. He does this by showing that ZFC + $V = L$ (the usual ZFC axiomatic set theory together with the axiom of constructibility, an axiomatic system already proven to be consistent assuming consistency of ZFC) implies (WH); then he showed that the (already proven to be consistent) axiomatic system ZFC + MA + $2^{\aleph_0} > \aleph_1$ (where MA = Martin's axiom) implies that there are non-free W-groups of cardinality $\aleph_1$, the negation of (WH). Therefore, (WH) must be independent of ZFC. For both of these results, he uses a categorization of a group G based on finite subsets and pure subgroups of it.

This first independence result was just the first step, however. A couple of years later, in [14] Shelah graced the mathematical community with his much celebrated Singular Compactness Theorem. This version uses lemmas about free algebras (a very set-theoretic version of 'free' that is later generalized by Hodges) to prove that:

Theorem (Shelah's Singular Compactness Theorem)**.** *If $G = \lambda$-free group, where λ is a singular cardinal, then it is λ^+-free.*

Here, λ-free means that every subgroup of cardinality $< \lambda$ is free. The same holds for free abelian groups, and those two prove compactness for these cases of 'free' structures. To prove compactness for free algebras in general, he uses the notion of E-freeness, where E is a filter. He then went on ([15], [16]) to provide a more general independence result to (WH), even assuming the Continuum Hypothesis.

Hodges in [10] refined Shelah's proof of the Singular Compactness Theorem in the case of abelian groups. He uses the notions of *fully closed unbounded subsets* and *free factor of an abelian* *group* (tightly coupled with the basis of the group), and presents a proof which revolves around a two-player game where subgroups of a group A are chosen so that every other subgroup B_i is a free factor of B_{i+2}. He then generalized the proof by giving a set of axioms that abstract the notions involved: those of *abelian groups* (or other mathematical structures in their stead) and of *free*. Note that this is different from the axiomatization that Shelah did in his original work, which only regards algebras.

It is this latter system of axioms that David relaxed in [2] by eliminating the first axiom, and Shelah himself expanded upon in [17] for a general theorem on lifting incompactness.

In Eklof and Fuchs [7] the version of the Singular Compactness Theorem that appears in Hodges is used to fully characterize Baer modules over arbitrary valuation domains (so that all such modules are free). This work is furthered by a subsequent paper ([8]) to Baer modules over arbitrary domains, in which a stronger version of the Singular Compactness Theorem from the first paper is developed by confining it to just modules and formulating it in such a way that it applies below the cardinality of the ring.

In the following years, not much was done pertaining to new proofs of Shelah's theorem. The version that appeared in Hodges was the one most referred-to by authors to apply and the one that saw the most re-prints and refinements of its proof. The most notable literature from 1990 to 2010 is as follows:

◦ Eklof [4]: it is an introductory article to the (WH) problem and the work of Shelah. A good starting point that contains definitions, methods and tools used throughout the literature on the subject.

◦ Eklof [5]: it contains a detailed history of the connection between set theory and algebra, how the former has helped solve some long-standing problems of the latter, as well as a lengthy bibliography for further study.

◦ Eklof and Mekler [9]: the second version of this book is a solid read for anyone who wishes to start small (from homomorphisms and extensions in algebra, from filters and large cardinals in set theory) and work their way up to more advanced theories (cotorsion theory, dual groups and topological tools, just to name a few of its chapter titles). The historical notes at the end of each chapter have been especially helpful in the compilation of this here section. Shelah's Singular Compactness Theorem (the version for modules that is inspired from Hodges) is an improved attempt at an accessible proof, without losing generality.

◦ Eklof [6]: this small paper has again some history on the Whitehead problem, some preliminary definitions, the notion of 'freeness' for modules, and finally it explains a self-contained proof of the Singular Compactness Theorem that appears in [9].

The next big breakthrough comes almost twenty five years after the last one in the form of a paper by Beke and Rosicky ([1]). The basic aim of this is to reformulate Shelah's theorem in categorical terms, state its new 'functorial' form that is broad enough to encapsulate all known applications of the theorem, then present the novel proof of it (which is based on the one by Hodges). The theorem states:

Theorem (Singular compactness theorem (functorial form))**.** *Let $\mathscr{A}$ be an accessible category with filtered colimits, $\mathscr{B}$ a finitely accessible category and $F : \mathscr{A} \longrightarrow \mathscr{B}$ a functor preserving filtered colimits. Assume that F-structures extend along morphisms. Let $X \in \mathscr{B}$ be an object whose size μ is a singular cardinal. If all subobjects of X of size less than μ are in the image of F, then X itself is in the image of F.*

Special attention is given to cellular objects, which, according to the authors, provide "the most elegant version of singular compactness" without losing much generality. It is here that the notions of 'structure' and 'free' are put into test: how much can one relax them while still keeping the result of the theorem intact.

This brings us neatly along to the last article that we will cite, and it is the one that this essay has set out to explain. The article is by J. Šaroch and J. Št'ovíček ([18]), was published in 2020, and concerns itself with $\sum$-cotorsion modules. To show that a module being $\sum$-cotorsion is a property of the complete theory of the module, the authors develop a general (set-theoretic) proof of Shelah's Singular Compactness theorem, which enhances the ideas found in [9].

We will now conclude this essay with an attempt to analyze the aforementioned paper by Šaroch and Št'ovíček, in order to fully appreciate this new proof of the Singular Compactness Theorem. But before we do so, let's make sure that we clearly state the problem we are trying to solve: What is the connection between co-limits and the roots of the functor $\mathrm{Ext}^1(-,C)$, where C is an object of a category $\mathscr{C}$? More specifically, if $\mathscr{C}$ is the class of all left R-modules and we define ${}^{\perp}\mathscr{C} = \{M \in \mathscr{C} \mid \forall C \in \mathscr{C}\ (\mathrm{Ext}^1(M,C) = 0)\}$, then ${}^{\perp}\mathscr{C}$ is the class of projective modules; when can one decompose a projective module into "smaller" projective modules, and, vice versa, when is the co-limit of projective modules also a projective module?

Theorem (SSCT as appears in Šaroch and Št'ovíček)**.** *Let R be a ring with enough idempotents. Let κ be a singular cardinal, M be a κ-presented module and $\mathscr{C}$ a filter-closed class of modules. Assume that there is an infinite cardinal ν such that, for all successor cardinals $\nu < \lambda < \kappa$, there is a system S_λ witnessing that M is almost $(\mathscr{C},\lambda)$-projective. Then $M \in^{\perp} \mathscr{C}$.*

3.2 The regular-cardinal case

In the rest of this essay, by R we mean an associative ring with a multiplicative identity element, and by "module" we will mean a left R-module (i.e. a module in the class R-Mod).

The following definitions are a first bridge between homological algebra and set theory:

Definition 3.2.1. Let λ be a regular (uncountable) cardinal. A direct system $\big((M_i)_{i\in I}\, ,\, (f_{ji})_{i\leq j}\big)$ of R-modules is called *λ-continuous* if for every $J \subseteq I$ which is linearly ordered and has cardinality $|J| < \lambda$ there exists $x \in I$ such that $x = \sup J$ and $M_x = \varinjlim_{j\in J} M_j$.

Remark 3.2.2. It follows immediately from the last definition and Definition 2.4.46 that if a direct system $\big((M_i)_{i\in I}\, ,\, (f_{ji})_{i\leq j}\big)$ is λ-continuous, then the poset $(I, \leq)$ is λ-directed.

Definition 3.2.3. Let τ be an ordinal, and M be an R-module. Then a direct system $\{M_\alpha \mid \alpha < \tau\}$ of modules is called *a well-ordered continuous filtration* of M if:

- $M_0 = 0$
- $M = \varinjlim_{\alpha<\tau} M_\alpha$
- every map $M_\alpha \longrightarrow M$ is 1-1
- for every limit ordinal $\alpha < \tau$ we have $M_\alpha = \varinjlim_{\beta<\alpha} M_\beta$

Let us now extend the definition of a finitely-presented module (Definition 1.1.64) to infinite cardinal representation:

Definition 3.2.4. Let κ be any cardinal, and M be an R-module. We say that M is *κ-presented* if there exists a short exact sequence $R^{(\kappa)} \longrightarrow R^{(\kappa)} \longrightarrow M \longrightarrow 0$, where by $R^{(\kappa)}$ we denote the direct sum of κ copies of R.
We also say that M is $<\kappa$-presented if there exists a cardinal $\lambda < \kappa$ such that M is λ-presented.

Remark 3.2.5. The following hold:
i) If M' is a quotient of a κ-presented module M, then M' is κ-generated.
(*Proof.* Take the exact sequence $R^{(\kappa)} \longrightarrow R^{(\kappa)} \xrightarrow{g} M \longrightarrow 0$ and let $M' = M/K$ for some submodule $K \leq M$. Put $r_\mu \in R^{(\kappa)}$, $\mu < \kappa$, be the element with 1 for its μ-th coordinate, and 0 elsewhere. Then M is generated by the set $\{g(r_\mu) \mid \mu < \kappa\}$ (since $g : R^{(\kappa)} \longrightarrow M$ is surjective), so M' is generated by the set $\{g(r_\mu) + K \mid \mu < \kappa\}$.)

ii) If M' is a direct summand of a κ-presented module M, then M' is κ-presented.
(*Proof.* Write $M = M' \oplus K$, whence we get that $M' = M/K$ like above. Then M' is the quotient of a κ-presented module over a κ-generated module (K is κ-generated due to the previous point), giving us that the relations in M' are the relations in M plus the generators of K, which total κ. Hence M' is κ-presented itself.)

The following proposition will be used several times in the pages to come. It itself is a generalization of a well-known result about countably-presented modules and their representation by finitely-presented modules.

Proposition 3.2.6. *Let $\lambda \leq \kappa$ be cardinals with λ=regular, and M be a κ-presented module. Then there exists a representation $M = \varinjlim_{i\in I} M_i$ of $<\lambda$-presented modules, where the direct system $\big((M_i)_{i\in I}\ ,\ (f_{ji})_{i\leq j}\big)$ is λ-continuous.*

Proof. Since M is κ-presented, by definition there exists a short exact sequence $R^{(\kappa)} \xrightarrow{f} R^{(\kappa)} \longrightarrow M \longrightarrow 0$. Define

$$I = \{(A,B) \mid A,B \subseteq \kappa \text{ with } |A|,|B| < \lambda \text{ and } f(R^{(A)}) \subseteq R^{(B)}\}$$

and let $\leq$ be an ordering on that set, where $(A,B) \leq (A',B')$ iff $A \subseteq A'$ and $B \subseteq B'$.

For every cardinal τ and every $J \subseteq I$ with $J = \{(A_\sigma, B_\sigma) \mid \sigma < \tau\}$ and $|J| < \lambda$ we have $(\bigcup\limits_{\sigma<\tau} A_\sigma, \bigcup\limits_{\sigma<\tau} B_\sigma) \in I$ (this is easy to verify using Proposition 2.4.45, since λ is regular); furthermore, $\forall \rho < \tau \left((A_\rho, B_\rho) \leq (\bigcup\limits_{\sigma<\tau} A_\sigma, \bigcup\limits_{\sigma<\tau} B_\sigma)\right)$.

Now for every $(A,B) \in I$ define $M_{(A,B)} = \operatorname{coker}(R^{(A)} \xrightarrow{f_|} R^{(B)})$. We have in this manner constructed the modules of the wanted direct system, hence leaving us to define the homomorphisms between them to finish the proof. We are able to do this by defining the homomorphism $can_{(A',B'),(A,B)} = g : M_{(A,B)} \longrightarrow M_{(A',B')}$ such that the following diagram is commutative:

$$\begin{array}{ccccccc}
R^{(A)} & \xrightarrow{f_|} & R^{(B)} & \longrightarrow & M_{(A,B)} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow & & \big\downarrow g & & \\
R^{(A')} & \xrightarrow{f_|} & R^{(B')} & \longrightarrow & M_{(A',B')} & \longrightarrow & 0
\end{array}$$

It is easy to prove that:
i) if $J = \{(A_\sigma, B_\sigma) \mid \sigma < \tau\} \subseteq I$ is a chain and $(A,B) = (\bigcup\limits_{\sigma<\tau} A_\sigma, \bigcup\limits_{\sigma<\tau} B_\sigma) \in I$, then $(A,B) = \sup J$

and $M_{(A,B)} = \varinjlim_{\sigma<\tau} M_{(A_\sigma,B_\sigma)}$
(*Proof.* To prove that $(A,B) = \sup J$ simply observe that for all $\sigma < \tau$ we have $A_\sigma \subseteq A$ (likewise for B); this means that $(A_\sigma,B_\sigma) \leq (A,B)$. Furthermore, for any other upper bound (A',B') of J we must have $A_\sigma \subseteq A'$ and therefore $\bigcup_{\sigma<\tau} A_\sigma \subseteq A'$ (likewise for B'), i.e. $A \subseteq A'$ and $B \subseteq B'$, giving us $(A,B) \leq (A',B')$. In order to prove the limit equality, observe that cokernels commute with colimits, therefore we have: $M_{(A,B)} = \operatorname{coker}\big(R^{(A)} \xrightarrow{f_|} R^{(B)}\big) = \operatorname{coker}\big(\varinjlim_{\sigma<\tau}(R^{(A_\sigma)} \xrightarrow{f_|} R^{(B_\sigma)})\big) = \varinjlim_{\sigma<\tau} \operatorname{coker}\big(R^{(A_\sigma)} \xrightarrow{f_|} R^{(B_\sigma)}\big) = \varinjlim_{\sigma<\tau} M_{(A_\sigma,B_\sigma)}$.)
ii) we have $M = \operatorname{coker}(R^{(\kappa)} \xrightarrow{f} R^{(\kappa)}) = \varinjlim_{(A,B)\in I} M_{(A,B)}$.
(*Proof.* Use the limits argument from above.)

