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Deformation Theory of Curves with Automorphisms

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To my brother George

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Introduction in English

The deformation theory of curves with automorphisms is an important generalization of the classical deformation theory of curves. This theory is related to the lifting problem of curves with automorphisms, since one can consider liftings from characteristic $p > 0$ to characteristic zero in terms of a sequence of local Artin-rings.

J. Bertin and A. Mézard in [10], following Schlessinger's [68] approach, introduced a deformation functor D_{gl} and studied it in terms of Grothendieck's equivariant cohomology theory [31]. In Schlessinger's approach to deformation theory, we want to know the tangent space to the deformation functor $D_{gl}(k[\epsilon])$ and the possible obstructions to lift a deformation over an Artin local ring Γ to a small extension $\Gamma' \to \Gamma$. The reader who is not familiar with deformation theory is referred to section 2.2a for terminology and references to the literature. Let X be a non-singular complete algebraic curve defined over an algebraically closed field k of characteristic $p > 0$. The tangent space of the global deformation functor $D_{gl}(k[\epsilon])$ can be identified as Grothendieck's equivariant cohomology group $H^1(G, X, \mathscr{T}_X)$, which is known to be equal to the invariant space $H^1(X, \mathscr{T}_X)^G$. Moreover, a local-global theorem is known, which can be expressed in terms of the short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & H^1(X/G, \pi_*^G(\mathscr{T}_X)) \arrow[r] & H^1(G, X, \mathscr{T}_X) \arrow[r] & H^0(X/G, R^1\pi_*^G(\mathscr{T}_X)) \arrow[r] \arrow[d, "\wr\|"] & 0 \\ & & & \bigoplus_{i=1}^{r} H^1\left(G_{x_i}, \widehat{\mathscr{T}}_{X,x_i}\right) & \end{tikzcd} \tag{1}$$

The lifting obstruction can be seen as an element in

$$H^2(G, X, \mathscr{T}_X) \cong \bigoplus_{i=1}^{r} H^2\left(G_{x_i}, \widehat{\mathscr{T}}_{X,x_i}\right).$$

In the above equation $x_1, \ldots, x_r \in X$ are the ramified points, G_{x_i} are the corresponding isotropy groups and $\widehat{\mathscr{T}}_{X,x_i}$ are the completed local tangent spaces, that is $\widehat{\mathscr{T}}_{X,x_i} = k[[t_i]]\frac{d}{dt_i}$, where t_i is a local uniformizer at x_i. The space $k[[t_i]]\frac{d}{dt_i}$ is seen as G_{x_i}-module by the adjoint action, see [22, 2.1], [49, 1.5]. Bertin and Mézard reduced the computation of obstruction to the infinitesimal lifting problem of representations of the isotropy group G_{x_i} to the difficult group $\mathrm{Aut}k[[t]]$, where $\mathrm{Aut}k[[t]]$ denotes the group of continuous automorphisms of $k[[t]]$.

Let now G be a finite group, and consider the homomorphism

$