This concludes the proof. □

Remark 3.2.7. In the above proposition, if $\lambda = \kappa$, then we can take I to be linearly ordered, namely $M = \varinjlim_{\alpha<\lambda} M_\alpha$. Indeed, for every $\alpha < \lambda$ pick $\alpha \leq \beta_\alpha < \lambda$ such that $f(R^{(\alpha)}) \subseteq R^{(\beta_\alpha)}$, and put $M_\alpha = \operatorname{coker}\big(R^{(\alpha)} \xrightarrow{f_|} R^{(\beta_\alpha)}\big)$.

The connection between λ-continuity and λ-representation can be further strengthened; for that, we turn to a similar notion for functors.

Definition 3.2.8. Let λ be a regular cardinal. The functor $F : R\text{-Mod} \longrightarrow \text{Ab}$ is called *λ-continuous* if for every λ-continuous direct system $\big((M_i)_{i\in I}\ ,\ (f_{ji})_{i\leq j}\big)$ with $M = \varinjlim_{i\in I} M_i$ the natural map $\varinjlim_{I\in I} FM_i \longrightarrow FM$ is an isomorphism.

The next proposition illustrates the importance of the above definition using the $\operatorname{Hom}(M,-)$ functor:

Proposition 3.2.9. *Let λ be a regular cardinal. The module M is $< \lambda$-presented if-f the functor $Hom(M,-) : R\text{-}Mod \longrightarrow Ab$ is λ-continuous.*

Proof. $\underline{\Rightarrow)}$ Let $\big((N_i)_{i\in I}\ ,\ (f_{ji})_{i\leq j}\big)$ be a λ-continuous direct system, and $N = \varinjlim_{i\in I} N_i$. We will show that the canonical map $can : \varinjlim_{i\in I} \operatorname{Hom}(M,N_i) \longrightarrow \operatorname{Hom}(M,N)$ is one-to-one and onto.
onto: Pick a homomorphism $g : M \longrightarrow N$ and let $\{m_\alpha \mid \alpha < \tau\}, \{r_\alpha \mid \alpha < \tau\}$ be the generators and the relations of M (such exist for some $\tau < \lambda$, since M is $< \lambda$-presented). Consider the set $\{g(m_\alpha) \mid \alpha < \tau\} \subseteq N$. By definition of the direct limit and λ-continuity of the system: $\exists i \in I\ \forall \alpha < \tau\ \big(g(m_\alpha) \in \operatorname{im}(N_i \xrightarrow{f_i} N)\big)$, where the $\{f_k\ ,\ k \in I\}$ are the canonical homomorphisms. Now, for every relation r_α we have $g(r_\alpha) = 0$, therefore there exists $r'_\alpha \in N_i$ with $f_i(r'_\alpha) = 0$. Again

from the direct limit there exists a $j \in I$ such that $\{r'_\alpha \mid \alpha < \tau\} \subseteq \ker\big(N_i \xrightarrow{f_{ji}} N_j\big)$. Therefore the function g can be written as the composition $f_j \circ \tilde{g} = (f_j)_*(\tilde{g})$, for a homomorphism $\tilde{g} : M \longrightarrow N_j$. Since the following diagram is commutative, we have proved the "onto" property of the *can* map.

$$\begin{array}{ccc} \varinjlim_{i\in I} \operatorname{Hom}(M,N_i) & \xrightarrow{can} & \operatorname{Hom}(M,N) \\ \uparrow & \nearrow_{(f_j)_*} & \\ \operatorname{Hom}(M,N_j) & & \end{array}$$

<u>one-to-one:</u> Pick a $\xi \in \ker\big(\varinjlim_{i\in I} \operatorname{Hom}(M,N_i) \xrightarrow{can} \operatorname{Hom}(M,N)\big)$. Then there exists an $i \in I$ and a $g \in \operatorname{Hom}(M,N_i)$ such that $\xi = [g]$. We have: $0 = can(\xi) = (f_i)_*(g) = f_i \circ g : M \longrightarrow N$, and so $\forall \alpha < \tau \ \Big(f_i(g(m_\alpha)) = 0 \in N\Big)$. But then there exists $j \in I$ such that $\forall \alpha < \tau \ \Big(f_{ji}(g(m_\alpha)) = 0\Big)$. Since the set $\{m_\alpha \mid \alpha < \tau\}$ is a generating set of the module M, this tells us that $f_{ji} \circ g \equiv 0 : M \longrightarrow N_j$, and so $\xi = [g] = [f_{ji} \circ g] = [0] = 0$.

<u>$\Leftarrow$)</u> By Proposition 3.2.6 we can write the module M as $\varinjlim_{i\in I} M_i$ for a λ-continuous direct system of $< \lambda$-presented modules $(M_i)_{i\in I}$.
By hypothesis, the canonical map $can : \varinjlim_{i\in I} \operatorname{Hom}(M,M_i) \longrightarrow \operatorname{Hom}(M,M)$ is an isomorphism, and therefore onto. Hence, for the identity homomorphism Id_M there exists $\xi \in \varinjlim_{i\in I} \operatorname{Hom}(M,M_i)$ such that $Id_M = can(\xi)$. By definition of the direct limit, there exist $i \in I$, $g \in \operatorname{Hom}(M,M_i)$ such that $\xi = [g]$, and so: $Id_M = can(\xi) = (f_i)_*(g) = f_i \circ g$. We will now show that $M_i \simeq \operatorname{im}(g) \oplus \ker(f_i)$ and $M \simeq \operatorname{im}(g)$. Then, M will be a direct summand of the $< \lambda$-presented module M_i, and so will itself be $< \lambda$-presented by Remark 3.2.5.
The second claim, $M \simeq \operatorname{im}(g)$, is a direct consequence of the First Isomorphism Theorem and the fact that g is one-to-one (since $Id_M = f \circ g$ is one-to-one). For the other claim, choose any $m \in M_i$. Then $f_i(m) \in M$ and $m - g(f_i(m)) \in \ker(f_i)$, since $f_i(m - g(f_i(m))) = f_i(m) - f_i(g(f_i(m))) \overset{Id_M = f_i \circ g}{=} f_i(m) - f_i(m) = 0$. So $m = g(f_i(m)) + \big(m - g(f_i(m))\big) \in \operatorname{im}(g) + \ker(f_i)$, which shows that $M_i = \operatorname{im}(g) + \ker(f_i)$. It remains to show that $\operatorname{im}(g) \cap \ker(f_i) = \{0\}$. For that, pick an element m in $\operatorname{im}(g) \cap \ker(f_i)$. Since $m \in \operatorname{im}(g)$, consider $m' \in M \ \big(m = g(m')\big)$. Now, $m \in \ker(f_i)$, therefore $f_i(m) = 0$. So, $m' = Id_M(m') = f_i(g(m')) = f_i(m) = 0$, giving us $m = 0$. The proof is completed. □

We now give a definition that will play a crucial role to the rest of our analysis. Notice the word *almost*.

Definition 3.2.10. Let $\mathscr{C} \subseteq R$-Mod be a class of modules, and λ be a regular cardinal. A module M is called *almost-$(\mathscr{C},\lambda)$-projective* if there exists a λ-continuous direct system of $< \lambda$-presented modules $(M_i)_{i\in I}$ such that $M = \varinjlim_{i\in I} M_i$ and $\forall i \in I \ (M_i \in {}^{\perp}\mathscr{C})$. In this case, we say that the direct system *witnesses* the almost-$(\mathscr{C},\lambda)$-projectivity of M.

We can already see the importance of this definition:

Remark 3.2.11. If M is $<\lambda$-presented and almost-$(\mathscr{C},\lambda)$-projective (where λ is regular), then $M \in {}^{\perp}\mathscr{C}$.

Proof. Let $\big((M_i)_{i\in I}\ ,\ (f_{ji})_{i\leq j}\big)$ be the direct system that witnesses the almost-$(\mathscr{C},\lambda)$-projectivity of M. The map $\varinjlim_{i\in I}\mathrm{Hom}(M,M_i) \longrightarrow \mathrm{Hom}(M,\varinjlim_{i\in I} M_i) = \mathrm{Hom}(M,M)$ is onto. Therefore, the identity map Id_M factors through M_i for some $i\in I$, meaning that there exists $g \in \mathrm{Hom}(M,M_i)$ with $Id_M = f_i \circ g$ (where the $f_j : M_j \longrightarrow M$ are the canonical homomorphisms). But then M is a direct summand of $M_i \in {}^{\perp}\mathscr{C}$, giving us that $\mathrm{Ext}^1(M,C)$ is a direct summand of $\mathrm{Ext}^1(M_i,C) = 0$ for all $C \in \mathscr{C}$. □

The above remark is very close to what we set out to do in this chapter. However, we can do better than *almost*; for that, we have to introduce a new concept:

Definition 3.2.12. The homomorphism $f : M \longrightarrow N$ is called $\mathscr{C}$*-monomorphism* if for every $C\in\mathscr{C}$ the homomorphism $f^* : \mathrm{Hom}(N,C) \longrightarrow \mathrm{Hom}(M,C)$ is onto.

Let us familiarize ourselves with this new idea, by giving some remarks:

1) If $\mathscr{C}$ only contains injective modules, then every monomorphism is a $\mathscr{C}$-monomorphism.
(*Proof.* It follows immediately from the definition of injective modules.)

2) If the injective cogenerator $DR = \mathrm{Hom}(R_R,\mathbb{Q}/\mathbb{Z})$ is in $\mathscr{C}$, then every $\mathscr{C}$-monomorphism is a monomorphism.
(*Proof.* The exact sequence $0 \longrightarrow \ker f \longrightarrow M \xrightarrow{f} N$ of modules induces the exact sequence $\mathrm{Hom}(N,DR) \xrightarrow{f^*} \mathrm{Hom}(M,DR) \longrightarrow \mathrm{Hom}(\ker f,DR) \longrightarrow 0$, for every homomorphism $f : M \longrightarrow N$ and modules $M,N\in\mathscr{C}$. If $DR\in\mathscr{C}$ and f is a $\mathscr{C}$-monomorphism, then f^* is onto, meaning that $\mathrm{Hom}(\ker f,DR) = 0$. Hence $\ker f = 0$ and f is a monomorphism.)

3) The composition of two $\mathscr{C}$-monomorphisms is a $\mathscr{C}$-monomorphism.
(*Proof.* Remember that $(g\circ f)^* = f^*\circ g^*$. If both f^*,g^* are onto, so is their composition.)

4) If both $f\in\mathrm{Hom}(M,N)$, $g\in\mathrm{Hom}(N,L)$ are linear maps and $g\circ f$ is a $\mathscr{C}$-monomorphism, then f is also a $\mathscr{C}$-monomorphim.
(*Proof.* We want to show that $f^* : \mathrm{Hom}(N,C) \longrightarrow \mathrm{Hom}(M,C)$ is onto for any $C\in\mathscr{C}$. Pick

any $h \in \mathrm{Hom}(M,C)$. Since $g \circ f$ is a $\mathscr{C}$-monomorphism, there exists a $h' \in \mathrm{Hom}(L,C)$ such that $h' \circ (g \circ f) = h$. The morphism $h' \circ g \in \mathrm{Hom}(N,C)$ is then such that $f^*(h' \circ g) = h' \circ g \circ f = h$.)

Proposition 3.2.13. *Let $f : M \longrightarrow N$ be a $\mathscr{C}$-monomorphism, and D be a submodule of $\prod_{i\in I} C_i$ with $C_i \in \mathscr{C}$ for all $i \in I$. Then:*

- *Every $g : M \longrightarrow D$ factors through $\mathrm{im} f \longrightarrow D$.*
- *If $Ext^1(\mathrm{coker} f, D) = 0$, then f is a D-monomorphism.*

Proof. Let $\pi_i : \prod_{i\in I} C_i \longrightarrow C_i$ be the projection, and put $g_i = \pi_i \circ g : M \longrightarrow C_i$, i.e. for $m \in M$ we have $g(m) = (g_i(m))_{i\in I}$. Notice that $\ker g = \bigcap_{i\in I} \ker g_i$. Since f is a $\mathscr{C}$-monomorphism and $g_i \in \mathrm{Hom}(M,C_i)$, for every $i \in I$ there exists a homomorphism $h_i \in \mathrm{Hom}(N,C_i)$ with $g_i = h_i \circ f$, therefore $\ker f \subseteq \ker g_i$, which in turn implies that $\ker f \subseteq \ker g$. Now, since $\mathrm{im} f \simeq M/\ker f$ there exists $\bar{g} : \mathrm{im} f \longrightarrow D$ with $\bar{g} \circ \phi = g$ (where we have written $f = i \circ \phi$, $\phi : M \longrightarrow \mathrm{im} f$ and $i : \mathrm{im} f \longrightarrow N$ is the inclusion). This is what we were after.

For the second part of the proposition, consider the exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathrm{coker} f, D) \longrightarrow \mathrm{Hom}(N,D) \xrightarrow{i^*} \mathrm{Hom}(\mathrm{im} f, D) \longrightarrow \mathrm{Ext}^1(\mathrm{coker} f, D) = 0$$

This gives that i^* is onto, therefore there exists $\bar{\bar{g}} : N \longrightarrow D$ with $\bar{\bar{g}} \circ i = \bar{g}$. This concludes the proof. □

With this new tool, we can now give the following definition, a stronger notion than the one in Definition 3.2.10:

Definition 3.2.14. Let $\mathscr{C}$ be a class of modules, and λ be a regular cardinal. The module M is called $(\mathscr{C},\lambda)$-*projective* if it is almost-$(\mathscr{C},\lambda)$-projective and every canonical map $M_i \longrightarrow M$ from the direct system that witnesses it is a $\mathscr{C}$-monomorphism.

The importance of this definition can be seen in the next proposition, which is a partial answer to our original question:

Proposition 3.2.15. *Let λ be a regular cardinal, and M be a λ-presented module which is $(\mathscr{C},\lambda)$-projective. Then $M \in {}^{\perp}\mathscr{C}$.*

In order to prove Proposition 3.2.15 we first need to prove a special case of it in the form of a lemma, which is often called *Eklof's Lemma*.

Lemma 3.2.16 (Eklof's Lemma)**.** *Let $M = \varinjlim_{\alpha<\tau} M_\alpha$ be a module where τ is a regular cardinal, the direct system $\big((M_\alpha)_{\alpha<\tau}\ ,\ (f_{\beta\alpha})_{\beta\leq\alpha}\big)$ is τ-continuous, $\forall\alpha<\tau\ (M_\alpha \in {}^\perp\mathscr{C})$ and every $f_{\alpha+1,\alpha}$: $M_\alpha \longrightarrow M_{\alpha+1}$ is a $\mathscr{C}$-monomorphism. Then $M \in {}^\perp\mathscr{C}$.*

Proof. Pick any $C \in \mathscr{C}$. We want to show that $\mathrm{Ext}^1(M,C) = 0$, i.e. that every exact sequence $0 \longrightarrow C \xrightarrow{i} X \xrightarrow{p} M \longrightarrow 0$ splits. For every $\alpha < \tau$ we will construct a linear map $g_\alpha : M_\alpha \longrightarrow X$ such that:

i) $p \circ g_\alpha = f_\alpha$ (where $f_\alpha : M_\alpha \longrightarrow M$ is of course the canonical map)

ii) $\forall\beta < \alpha\ (g_\beta = g_\alpha \circ f_{\alpha\beta})$

Having done that, a homomorphism $g : M \longrightarrow X$ with $p \circ g = Id_M$ is induced, which is what we are after. The construction will be done via induction on the ordinal α:

$\circ\ \alpha = 0$: Since $M_0 \in {}^\perp\mathscr{C}$, the homomorphism $p_* : \mathrm{Hom}(M_0,X) \longrightarrow \mathrm{Hom}(M_0,M)$ is surjective. Hence, we can find $g_0 : M_0 \longrightarrow X$ with $p \circ g_0 = f_0$, as desired.

$\circ\ \alpha$=limit ordinal: By continuity of the direct system we have $M_\alpha = \varinjlim_{\beta<\alpha} M_\beta$, and so there exists a unique $g_\alpha : M_\alpha \longrightarrow X$ such that $g_\alpha \circ f_{\alpha\beta} = g_\beta$ for all $\beta < \alpha$.

$\circ\ \alpha = \beta + 1 < \tau$: As in the 0-th inductive step, since $M_\alpha \in {}^\perp\mathscr{C}$ there exists a homomorphism $h : M_\alpha \longrightarrow X$ with $p \circ h = f_\alpha$. Put $c = f_{\alpha\beta} = f_{\beta+1,\beta}$. We can calculate that

$$p \circ (g_\beta - h \circ c) = f_\beta - f_{\beta+1} \circ c = 0$$

i.e. $g_\beta - h \circ c \in \ker p_*$. Combining this with the fact that the sequence $0 \longrightarrow \mathrm{Hom}(M_\beta,C) \xrightarrow{i_*} \mathrm{Hom}(M_\beta,X) \xrightarrow{p_*} \mathrm{Hom}(M_\beta,M)$ is exact (which itself stems from $0 \longrightarrow C \xrightarrow{i} X \xrightarrow{p} M$ being exact and Theorem 1.1.46), we get a $t : M_\beta \longrightarrow C$ with $i \circ t = g_\beta - h \circ c$. But c is a $\mathscr{C}$-monomorphism, so there exists $s : M_{\beta+1} \longrightarrow C$ with $t = s \circ c$. Then $i \circ s \circ c = g_\beta - h \circ c \implies (i \circ s + h) \circ c = g_\beta$. Define $g_\alpha = g_{\beta+1} = i \circ s + h$. We can easily see that:

i) $p \circ g_\alpha = p(i \circ s + h) = p \circ i \circ s + p \circ h \overset{\mathrm{im}(i)=\ker(p)}{=} 0 + f_\alpha = f_\alpha$, and

ii) for any $\gamma < \alpha$, we have $g_\gamma = g_\beta \circ f_{\beta\gamma} = (i \circ s + h) \circ c \circ f_{\beta\gamma} = g_\alpha \circ f_{\alpha\beta} \circ f_{\beta\gamma} = g_\alpha \circ f_{\alpha\gamma}$

as desired. This concludes the proof. □

Continuing our work towards the proof of Proposition 3.2.15, we present the following lemma:

Lemma 3.2.17. *Let $\lambda > \aleph_0$ be a regular cardinal, $f \in Hom(M,N)$, $M = \varinjlim_{i\in I} M_i$, $N = \varinjlim_{j\in J} N_j$ where the direct systems $\big((M_i)_{i\in I}\ ,\ (a_{ji})_{i\leq j}\big), \big((N_i)_{j\in J}\ ,\ (b_{ji})_{i\leq j}\big)$ are λ-continuous and their modules are $< \lambda$-presented. Then:*
1) There exists a λ-continuous direct system $(u_k\ :\ M_{i_k} \longrightarrow N_{j_k} \mid k \in K)$ of homomorphisms with $f = \varinjlim_{k\in K} u_k$.
2) If f is an isomorphism, then every u_k above can also be taken to be an isomorphism.

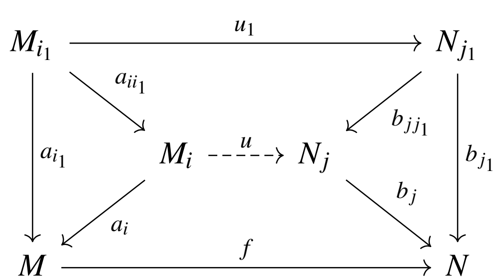
Proof. 1) We begin by defining the set

$$K = \{(i,j,u) \mid i \in I,\ j \in J,\ u\ :\ M_i \longrightarrow N_j \text{ with } b_j \circ u = f \circ a_i\}$$

as well as the following order on it:

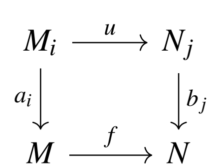
$$(i,j,u) \leq (i',j',u') \iff (i \leq i') \wedge (j \leq j') \wedge (b_{j'j} \circ u = u' \circ a_{i'i})$$

This makes K a directed set where every chain of length $< \lambda$ has a supremum. Indeed, for any two elements $(i_1,j_1,u_1),(i_2,j_2,u_2) \in K$ pick $i \geq i_1,i_2$ in I and $j \geq j_1,j_2$ in J; consider the below diagram:



where all of the already-completed subdiagrams are known to be commutative. Complete the diagram by finding $u\ :\ M \longrightarrow N$ (possibly by increasing the index j) such that the whole diagram is commutative. Repeat the process with the element (i_2,j_2,u_2), and we end up with $(i,j,u) \in K$ that is $\geq$ both $(i_1,j_1,u_1),(i_2,j_2,u_2)$.

We will now show that $f = \varinjlim_{k\in K}(M_{i_k} \xrightarrow{u_k} N_{j_k})$. Pick $m \in M$ and consider $i \in I$, $m_i \in M_i$ such that $m = a_i(m_i)$. Since the homomorphism $f \circ a_i\ :\ M_i \longrightarrow N$ factors through some N_j, for some large i,j we can find $u\ :\ M_i \longrightarrow N_j$ such that $(i,j,u) \in K$. Then the following diagram is commutative by construction of K:



2) Following the construction of K above, we will show that there exists a cofinal subposet K' of K such that for every $(i, j, u) \in K'$ we have u = isomorphism. Pick a random $(i_0, j_0, u_0) \in K$. We want to find an element $(i, j, u) > (i_0, j_0, u_0)$ such that u is an isomorphism. It is true that

$$\mathrm{Hom}(N_{j_0}, M) = \mathrm{Hom}(N_{j_0}, \varinjlim_{i \in I} M_i) = \varinjlim_{i \in I} \mathrm{Hom}(N_{j_0}, M_i)$$

and

$$f^{-1} \circ b_{j_0} \in \mathrm{Hom}(N_{j_0}, M)$$

and so $\exists v_0 : N_{j_0} \longrightarrow M_{i_1}$ for some $i_1 > i_0$ with $a_{i_1} \circ v_0 = f^{-1} \circ b_{j_0}$. We can pick i_1 large enough so that we can find $(i_1, j_1, u_1) > (i_0, j_0, u_0)$ in K. We repeat the argument for the new (i_1, j_1, u_1) element and find a larger $(i_2, j_2, u_2) \in K$ and so on.We can summarize our findings in the following diagram:

$$\begin{array}{ccccccccc}
M_{i_0} & \longrightarrow & M_{i_1} & \longrightarrow & M_{i_2} & \longrightarrow & \ldots M_{i_\infty} = \varinjlim_{n \in \mathbb{N}} M_{i_n} & \longrightarrow & M \\
\downarrow u_0 & \nearrow v_0 & \downarrow u_1 & \nearrow v_1 & \downarrow u_2 & & v_\infty \uparrow\downarrow u_\infty & & \downarrow f \\
N_{j_0} & \longrightarrow & N_{j_1} & \longrightarrow & N_{j_2} & \longrightarrow & \ldots N_{i_\infty} = \varinjlim_{n \in \mathbb{N}} N_{j_n} & \longrightarrow & N
\end{array}$$

where u_∞, v_∞ are the direct limits of their respective sequences. But now, making use of the commutative triangles that emerge, we have $u_\infty \circ v_\infty = Id_{N_\infty}$ and $v_\infty \circ u_\infty = Id_{M_\infty}$. This means that u_∞ is an isomorphism, and $(i_\infty, j_\infty, u_\infty) > (i_0, j_0, u_0)$. This concludes the proof. □

The above proof is fairly technical and a lot of the details have been skipped over. What we are interested in is the corollary that stems from the lemma.

Corollary 3.2.18. *If $\lambda > \aleph_0$ is regular, $M = \varinjlim_{i \in I} M_i = \varinjlim_{j \in J} N_j$ with the two direct systems being λ-continuous, and their modules $< \lambda$-presented, then there exist cofinal λ-continuous subsystems $(M_i \mid i \in I'), (N_j \mid j \in J')$ that are isomorphic to each other.*

Proof. Simply put $f = Id_M$ in the above lemma. □

This corollary essentially allows us to talk about an "intersection" of the two systems, something that will help us a lot in the theory to come. With that, we are finally able to prove Proposition 3.2.15.

Proof. Since M is λ-presented, we can write $M = \varinjlim_{\alpha < \lambda} M_\alpha$ for some λ-continuous direct system where each module M_α is $< \lambda$-presented (see Remark 3.2.7). But M is also $(\mathscr{C}, \lambda)$-projective, therefore we can write $M = \varinjlim_{j\in J} N_j$ for a λ-continuous system where each N_j is $< \lambda$-presented, each belongs to ${}^{\perp}\mathscr{C}$ and each $N_j \xrightarrow{can} M$ is a $\mathscr{C}$-monomorphism. We can now "merge" these two systems together by use of the last corollary to find a cofinal λ-continuous $J' \subseteq J$ that is linearly ordered. Apply Eklof's Lemma to this system, and we get our desired result: $M \in {}^{\perp}\mathscr{C}$. □

3.3 The singular-cardinal case

Proposition 3.2.15 has been satisfactory in answering our initial question when the module M is λ-presented, for some regular cardinal λ. Now, we turn our attention towards the case where M is κ-presented for some singular cardinal κ. The answer here will be the main theorem of our thesis, Shelah's Singular Compactness Theorem.

Let us first discuss filters.

Definition 3.3.1. Let X be a non-empty set. A family $\mathscr{F} \subseteq \mathscr{P}(X)$ is called *a filter* on X if:
i) $\forall A \in \mathscr{F}\ \Big(\forall B \subseteq X\ \big((A \subseteq B \longrightarrow B \in \mathscr{F})\big)\Big)$ ($\mathscr{F}$ is upwards closed) and
ii) $\forall A, B \in \mathscr{F}\ (A \cap B \in \mathscr{F})$ ($\mathscr{F}$ is closed under finite intersections)

Definition 3.3.2. If λ=regular cardinal and $\mathscr{F}$ is a filter on X, then $\mathscr{F}$ is called *λ-complete* if for every family $(A_i)_{i\in I}$ with $|I| < \lambda$ and $A_i \in \mathscr{F}$ we have $\bigcap_{i\in I} A_i \in \mathscr{F}$.

The following examples will help us familiarize ourselves with filters. Where it appears, X is a non-empty set:
i) Pick $x \in X$ and consider the family $\mathscr{F} = \{A \subseteq X \mid x \in A\}$. Then $\mathscr{F}$ is a filter on X (this type of filter is signified as $\mathscr{F}(x)$ and is called the principal filter of x).
ii) Put $\mathscr{F} = \{A \subseteq X \mid \text{card}(X \backslash A) < \infty\}$. Then $\mathscr{F}$ is a filter on X.
(*Proof.* Observe that if $A \subseteq B$, then $X \backslash B \subseteq X \backslash A$ and $|X \backslash B| \leq |X \backslash A|$. Also for any $A, B \in \mathscr{F}$, $|X \backslash (A \cap B)| = |(X \backslash A) \cup (X \backslash B)| \leq |X \backslash A| + |X \backslash B| < \infty$.)
iii) Let $(I, \leq)$ be a directed set. Define $\mathscr{F}_I = \big\{A \subseteq I \mid \exists i \in I\ (\{j \in I \mid j \geq i\} \subseteq A)\big\}$. Then $\mathscr{F}_I$ is a filter on I. Moreover, if $(I, \leq)$ is λ-directed then $\mathscr{F}_I$ is λ-complete.
(*Proof.* For any $i \in I$ put $F_i := \{j \in I \mid j \geq i\}$. First, observe that for any $A \subseteq B \subseteq X$ with $A \in \mathscr{F}_I$, if $F_i \subseteq A$, then $F_i \subseteq B$ as well. Second, take any two $A, B \in \mathscr{F}_I$ and let F_{i_0}, F_{i_1} be the sets that witness this. Since I is directed, we can find $i_2 \in I$ such that $i_0 \leq i_2$ and $i_1 \leq i_2$. But then F_{i_2} is a subset of both F_{i_0} and F_{i_1}, so $F_{i_2} \subseteq A \cap B$. This last argument can be adapted to show that λ-directedness of $(I, \leq)$ implies λ-completenss of the filter $\mathscr{F}_I$.)

iv) The filter $\mathscr{F}_I$ is a principal filter $\mathscr{F}(x)$ for some $x \in I$ if-f x is the maximum element of $(I, \leq)$. (*Proof.* For the $\Longrightarrow$ way, notice that $\{x\} \in \mathscr{F}(x)$, hence $\{x\} \in \mathscr{F}_I$ and that the sets F_i (see above proof) are non-empty; use these to show that x is a maximal element in I. Since I is directed, it follows that x is the maximum element. For the $\Longleftarrow$ way, use the definitions of the two filters to show that $\mathscr{F}_I \subseteq \mathscr{F}(x)$ and that $\mathscr{F}(x) \subseteq \mathscr{F}_I$ (for the last one, consider the set $F_x = \{x\}$).)

Symbolization: Let X be a non-empty set, $(M_x)_{x\in X}$ be R-modules and $M = \prod_{x\in X} M_x$. For $m = (m_x)_{x\in X} \in M$ we will write $z(m) = \{x \in X \mid m_x = 0\}$ to mean the set of coordinates of m that are zero.

Remark 3.3.3. For any $m, m' \in M$ and $r \in R$ we have: $z(m) \cap z(m') \subseteq z(m + m')$ and $z(m) \subseteq z(rm)$. Therefore, if for a filter $\mathscr{F}$ on X we define the subset $\sum_{\mathscr{F}} M = \{m \in M \mid z(m) \in \mathscr{F}\}$, then it is trivial to show that $\sum_{\mathscr{F}} M$ is a (R-)submodule of M. We will call this the *$\mathscr{F}$-product of* $(M_x)_{x\in X}$. Moreover the quotient $M/\sum_{\mathscr{F}} M$ will be called the *reduced $\mathscr{F}$-product of* $(M_x)_{x\in X}$.

If $m, m' \in M$ and $p : M \longrightarrow M/\sum_{\mathscr{F}} M$ is the quotient map, then $p(m) = p(m')$ if-f there exists an $A \in \mathscr{F}$ such that $m_x = m'_x$ for all $x \in A$ (we say that the coordinates of m and m' are equal *almost everywhere*).

Let us now tie these new ideas back to the theory we had started investigating. The next definition is pretty self-explanatory:

Definition 3.3.4. Let $\mathscr{C}$ be a class of R-modules. We say that $\mathscr{C}$ is *closed under filtered products* if for every family $(M_x)_{x\in X}$ of modules in $\mathscr{C}$ and every filter $\mathscr{F}$ on X we have $\sum_{\mathscr{F}} \left(\prod_{x\in X} M_x\right) \in \mathscr{C}$.

Remark 3.3.5. If a class $\mathscr{C} \subseteq R$-Mod is closed under products and directed unions, then it is closed under filtered products.
(*Proof.* Notice that $\sum_{\mathscr{F}} \left(\prod_{x\in X} M_x\right) = \bigcup\limits_{A\in\mathscr{F}} \prod\limits_{x\in A^c} M_x$.)

Proposition 3.3.6. *Let $\mathscr{C}$ be a class of modules that is closed under filtered products, and let $\lambda > \aleph_0$ be a regular cardinal. If $M \in {}^\perp\mathscr{C}$ and $M = \varinjlim_{i\in I} M_i$ for a λ-continuous directed system $\left((M_i)_{i\in I}\ ,\ (f_{ji})_{i\leq j}\right)$ of $< \lambda$-generated modules, then there exists a λ-closed cofinal subset $J \subseteq I$ such that for all $j \in J$ the canonical map $f_j : M_j \longrightarrow M$ is a $\mathscr{C}$-monomorphism.*

Proof. Consider the set

$$S = \{i \in I \mid f_i : M_i \longrightarrow M \text{ is not a } \mathscr{C}\text{-monomorphism}\}$$

Then for each $i \in S$ there exist a $C_i \in \mathscr{C}$ and a homomorphism $g_i : M_i \longrightarrow C_i$ such that $g_i \notin \mathrm{im}(f_i^*)$. For $i \in I \backslash S$ put $g_i = 0$ and pick any $C_i \in \mathscr{C}$. For every $i, j \in I$ define the map $h_{ji} : M_i \longrightarrow C_j$ as follows:

$\circ$ if $i \nleq j$ then $h_{ji} = 0$, or

$\circ$ if $i \leq j$ then $h_{ji} = g_j \circ f_{ji}$, so that the next diagram commutes:

$$\begin{array}{ccc} M_i & \xrightarrow{f_{ji}} & M_j \\ & \searrow^{h_{ji}} & \downarrow{\scriptstyle g_j} \\ & & C_j \end{array}$$

Now if $i \leq j \leq k$, then the following diagram also commutes:

$$\begin{array}{ccccc} M_i & \xrightarrow{f_{ji}} & M_j & \xrightarrow{f_{kj}} & M_k \\ & \overset{h_{kj} \circ f_{ji}}{\searrow} & & \searrow^{h_{kj}} & \downarrow{\scriptstyle g_k} \\ & & & & C_k \end{array}$$

But since $f_{kj} \circ f_{ji} = f_{ki}$ we have $h_{kj} \circ f_{ji} = g_k \circ f_{kj} \circ f_{ji} = h_{ki}$, and so for all $i \leq j$ it is true that $\{k \in I \mid h_{ki} = h_{kj} \circ f_{ji}\} \in \mathscr{F}_I \quad (1)$.

Consider the inclusions $v_i : M_i \longrightarrow \bigoplus_{i \in I} M_i$ and the projections $\pi_j : \prod_{k \in I} C_k \longrightarrow C_j$, and define the map $h : \bigoplus_{i \in I} M_i \longrightarrow \prod_{k \in I} C_k$ such that $h_{ji} = \pi_j \circ h \circ v_i$ for all $i, j \in I$. Also put $C = \prod_{j \in I} C_j$ and let $p : C \longrightarrow C / \sum_{\mathscr{F}_I} C$ be the usual quotient map. By definition of the direct limit and the direct sum of modules, there exists a unique $\phi : \bigoplus_{i \in I} M_i \longrightarrow M$ such that $\phi \circ v_i = f_i$ for all $i \in I$. Also, using the fact from (1) we get that $p \circ h \circ v_i = p \circ h \circ v_j \circ f_{ji}$, and so there exists a unique $u : M \longrightarrow C / \sum_{\mathscr{F}_I} C$ with $p \circ h = u \circ \phi$. We can summarize these in the following diagram:

$$\begin{array}{ccccc} M_i & \xrightarrow{v_i} & \bigoplus_{i \in I} M_i & \xrightarrow{\phi} & M \\ & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle u} \\ & & C & \xrightarrow{p} & C / \sum_{\mathscr{F}_I} C \\ & & \downarrow{\scriptstyle \pi_k} & & \\ & & C_k & & \end{array}$$

$\mathscr{C}$ is closed under filtered products, meaning $\sum_{\mathscr{F}_I} C \in \mathscr{C}$. Also $M \in {}^{\perp}\mathscr{C}$ by hypothesis, so $\mathrm{Ext}^1(M, \sum_{\mathscr{F}_I} C) = 0$. Therefore the map $p_* : \mathrm{Hom}(M, C) \longrightarrow \mathrm{Hom}(M, C / \sum_{\mathscr{F}_I} C)$ is surjective. Put $g : M \longrightarrow C$ such that $u = p_*(g) = p \circ g$. Then $p \circ h = u \circ \phi = p \circ g \circ \phi$, which gives us $p \circ h \circ v_i = p \circ g \circ \phi \circ v_i$ for all $i \in I$.

Now, consider $i \in I$ and let $\{m_\alpha \mid \alpha < \tau\}$ be a generating set of M_i (where obviously $\tau < \lambda$). Pick $\alpha < \tau$. Then the above equation gives $p \circ h \circ v_i(m_\alpha) = p \circ g \circ \phi \circ v_i(m_\alpha)$, and Remark 3.3.3 gives a set $A_{\alpha,i} \in \mathscr{F}_I$ such that $\pi_k \circ h \circ v_i(m_\alpha) = \pi_k \circ g \circ \phi \circ v_i(m_\alpha)$ for all $k \in A_{\alpha,i}$. The filter $\mathscr{F}_I$ is λ-complete, and so $\bigcap_{\alpha<\tau} A_{\alpha,i} \in \mathscr{F}_I$. But filters are "upwards-closed" and so for all $i \in I$ we have $\{k \in I \mid \pi_k \circ h \circ v_i = \pi_k \circ g \circ \phi \circ v_i\} \in \mathscr{F}_I \quad (2)$. Therefore for any $i \in I$ there exists $s(i) > i$ such that for all $k \geq s(i)$ we have $\pi_k \circ h \circ v_i = \pi_k \circ g \circ \phi \circ v_i$.

For $i \in I$ consider the sequence $i < s(i) < s(s(i)) = s^2(i) < s(s^2(i)) = s^3(i) < \dots$ and put $j = s^\infty(i) = \sup_{n\in\omega}\{s^n(i)\} \in I$. So $M_j = \varinjlim_{n\in\omega} M_{s^n(i)}$ and $\forall k \geq j \ \left(\pi_k \circ h \circ v_j = \pi_k \circ g \circ \phi \circ v_j\right)$. Therefore the subset $J = \{j \in I \mid \forall k \geq j \ \left(\pi_k \circ h \circ v_j = \pi_k \circ g \circ \phi \circ v_j\right)\}$ is cofinal in I and λ-closed. Finally, pick $j \in J$. Then (for $k = j$) it is true that

$$\pi_j \circ h \circ v_j = \pi_j \circ g \circ \phi \circ v_j \iff$$
$$h_{jj} = \pi_j \circ g \circ \phi \circ v_j \iff$$
$$g_j = \pi_j \circ g \circ f_j = f_j^*(\pi_j \circ g)$$

This means that $J \cap S = \emptyset$, which concludes the proof. □

We are now ready to introduce the main star of the show: Shelah's Singular Compactness Theorem. The proof of this needs to be broken down into bite-sized pieces, so one should look at this as its own chapter, rather than one big and hard to follow proof.

3.3.1 Shelah's Singular Compactness Theorem

Theorem 3.3.7. *Let κ=singular cardinal, M=κ-presented module and $\mathscr{C}$=class of modules which is closed under filtered products. Suppose that there exists an infinite cardinal ν such that for each successor cardinal λ with $\nu < \lambda < \kappa$ there exists a direct system S_λ of modules that witnesses the almost-$(\mathscr{C},\lambda)$-projectivity of M. Then $M \in {}^\perp\mathscr{C}$.*

We now begin the proof.

M is a κ-presented module, where κ is a singular uncountable cardinal, so by Proposition 3.2.6 we can write $M = \varinjlim_{i\in I} M_i$ for a ($\aleph_1$-continuous) direct system $\left((M_i)_{i\in I}\,,\,(f_{ji})_{i\leq j}\right)$ of countably presented modules. For each regular cardinal $\lambda \geq \aleph_1$ and each directed subset $J \subseteq I$ with $|J| < \lambda$, put $M_J = \varinjlim_{j\in J} M_j$. Define $I_\lambda = \{J \subseteq I \mid J = \text{ directed and } |J| < \lambda\}$ and $S^\lambda = \left((M_J)_{J\in I_\lambda}\right)$ together with the respective homomorphisms. The following lemma ensures that this is in fact a direct system of modules:

Lemma 3.3.8. *Let $(I,\leq)$ be a directed set, and $X \subseteq I$ with $|X| < \lambda$, for a regular uncountable cardinal λ. Then there exists a directed subset $Y \subseteq I$ with $|Y| < \lambda$ and $X \subseteq Y$.*

Proof. Since I is a directed set, we may define a function $f : I \times I \longrightarrow I$ such that for each input (i,j) it outputs an element $k \in I$ with $i \leq k$ and $j \leq k$. Now, put $X_0 = X$ and define recursively the sets $X_n = X_{n-1} \cup \{f(i,j) \in I \mid i,j \in X_{n-1}\}$ for all natural numbers $n > 0$. Then the set $Y := \bigcup_{n\in\omega} X_n$ is our sought-after subset.
Indeed, Y is directed, since any two elements y_n, y_m can be found in some X_n, X_m; put $l = \max\{n,m\}$, and notice that the set X_{l+1} contains the image $f(y_n,y_m)$ which is $\geq$ both y_n, y_m. Furthermore, the cardinality of Y is $< \lambda$, since each X_n has cardinality $< \lambda$ (this can be shown recursively, using the equations $\lambda + \lambda = \lambda$ and $\lambda \cdot \lambda = \lambda$ from Theorems 2.4.28 and 2.4.30) and the sequence of sets has cardinality $\aleph_0 < \lambda$; from the fact that λ is regular, Proposition 2.4.45 gives us that the cardinality of their union also has cardinality $< \lambda$. Finally, it is obvious that $X = X_0 \subseteq Y$. □

Next, we prove that the system S^λ is λ-continuous. First, notice that for any chain $\mathscr{X}$ in I_λ with $|\mathscr{X}| < \lambda$ we have $\bigcup \mathscr{X} \in I_\lambda$. Indeed, since λ is regular and $\mathscr{X}$ is a family of $< \lambda$ sets each of cardinality $< \lambda$, this means that $|\bigcup \mathscr{X}| < \lambda$. Furthermore, take any two elements $j_0, j_1 \in \bigcup \mathscr{X}$; then there exist $J_0, J_1 \in \mathscr{X}$ such that $j_0 \in J_0, j_1 \in J_1$. But $\mathscr{X}$ is a chain, therefore we may assume that $J_0 \subseteq J_1$. Then by directedness of J_1 we can find an element $j \in J_1$ such that $j \geq j_0$ and $j \geq j_1$. This j also belongs to $\bigcup \mathscr{X}$, which shows that $\bigcup \mathscr{X}$ is a directed set.
But then for any such chain we also have:

$$M_{\bigcup \mathscr{X}} = \varinjlim_{i \in \bigcup \mathscr{X}} M_i = \varinjlim_{J \in \mathscr{X}} \big(\varinjlim_{i\in J} M_i\big) = \varinjlim_{J\in\mathscr{X}} M_J \in S^\lambda$$

which gives us the λ-continuity. Finally, since we can write

$$M_J = \varinjlim_{i\in J} M_i = \operatorname{coker}\big(\bigoplus_{i<j\in J} M_i \longrightarrow \bigoplus_{i\in J} M_i\big)$$

we have that every M_J in S^λ is $< \lambda$-presented.

For every successor cardinal λ (strictly between ν and κ), consider the λ-continuous direct systems S_λ (given by hypothesis) and S^λ (constructed). By Corollary 3.2.18 we can replace the S_λ with the "intersection" of the aforementioned systems. Then, each $M_J \in S^\lambda$ that appears in our new system is itself the direct limit of a direct subsystem of our original $\big((M_i)_{i\in I}\ ,\ (f_{ji})_{i\le j}\big)$ system; write $\mathfrak{I}_\lambda$ for the set containing the directed subsets $J \subseteq I$ that index these subsystems and put $\mathfrak{I} = \bigcup_{\nu<\lambda<\kappa} \mathfrak{I}_\lambda$. It is now obvious that each $\mathfrak{I}_\lambda$ has the following properties:

- $\emptyset \in \mathfrak{I}_\lambda$
  (*Proof.* The zero module $\{0\}$ is the direct limit of an empty directed system of modules and exists in S_λ as well.)

- if $A \subseteq I$ and $|A| < \lambda$, then $\exists B \in \mathfrak{I}_\lambda$ such that $A \subseteq B$
  (*Proof.* This is exactly Lemma 3.3.8.)

- if $\mathscr{X} \subseteq \mathfrak{I}_\lambda$ is a chain with $|\mathscr{X}| < \lambda$, then $\bigcup \mathscr{X} \in \mathfrak{I}_\lambda$
  (*Proof.* Both the old S_λ (by definition) and the constructed S^λ (as shown above) are λ-continuous. So the module $M_{\bigcup \mathscr{X}}$ will also exist in the new direct system.)

Now, let us define the set $\mathscr{W} = \{(A,B) \mid A \subseteq B \subseteq I \text{ with } A,B = \text{ directed}\}$, and more specifically its subset $\mathscr{W}_0 = \{(\emptyset,B) \in \mathscr{W}\}$. These become ordered sets if we define the order:

$$(A,B) \le (A',B') \iff (A \subseteq A') \wedge (B \subseteq B')$$

For every $A \subseteq B \subseteq I$ directed sets, consider the canonical map $c_{BA} : \varinjlim_{i\in A} M_i \longrightarrow \varinjlim_{i\in B} M_i$ and put $\Phi(A,B) = \operatorname{coker}(c_{BA})$. More specifically we have $\Phi(\emptyset,B) = \varinjlim_{i\in B} M_i$, and $\Phi(\emptyset,\emptyset) = \{0\}$. Finally, for every $(A,B),(C,D) \in \mathscr{W}$ with $(A,B) \le (C,D)$ define the map $\Phi(A,B) \longrightarrow \Phi(C,D)$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
\Phi(\emptyset,A) & \xrightarrow{c_{BA}} & \Phi(\emptyset,B) & \longrightarrow & \Phi(A,B) & \longrightarrow & 0 \\
\downarrow{\scriptstyle c_{CA}} & & \downarrow{\scriptstyle c_{DB}} & & \vdots & & \\
\Phi(\emptyset,C) & \xrightarrow{c_{DC}} & \Phi(\emptyset,D) & \longrightarrow & \Phi(C,D) & \longrightarrow & 0
\end{array}$$

We make the following observations:

1. If $A \in \mathfrak{I}$ then $\Phi(\emptyset,A) \in {}^\perp\mathscr{C}$.

2. If $Y \in {}^\perp\mathscr{C}$ and the homomorphism $f : X \longrightarrow Y$ is a $\mathscr{C}$-monomorphism, then $\operatorname{coker}(f) \in {}^\perp\mathscr{C}$.

3. If $(A,B) \in \mathscr{W}, B \in \mathfrak{I}$ and the natural map $\Phi(\emptyset,A) \longrightarrow \Phi(\emptyset,B)$ is a $\mathscr{C}$-monomorphism, then $\Phi(A,B) \in {}^\perp\mathscr{C}$.

From these three, we will prove the second observation (the first is simply Remark 3.2.11 applied to the module $\varinjlim_{i\in A} M_i = \Phi(\emptyset, A)$, whereas the third one is derived from the first two observations).

Proof. Pick $C \in \mathscr{C}$ and a homomorphism $g : \operatorname{im}(f) \longrightarrow C$. Write $f = i \circ \pi$ where $i : \operatorname{im}(f) \longrightarrow Y$ is the injection as usual. Then there exists $h : Y \longrightarrow C$ such that $g \circ \pi = h \circ f = h \circ i \circ \pi$. But $\pi : X \longrightarrow \operatorname{im}(f)$ is surjective, therefore $g = h \circ i = i^*(h)$. We have thus shown that the map $i^* : \operatorname{Hom}(Y, C) \longrightarrow \operatorname{Hom}(\operatorname{im}(f), C)$ is surjective. Combining this with the fact that the sequence

$$0 \longrightarrow \operatorname{Hom}(\operatorname{coker}(f), C) \longrightarrow \operatorname{Hom}(Y, C) \xrightarrow{i^*} \operatorname{Hom}(\operatorname{im}(f), C) \longrightarrow \operatorname{Ext}^1(\operatorname{coker}(f), C) \longrightarrow 0$$

is exact gives us that $\operatorname{Ext}^1(\operatorname{coker}(f), C) = 0$. This concludes the proof: $\operatorname{coker}(f) \in {}^{\perp}\mathscr{C}$. □

Let us now define a weaker order on the set $\mathscr{W}$. For $(A, B), (C, D) \in \mathscr{W}$ with $(A, B) \leq (C, D)$ write $(A, B) \preceq (C, D)$ if-f the map $\Phi(A, B) \longrightarrow \Phi(C, D)$ is a $\mathscr{C}$-monomorphism.

Some properties of this new order which we will later use:

i) $(\emptyset, \emptyset) \in \mathscr{W}$ is the minimum element

ii) if $(A, B) \preceq (C, D)$ then $(A, B) \leq (C, D)$

iii) for all $V, W, X \in \mathscr{W}$ with $V \leq W \leq X$ and $V \preceq X$, we have that $V \preceq W$

iv) let $(A, B), (C, D) \in \mathscr{W}$ be such that $(A, B) \leq (C, D)$ and $(A, C) \preceq (B, D)$ and $(\emptyset, B) \preceq (\emptyset, D)$. Then $(A, B) \preceq (C, D)$

v) let $\big((A_n, B_n)\big)_{n\in\omega}$ be a $\preceq$-increasing sequence in $\mathscr{W}$. Put $(A_\omega, B_\omega) = \big(\bigcup_{n\in\omega} A_n, \bigcup_{n\in\omega} B_n\big) \in \mathscr{W}$ and suppose that $(\emptyset, A_n) \preceq (\emptyset, B_\omega)$ for all $n \in \omega$. If $B_n \in \mathfrak{I}$ for all $n \in \omega$, then $(\emptyset, A_\omega) \preceq (\emptyset, B_\omega)$.

Let's discuss their proofs:

Proof. (i) Obviously, for any (A, B) we have $(\emptyset, \emptyset) \leq (A, B)$. Furthermore, the map $\Phi(\emptyset, \emptyset) = \{0\} \longrightarrow \Phi(A, B)$ is the zero map, which is easily seen to be a $\mathscr{C}$-monomorphism. □

Proof. (ii) This is immediate from the definition of the new order. □

Proof. (iii) Take any three elements $(A_1, B_1) \leq (A_2, B_2) \leq (A_3, B_3)$ in $\mathscr{W}$ such that $(A_1, B_1) \preceq (A_3, B_3)$. Consider the three commutative diagrams:

$$(1)\qquad \begin{array}{ccccccc}
\Phi(\emptyset, A_1) & \longrightarrow & \Phi(\emptyset, B_1) & \longrightarrow & \Phi(A_1, B_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\Phi(\emptyset, A_2) & \longrightarrow & \Phi(\emptyset, B_2) & \longrightarrow & \Phi(A_2, B_2) & \longrightarrow & 0
\end{array}$$

$$(2)\qquad \begin{array}{ccccccc}
\Phi(\emptyset,A_2) & \longrightarrow & \Phi(\emptyset,B_2) & \longrightarrow & \Phi(A_2,B_2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
\Phi(\emptyset,A_3) & \longrightarrow & \Phi(\emptyset,B_3) & \longrightarrow & \Phi(A_3,B_3) & \longrightarrow & 0
\end{array}$$

$$(3)\qquad \begin{array}{ccccccc}
\Phi(\emptyset,A_1) & \longrightarrow & \Phi(\emptyset,B_1) & \longrightarrow & \Phi(A_1,B_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle h} & & \\
\Phi(\emptyset,A_3) & \longrightarrow & \Phi(\emptyset,B_3) & \longrightarrow & \Phi(A_3,B_3) & \longrightarrow & 0
\end{array}$$

that arise from the definitions of f,g,h. Splice them together into one:

$$\begin{array}{ccccccc}
\Phi(\emptyset,A_1) & \longrightarrow & \Phi(\emptyset,B_1) & \longrightarrow & \Phi(A_1,B_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & {\scriptstyle h} & \\
\Phi(\emptyset,A_2) & \longrightarrow & \Phi(\emptyset,B_2) & \longrightarrow & \Phi(A_2,B_2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle g} & & \\
\Phi(\emptyset,A_3) & \longrightarrow & \Phi(\emptyset,B_3) & \longrightarrow & \Phi(A_3,B_3) & \longrightarrow & 0
\end{array}$$

Now this is also easily seen to be a commutative diagram (using the properties of the canonical homomorphisms of the direct system). In particular, we have $h = g \circ f$. Since h is a $\mathscr{C}$-monomorphism (by the weak order $(A_1,B_1) \preceq (A_3,B_3)$), from the 4th remark under Definition 3.2.12 we get that f is also a $\mathscr{C}$-monomorphism, which was desired. □

Proof. (iv) Consider the following diagram, which has exact rows and columns:

$$\begin{array}{ccccccc}
\Phi(\emptyset,A) & \longrightarrow & \Phi(\emptyset,B) & \longrightarrow & \Phi(A,B) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \alpha} & & \\
\Phi(\emptyset,C) & \longrightarrow & \Phi(\emptyset,D) & \longrightarrow & \Phi(C,D) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\Phi(A,C) & \xrightarrow{\gamma} & \Phi(B,D) & & & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Pick $N \in \mathscr{C}$. We will show that the induced map $\mathrm{Hom}(\alpha,N) : \mathrm{Hom}(\Phi(C,D),N) \longrightarrow \mathrm{Hom}(\Phi(A,B),N)$ is onto. Apply the $\mathrm{Hom}(-,N)$ functor to the above diagram. By the Snake Lemma (1.2.29) we get an exact sequence

$$\ker(\gamma^*) \longrightarrow \mathrm{Hom}(\Phi(C,D),N) \xrightarrow{\alpha^*} \mathrm{Hom}(\Phi(A,B),N) \longrightarrow \mathrm{coker}(\gamma^*) \longrightarrow \cdots \longrightarrow \cdots$$

and since γ^* is onto by hypothesis (γ is a $\mathscr{C}$-monomorphism), we finally get that α^* is onto, meaning that α is itself a $\mathscr{C}$-monomorphism. $\square$

Proof. (v) From the relations $A_n \subseteq B_n \subseteq B_\omega$ we get $(\emptyset, A_n) \leq (\emptyset, B_n) \leq (\emptyset, B_\omega)$ for all $n \in \omega$. But we also have $(\emptyset, A_n) \preceq (\emptyset, B_\omega)$, therefore by property (iv) we get $(\emptyset, A_n) \preceq (\emptyset, B_n)$ for all $n \in \omega$. Since $B_n \in \mathfrak{I}$ by observation (1) above we have $\Phi(\emptyset, B_n) \in {}^\perp\mathscr{C}$, and so $\Phi(A_n, B_n) \in {}^\perp\mathscr{C}$ for all $n \in \omega$. By Eklof's Lemma we get: $\Phi(A_\omega, B_\omega) = \varinjlim_{n\in\omega} \Phi(A_n, B_n) \in {}^\perp\mathscr{C}$.

We want to prove that $f : \Phi(\emptyset, A_\omega) \longrightarrow \Phi(\emptyset, B_\omega)$ is a $\mathscr{C}$-monomorphism. Write f in the usual form $f = i \circ \phi$, where $i : \mathrm{im}(f) \longrightarrow \Phi(\emptyset, B_\omega)$ is the injection. Also, pick $C \in \mathscr{C}$ and a homomorphism $g : \Phi(\emptyset, A_\omega) \longrightarrow C$. The proof will be complete once we finish proving the 3 steps below:

1. $\ker(f) \subseteq \ker(g)$
2. $\exists \gamma : \mathrm{im}(f) \longrightarrow C$ with $\gamma \circ \phi = g$
3. $\exists h : \Phi(\emptyset, B_\omega) \longrightarrow C$ with $h \circ i = \gamma$, meaning: $f^*(h) = h \circ f = \gamma \circ \phi = g$.

Step 2 follows immediately from step 1: Define the morphism $\tilde{\gamma} : \Phi(\emptyset, A_\omega)/\ker(\phi) \longrightarrow C$ to be $\tilde{\gamma}(x + \ker(\phi)) = g(x)$. The fact that $\ker(\phi) = \ker(f) \subseteq \ker(g)$ means that $\tilde{\gamma}$ is well-defined. From the First Isomorphism Theorem we get the isomoprhism $\tilde{\phi} : \mathrm{im}(f) \longrightarrow \Phi(\emptyset, A_\omega)/\ker(\phi)$ with $\tilde{\phi}(\phi(x)) = x + \ker(\phi)$. The desired γ is simply $\tilde{\gamma} \circ \tilde{\phi}$.
For step 3 consider the exact sequence $0 \longrightarrow \mathrm{im}(f) \xrightarrow{i} \Phi(\emptyset, B_\omega) \longrightarrow \Phi(A_\omega, B_\omega) \longrightarrow 0$ and apply the $\mathrm{Hom}(-, C)$ functor to it. Since $\Phi(A_\omega, B_\omega) \in {}^\perp\mathscr{C}$, we have $\mathrm{coker}(i^*) = \mathrm{Ext}^1(\Phi(A_\omega, B_\omega), C) = 0$, therefore the induced morphism $i^* : \mathrm{Hom}(\Phi(\emptyset, B_\omega), C) \longrightarrow \mathrm{Hom}(\mathrm{im}(f), C)$ is onto.

It remains to prove step 1. Pick $t \in \Phi(\emptyset, A_\omega)$ with $f(t) = 0$. Then $\exists n \in \omega, t_n \in \Phi(\emptyset, A_n)$ such that $t = \lambda_n(t_n)$ where $\lambda_n : \Phi(\emptyset, A_n) \longrightarrow \Phi(\emptyset, A_\omega)$ is the canonical map. By hypothesis, the homomorphism $f \circ \lambda_n : \Phi(\emptyset, A_n) \longrightarrow \Phi(\emptyset, B_\omega)$ is a $\mathscr{C}$-monomorphism, and so the map $g \circ \lambda_n$ factors through $f \circ \lambda_n$. This means that there exists a homomorphism $\xi : \Phi(\emptyset, B_\omega) \longrightarrow C$ with $g \circ \lambda_n = \xi \circ f \circ \lambda_n$. This gives us: $g(t) = g \circ \lambda_n(t_n) = \xi \circ f \circ \lambda_n(t_n) = \xi \circ f(t) = 0$, and so $t \in \ker(g)$. $\square$

We will now present a short sequence of propositions which will help us with the proof of the main theorem.

Proposition 3.3.9. *Let $\mathscr{C}$ be a class of modules that is closed under filtered products, and $M \in {}^{\perp}\mathscr{C}$. Let also λ be a regular uncountable cardinal, and $R^{(X)} \xrightarrow{f} R^{(Y)} \longrightarrow M \longrightarrow 0$ be a short exact sequence. Put $S(\lambda) = \{(X',Y') \mid X' \subseteq X, Y' \subseteq Y, f(R^{(X')}) \subseteq R^{(Y')}$ and $|X'|+|Y'| < \lambda\}$. Then there exists a $V_A(\lambda) \subseteq S(\lambda)$ such that:*
i) for $(X',Y') \in V_A(\lambda)$ we have that the map $M_{(X',Y')} \longrightarrow M$ is a $\mathscr{C}$-monomorphism , where we have put $M_{(X',Y')} = coker\big(R^{(X')} \xrightarrow{f|} R^{(Y')}\big)$
ii) $\bigcup_{(X',Y')\in V_A(\lambda)} X' = X$ and $\bigcup_{(X',Y')\in V_A(\lambda)} Y' = Y$
iii) $V_A(\lambda)$ is a directed set
iv) $V_A(\lambda)$ is λ-closed.

Proof. We have already proven that $S(\lambda)$ is λ-continuous, that $M_{(X',Y')} = \mathrm{coker}\big(R^{(X')} \xrightarrow{f|} R^{(Y')}\big)$ is $< \lambda$-presented and that $\varinjlim_{(X',Y')\in S(\lambda)} M_{(X',Y')} = M \in {}^{\perp}\mathscr{C}$ (see the proof of Proposition 3.2.6). By Proposition 3.3.6 we also get that there exists a cofinal, λ-closed, directed subsystem $V_A(\lambda) \subseteq S(\lambda)$ such that property (i) holds. But $V_A(\lambda)$ being cofinal implies property (ii), which concludes the proof. □

Corollary 3.3.10. *Let $A \in \mathfrak{I}$ and $\lambda > \aleph_0$ be a successor cardinal with $\nu < \lambda < \kappa$. Then there exists a $V_A(\lambda) \subseteq \{(\emptyset,A') \in \mathscr{W} \mid A' \subseteq A$ and $|A'| < \lambda\}$ such that:*
i) $(\emptyset,A') \preceq (\emptyset,A)$ for all $(\emptyset,A') \in V_A(\lambda)$
ii) $\bigcup_{(\emptyset,A')\in V_A(\lambda)} A' = A$
iii) $V_A(\lambda)$ = directed
iv) $V_A(\lambda)$ = λ-closed.

Proof. Apply the previous Proposition for $M = \Phi(\emptyset,A) = \varinjlim_{i\in A} M_i$. □

Proposition 3.3.11. *Consider $A,B \in \mathfrak{I}$ with $V = (A,B) \in \mathscr{W}$ and $(\emptyset,A) \preceq (\emptyset,B)$. Also pick λ a successor cardinal with $\nu < \lambda < \kappa$. Then there exists a $V(\lambda) \subseteq \{(A',B') \in \mathscr{W} \mid A' \subseteq A$, $B' \subseteq B$, $A' \subseteq B'$ and $|B'| < \lambda\}$ such that:*
i) $\forall (A',B') \in V(\lambda)$ $\Big((A',B') \preceq (A,B) \wedge (\emptyset,A') \preceq (\emptyset,A) \wedge (\emptyset,B') \preceq (\emptyset,B)\Big)$
ii) $\bigcup_{(A',B')\in V(\lambda)} A' = A$ and $\bigcup_{(A',B')\in V(\lambda)} B' = B$
iii) $V(\lambda)$ is directed
iv) $V(\lambda)$ is λ-closed.

Proof. Consider the sets $V_A(\lambda), V_B(\lambda)$ which we get from the previous Corollary, and put σ to be the homomorphism $\Phi(\emptyset,A) \longrightarrow \Phi(\emptyset,B)$. We can find a cofinal subsystem

$$\tilde{V}(\lambda) \subseteq \{(A',B') \mid A' \in V_A(\lambda) \ , \ B' \in V_B(\lambda) \ , \ A' \subseteq B'\}$$

that is λ-continuous such that $\sigma = \varinjlim_{(A',B')\in\tilde{V}(\lambda)} \sigma'$ and $\Phi(A,B) = \varinjlim_{(A',B')\in\tilde{V}(\lambda)} \Phi(A',B')$, where σ' is the homomorphism $\Phi(\emptyset,A') \longrightarrow \Phi(\emptyset,B')$.

Now σ is a $\mathscr{C}$-monomorphism and $\Phi(\emptyset,B) \in {}^{\perp}\mathscr{C}$, so $\Phi(A,B) \in {}^{\perp}\mathscr{C}$. Hence, there exists a cofinal, λ-closed, directed subset $V(\lambda) \subseteq \tilde{V}(\lambda)$ such that the homomorphism $\Phi(A',B') \longrightarrow \Phi(A,B)$ is a $\mathscr{C}$-monomorphism for all $(A',B') \in V(\lambda)$. □

We have almost reached the end of the proof. One last theorem remains between us and a complete proof of Shelah's Singular Compactness Theorem.

Theorem 3.3.12. *There exists a continuous ascending $\preceq$-chain $\{(\emptyset,C_\alpha) \mid \alpha < cf(\kappa)\}$ such that $I = \bigcup_{\alpha<cf(\kappa)} C_\alpha$ and $C_\alpha \in \mathfrak{I}$ for all $\alpha < cf(\kappa)$.*

Before proving this last theorem, let us first see how we can complete the proof of our main theorem with it. We will give an outline of what we have done so far, and finish it off with the last 3 bullet-points:

1. We started with a κ-presented module M, where $\kappa > \aleph_0$ is a singular cardinal.
2. We wrote M as the direct limit $\varinjlim_{i\in I} M_i$, with each M_i being countably presented.
3. We defined the λ-continuous, direct system of modules $S^\lambda = \big((M_J)_{J\in I_\lambda}\big)$ for every successor cardinal $\nu < \lambda < \kappa$, where $I_\lambda = \{J \subseteq I \mid J = \text{directed and } |J| < \lambda\}$ and each $M_J = \varinjlim_{j\in J} Mj$ is $< \lambda$-presented.
4. From the almost-$(\mathscr{C},\lambda)$-projectivity of M we got a system S_λ which we intersected with our constructed S^λ to get a new system S_λ with the desired properties of both.
5. We put $\mathfrak{I}_\lambda$ to be the set of indices of this new system, and $\mathfrak{I} = \bigcup_{\nu<\lambda<\kappa} \mathfrak{I}_\lambda$.
6. We defined the family $\mathscr{W} = \{(A,B) \mid A \subseteq B \subseteq I\ ,\ A,B = \text{directed}\ \}$ and a $\leq$-order on it.
7. We defined the modules $\Phi(A,B) = \operatorname{coker}(M_A \longrightarrow M_B)$ for $(A,B) \in \mathscr{W}$.
8. We showed that for any such pair, if $B \in \mathfrak{I}$ and the morphism $\Phi(\emptyset,A) \longrightarrow \Phi(\emptyset,B)$ is a $\mathscr{C}$-monomorphism, then $\Phi(A,B) \in {}^{\perp}\mathscr{C}$. More specifically, $\Phi(\emptyset,B) \in {}^{\perp}\mathscr{C}$.
9. We defined the weaker $\preceq$-order on $\mathscr{W}$ such that $(A,B) \preceq (A',B') \iff (A,B) \leq (A',B') \wedge \Phi(A,B) \longrightarrow \Phi(A',B')$ is a $\mathscr{C}$-monomorphism.
10. With this last theorem, we found a continuous ascending $\preceq$-chain $\{(\emptyset,C_\alpha) \mid \alpha < \mathrm{cf}(\kappa)\}$ such that $I = \bigcup_{\alpha<\mathrm{cf}(\kappa)} C_\alpha$ and $C_\alpha \in \mathfrak{I}$ for all $\alpha < \mathrm{cf}(\kappa)$.
11. Now, since $C_\alpha \in \mathfrak{I}$ we get that $\Phi(\emptyset,C_\alpha) \in {}^{\perp}\mathscr{C}$ (see (8)). Furthermore, (10) also gives us that each $\Phi(\emptyset,C_\alpha) \longrightarrow \Phi(\emptyset,C_{\alpha+1})$ is a $\mathscr{C}$-monomorphism (see (9)).
12. By Eklof's Lemma we have: $\varinjlim_{\alpha<\mathrm{cf}(\kappa)} \Phi(\emptyset,C_\alpha) \in {}^{\perp}\mathscr{C}$.

13. However $I = \bigcup_{\alpha<\mathrm{cf}(\kappa)} C_\alpha$ and so $\varinjlim_{\alpha<\mathrm{cf}(\kappa)} \Phi(\emptyset, C_\alpha) = M$, completing the proof.

In order to prove Theorem 3.3.12 we will weed out all the unnecessary noise from before, and keep just the points we will need. Some cross-over with the bullet-points above cannot be avoided:

For a directed set $(I,\leq)$ with cardinality κ we have defined $\mathfrak{I} = \bigcup_{\nu<\lambda<\kappa} \mathfrak{I}_\lambda$, where the $\mathfrak{I}_\lambda$ are sets containing as elements directed subsets of I with cardinality $< \lambda$ and λ=successor cardinal such that:

- $\emptyset \in \mathfrak{I}_\lambda$
- $\mathfrak{I}_\lambda$ is λ-closed
- for any $A \subseteq I$ with $|A| < \lambda$ there exists $B \in \mathfrak{I}_\lambda$ such that $A \subseteq B$

On the set $\mathscr{W} = \{(A,B) \mid A \subseteq B \subseteq I \text{ directed subsets}\}$ we have defined the order $\preceq$ with the following properties:

1. $(\emptyset,\emptyset)$ is the smallest element
2. if $(A,B) \preceq (C,D)$ then $(A,B) \leq (C,D)$
3. if $(A,B) \leq (C,D) \leq (E,F)$ and $(A,B) \preceq (E,F)$ then $(A,B) \preceq (C,D)$
4. Proposition 3.3.11
5. if $(A,B),(C,D) \in \mathscr{W}$ and $(A,B) \leq (C,D)$ and $(A,C) \preceq (B,D)$ and $(\emptyset,B) \preceq (\emptyset,D)$ then $(A,B) \preceq (C,D)$
6. if $\{(A_n,B_n)\}_{n\in\omega}$ is an ascending $\preceq$-sequence in $\mathscr{W}$ and $(A_\omega,B_\omega) = (\bigcup_{n\in\omega} A_n, \bigcup_{n\in\omega} B_n)$ and $(\emptyset,A_n) \preceq (\emptyset,B_\omega)\ \forall n \in \omega$ and $B_n \in \mathfrak{I}\ \forall n \in \omega$, then $(\emptyset,A_\omega) \preceq (\emptyset,B_\omega)$.

Finally, we have also defined the subset $\mathscr{W}_0 = \{(A,B) \in \mathscr{W} \mid A = \emptyset\}$.

We will need the following lemma. It is here that the idea from Hodges regarding a proof by a two-player game is utilized:

Lemma 3.3.13. *For every $X \in \mathscr{W}_0$ with $|X| \geq \nu$ there exists $N \in \mathscr{W}_0 \cap (\mathfrak{I}_{|X|^+})^2$ such that $X \leq N$ and $N \preceq Y$ for all $Y \in \mathscr{W}_0$ with $|Y| = |N|$ and $N \leq Y$.*

Proof. Put $\mu = |X|$ and pick any $N \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2$. Define the N-Shelah game as follows:
"The game consists of two players who take turns after one another. Player 1 begins the game and during his turn picks any $X_n \in \mathscr{W}_0$ with $|X_n| \leq \mu$, where n is the number of the current round. Player 2 in his turn picks $N_n \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2$ with $X_n \leq N_n$ and $N_{n-1} \preceq N_n$, where we have put $N_{-1} = N$. At most ω rounds are played. Player 2 wins if-f he can play for ω rounds, otherwise Player 1 wins."

Define the set

$$S = \{N \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2 \mid \text{ Player 1 does not have a winning strategy in the } N\text{-Shelah game } \}$$

We will show that $(\emptyset,\emptyset) \in S$. This will mean that Player 1 does not possess a winning strategy for the $(\emptyset,\emptyset)$-Shelah game where $X_0 = X$, and so Player's 2 first pick will serve as the N that satisfies the conditions of the statement we are after.

For every $K \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^{++}})^2$ fix an ascending $\leq$-chain $\{K^\alpha \in \mathscr{W}_0 \mid \forall \alpha < \mu^+ \left(|K^\alpha| \leq \mu\right) \}$ such that $\bigcup_{\alpha<\mu^+} K^\alpha = K$. Consider a strategy s for Player 1, i.e. a function $s(N_0, N_1, \dots, N_{n-1}) = X_n$, for the $(\emptyset,\emptyset)$-Shelah game. We will show that Player 2 can beat strategy s.

We will inductively define two ascending $\leq$-sequences $\{M_\alpha \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2 \mid \alpha < \mu^+\}$ and $\{K_\alpha \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^{++}})^2 \mid \alpha < \mu^+\}$ such that:
(0) $X_0 \leq M_0$
(1) $\forall \alpha < \mu^+$ limit ordinal we have $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$
(2) $\forall \alpha < \mu^+$ we have $M_\alpha \leq K_\alpha$
(3) $\forall \alpha < \mu^+$ we have $M_{\alpha+1} > M_\alpha \cup \bigcup_{\beta\leq\alpha} (K_\beta)^\alpha$
(4) $\forall \alpha < \mu^+$ $\forall n \in \omega$ $\forall \alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \alpha$ with $M_{\alpha_0} \preceq M_{\alpha_1} \preceq \dots \preceq M_{\alpha_n}$ being valid choices for Player 2 we have $s(M_{\alpha_0}, M_{\alpha_1}, \dots, M_{\alpha_n}) \leq M_{\alpha+1}$.

We begin the induction.
<u>$\circ n = 0$:</u> Since $X_0 = (\emptyset, B_0) \in \mathscr{W}_0$, by the third property of $\mathfrak{I}_{\mu^+}$ (notice how $\mu^+ < \kappa$, because κ is singular) there exists $A_0 \in \mathfrak{I}_{\mu^+}$ with $B_0 \subseteq A_0$. Define $M_0 = (\emptyset, A_0)$, so (0) holds. Likewise (since $\mu^{++} < \kappa$ as well) define $K_0 \geq M_0$.
<u>$\circ n = \alpha + 1$:</u> Put $\tilde{X}_\alpha = \{s(M_{\alpha_0}, M_{\alpha_1}, \dots, M_{\alpha_n}) \in \mathscr{W}_0 \mid \phi(\alpha)\}$, where $\phi(\alpha)$ stands for "$n \in \omega$, $\alpha_0 \leq \alpha_1 \leq \dots \leq \alpha_n \leq \alpha$ with $M_{\alpha_0} \preceq M_{\alpha_1} \preceq \dots \preceq M_{\alpha_n}$ being valid choices for Player 2". Notice that $|\bigcup \tilde{X}_\alpha| < \mu^+$ (we have at most ω choices for n, and at most $\alpha^\omega = \alpha$ for the α_i, so at most $\omega \cdot \alpha = \alpha$ choices for $s(M_{\alpha_0}, M_{\alpha_1}, \dots, M_{\alpha_n})$, and each such set has $< \mu^+$ elements since it is a Player 1 choice). Since $|(K_\beta)^\alpha| \leq \mu$ for all $\beta \leq \alpha$ and $|M_\alpha| \leq \mu$, by adding yet another element to the set $M_\alpha \cup \bigcup_{\beta\leq\alpha} (K_\beta)^\alpha \cup \bigcup \tilde{X}_\alpha$ we get that $|M_\alpha \cup \bigcup_{\beta\leq\alpha} (K_\beta)^\alpha \cup \bigcup \tilde{X}_\alpha| < \mu^+$. By the third property of

$\mathfrak{I}_{\mu^+}$ again we can find a $M_{\alpha+1} \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2$ such that (3) holds. Likewise define $K_{\alpha+1} \geq M_{\alpha+1}$.
<u>$\circ n = \alpha$ =limit ordinal:</u> Let $\alpha < \mu^+$ be a limit ordinal and suppose that we have defined M_β, K_β for all $\beta < \alpha$. Define $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ and K_α as in the previous steps. Notice that $M_\alpha \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2$ since M_β form a chain, and (1) holds.
By construction, (2) and (4) also hold. The induction has been completed.

Let us now put $M = \bigcup_{\alpha<\mu^+} M_\alpha$. We have: $|M| = \mu^+$ and $M = \bigcup_{\beta<\mu^+} K_\beta \in \mathfrak{I}^2$ by (3) above (notice: $\bigcup_{\beta<\mu^+} K_\beta = \bigcup_{\beta<\mu^+}(\bigcup_{\alpha<\mu^+}(K_\beta)^\alpha) = \bigcup_{\alpha<\mu^+}(\bigcup_{\beta<\mu^+}(K_\beta)^\alpha) = \bigcup_{\alpha<\mu^+}(\bigcup_{\beta\leq\alpha}(K_\beta)^\alpha) \leq \bigcup_{\alpha<\mu^+} M_{\alpha+1} = M$). Use Proposition 3.3.11 and consider the system $M(\mu^+)$ that it gives. Put $A = \{\beta < \mu^+ \mid M_\beta \in M(\mu^+)\}$. Then A is cofinal in μ^+. Indeed, pick $\beta < \mu^+$. By definition we have $|M_\beta| < \mu^+$, therefore there must exist a $V_0 \in M(\mu^+)$ such that $M_\beta \subseteq V_0$. But then $|V_0| < \mu^+$ whereas $|\bigcup M(\mu^+)| = |M| = \mu^+$, and $M = \bigcup_{\alpha<\mu^+} M_\alpha$, so there must exist a $\beta_1 > \beta$ such that $M_{\beta_1} \supseteq V_0$. Apply the same argument to M_{β_1} in order to get a $V_1 \in M(\mu^+)$, $\beta_2 > \beta_1$ and $M_{\beta_2} \supseteq V_1$, and continue inductively. Put $\bar{\beta} = \sup_{n\in\omega}\beta_n$. Since $\bar{\beta}$ is a limit ordinal, property (1) of the sequence $\{M_\alpha \in \mathscr{W}_0 \cap (\mathfrak{I}_{\mu^+})^2 \mid \alpha < \mu^+\}$ gives us $M_{\bar{\beta}} = \bigcup_{n\in\omega} M_{\beta_n}$ and it is easy to see that $\bigcup_{n\in\omega} M_{\beta_n} = \bigcup_{n\in\omega} V_n \in M(\mu^+)$. Therefore $M_{\bar{\beta}} \in M(\mu^+)$ and $\bar{\beta} \in A$, proving that A is cofinal in μ^+.

Now Player 2 defeats strategy s : For every X_n he can (by (4) above) choose $N_n = M_\beta \in M(\mu^+)$ for a large enough $\beta \in A$, since A is cofinal. The proof of the lemma is concluded. □

We are now on the home stretch. Unfortunately, this next bit of the proof is fairly technical in its nature. Nonetheless, we present the proof of Theorem 3.3.12.

Proof. Put $\mu = \mathrm{cf}(\kappa)$, and fix a continuous, strictly-increasing sequence of cardinals $(\nu_\alpha \mid \alpha < \mu)$ such that $\nu_0 > \mu + \nu$ and the set of its terms is cofinal in κ (we can always find such a sequence by definition of κ being a singular cardinal). For each $n \in \omega$ we inductively define a strictly-increasing $\preceq$-sequence $(V^n_\alpha \in \mathscr{W}_0 \mid \alpha < \mu)$ and arbitrarily picked enumerations $A^n_\alpha = \{a^n_{\alpha,\beta} \in I \mid \beta < \nu_\alpha\}$ with $V^n_\alpha = (\emptyset, A^n_\alpha)$ as described below:

<u>$\circ n = 0$:</u> For any $\alpha < \mu$ choose $V^0_\alpha \in \mathscr{W}_0 \cap (\mathfrak{I}_{\nu^+_\alpha})^2$ with cardinality ν_α and $\bigcup_{\beta<\alpha} V^0_\beta \leq V^0_\alpha$ such that $\forall Y \in \mathscr{W}_0$ with $|Y| = |V^0_\alpha| = \nu_\alpha$ and $V^0_\alpha \leq Y$ we have $V^0_\alpha \preceq Y$ (such is always possible due to Lemma 3.3.13 above). Also, we may pick the enumerations such that $\bigcup_{\alpha<\mu} A^0_\alpha = I$.

<u>$\circ n = 1$:</u> For each $\alpha < \mu$ pick arbitrary $V^1_\alpha \in V^0_{\alpha+1}(\nu^+_\alpha)$ with $V^0_\alpha \leq V^1_\alpha$. Also, define the families of subsets of I: $\mathscr{B}^1_\alpha = \{B \subseteq I \mid (\emptyset, B) \in V^0_{\alpha+1}(\nu^+_\alpha) \wedge A^1_\alpha \subseteq B\}$.

$\circ$<u>$n > 0$ even:</u> Using again Lemma 3.3.13 we choose the sets $V_\alpha^n \in \mathscr{W}_0 \cap (\mathfrak{I}_{\nu_\alpha^+})^2$ as in the $n = 0$ case. Also, since $\nu_0 > \mu = \mathrm{cf}(\kappa)$, we can pick the enumerations such that:

$$A_\alpha^n \supseteq \{a_{\gamma,\beta}^{n-1} \mid \gamma < \mu \ \wedge \ \beta < \min\{\nu_\gamma, \nu_\alpha\}\} \quad (1)$$

We are left to do the induction on the odd numbers. This will require some more work:
$\circ$<u>$n > 1$ odd:</u> By construction we have: $A_{\alpha+1}^{n-3}, A_{\alpha+1}^{n-1} \in \mathfrak{I}$ and $V_{\alpha+1}^{n-3} \preceq V_{\alpha+1}^{n-1}$. Also, we may define the sets $\mathscr{B}_\alpha^i$ for $i < n$ odd to be upwards directed, closed under unions of chains of length $\leq \nu_\alpha$, and such that $\bigcup \mathscr{B}_\alpha^i = A_{\alpha+1}^{i-1}$.
By Property (4) of the $\preceq$-order, we can choose arbitrary $(A_\alpha^{n-2,n}, A_\alpha^n) \in (A_{\alpha+1}^{n-3}, A_{\alpha+1}^{n-1})(\nu_\alpha^+)$ such that $A_\alpha^{n-2,n} \in \mathscr{B}_\alpha^{n-2}$ and $A_\alpha^{n-1} \subseteq A_\alpha^n$. Now, by re-definition of $\mathscr{B}_\alpha^{n-2}$ (see (!) further down) there exists an induced chain $(\emptyset, A_\alpha^{1,n}) \preceq (\emptyset, A_\alpha^{3,n}) \preceq \cdots \preceq (\emptyset, A_\alpha^{n-2,n})$ that satisfies the following statement:

$$\forall i < n \text{ odd} \left(A_\alpha^{i,n} \in \mathscr{B}_\alpha^i \ \wedge \ (A_\alpha^{i,n}, A_\alpha^{i+2,n}) \in (A_{\alpha+1}^{i-1}, A_{\alpha+1}^{i+1})(\nu_\alpha^+) \right) \quad (2)$$

where $A_\alpha^{n,n} := A_\alpha^n$. Put $V_\alpha^n = (\emptyset, A_\alpha^n)$.

(!) For every odd $i < n$ we recursively replace the $\mathscr{B}_\alpha^i$ with their subsets $\{B \in \mathscr{B}_\alpha^i \mid A_\alpha^{i,n} \subseteq B\}$. This change will not affect the desired properties of $\mathscr{B}_\alpha^i$. Furthermore, define $\mathscr{B}_\alpha^n = \{B \subseteq I \mid \exists A \in \mathscr{B}_\alpha^{n-2} \Big((A, B) \in (A_{\alpha+1}^{n-3}, A_{\alpha+1}^{n-1})(\nu_\alpha^+) \ \wedge \ A_\alpha^n \subseteq B \Big)\}$ (this secures the existence of the chain mentioned in equation (2) above; also it equips $\mathscr{B}_\alpha^n$ with the properties necessary to carry out the induction).

The induction has now been completed. All we have to do is show that the chain $\mathscr{S} = \left\{ \bigcup_{n \in \omega} V_\alpha^n \mid \alpha < \mu \right\}$ is the chain we were looking for in the statement of the theorem.

First, we have $\bigcup_{n\in\omega} V_\alpha^n = \bigcup_{k\in\omega} V_\alpha^{2k} \in (\mathfrak{I}_{\nu_\alpha^+})^2$ since the sequence $\left\{ V_\alpha^n \mid n \in \omega \right\}$ is increasing. Second, equation (1) above gives us immediately that $\mathscr{S}$ is continuous. It remains to show that it is also a $\preceq$-chain.

Pick $\alpha < \mu$. For every $k \in \omega$ define the following:

- $A_k = \bigcup_{k \leq j < \omega} A_\alpha^{2k+1, 2j+1}$
- $B_k = A_{\alpha+1}^{2k}$
- $V_k = (\emptyset, A_k)$
- $W_k = (\emptyset, B_k)$

Observe that for each $k \in \omega$ we have $V_k \preceq W_k$ by Property (4), $W_k \preceq W_{k+1}$ by the construction of $A^n_{\alpha+1}$ where n is even, and $V_k \leq V_{k+1} \leq W_{k+1}$. By Property (3) we get $V_k \preceq V_{k+1}$ for all $k \in \omega$, meaning that the $(V_n)_{n\in\omega}$ form a $\preceq$-chain. Furthermore, we have $(A_k, A_{k+1}) \in (B_k, B_{k+1})(\nu^+_\alpha)$ for all $k \in \omega$ and $\bigcup_{k\in\omega} A_k = \bigcup_{j\in\omega} A^j_\alpha$ which easily follows from definition.

Now, each B_k belongs to $\mathfrak{I}$. From $(A_k, A_{k+1}) \in (B_k, B_{k+1})(\nu^+_\alpha)$ we get $(\emptyset, A_k) \preceq (\emptyset, B_k)$. Furthermore, we have $(\emptyset, B_k) \preceq (\emptyset, B_\omega) = (\emptyset, \bigcup_{n\in\omega} B_n)$ by the choice of $(\emptyset, B_k) = V^{2k}_{\alpha+1}$, and thus $(\emptyset, A_k) \preceq (\emptyset, B_\omega)$ $\forall k \in \omega$. Lastly, from $(A_k, A_{k+1}) \preceq (B_k, B_{k+1})$ and Property (5) we conclude that $(A_k, B_k) \preceq (A_{k+1}, B_{k+1})$.

We have just shown that the conditions to apply Property (6) on the sequence $(A_k, B_k)_{k\in\omega}$ hold. We can now conclude that $\bigcup_{j\in\omega} V^j_\alpha \preceq \bigcup_{j\in\omega} V^j_{\alpha+1}$, proving that $\mathscr{S}$ is a chain.

The proof of Theorem 3.3.12 is now complete. □

Bibliography

[1] Beke, T. and Rosicky, J. (2014). Cellular objects and Shelah's singular compactness theorem.

[2] David, B. (1978). On Shelah's Compactness of Cardinals. *Israel Journal of Mathematics*, 31.

[3] Ehrenfeucht, A. (1955). On a problem of JHC Whitehead concerning Abelian groups. *Bull. Acad. Polon. Sci., Cl. III*, 3:127–128.

[4] Eklof, P. C. (1997). Set Theory Generated by Abelian Group Theory. *The Bulletin of Symbolic Logic*, 3(1).

[5] Eklof, P. C. (2002). The Affinity of Set Theory and Abelian Group Theory. *The Rocky Mountain Journal of Mathematics*, 32(4).

[6] Eklof, P. C. (2008). Shelah's Singular Compactness Theorem. *Publicacions Matemàtiques*, 52(1).

[7] Eklof, P. C. and Fuchs, L. (1988). Baer modules over valuation domains. *Annali di Matematica pura ed applicata*, 150.

[8] Eklof, P. C., Fuchs, L., and Shelah, S. (1990). Baer Modules over Domains. *Transactions of the American Mathematical Society*, 322.

[9] Eklof, P. C. and Mekler, A. (2002). Almost free modules. Set-theoretic methods. *The Journal of Symbolic Logic*, 60(2).

[10] Hodges, W. (1981). In singular cardinality, locally free algebras are free. *Algebra Universalis*, 12.

[11] Moschovakis, Y. (1994). *Notes on Set Theory*. Springer New York, NY.

[12] Rotman, J. J. (2008). *An Introduction to Homological Algebra*. Universitext. Springer New York.

[13] Shelah, S. (1974). Infinite abelian groups, Whitehead problem and some constructions. *Israel Journal of Mathematics*, 18:243–256.

[14] Shelah, S. (1975). A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals. *Israel Journal of Mathematics*, 21(4).

[15] Shelah, S. (1977). Whitehead groups may be not free, even assuming CH, i. *Israel Journal of Mathematics*, 28.

[16] Shelah, S. (1980). Whitehead groups may be not free, even assuming CH, ii. *Israel Journal of Mathematics*, 35.

[17] Shelah, S. (1985). Incompactness in Regular Cardinals. *Notre Dame Journal of Formal Logic*, 26.

[18] Šaroch, J. and Št'ovíček, J. (2020). Singular compactness and definability for Σ-cotorsion and Gorenstein modules. *Selecta Mathematica*, 26(2).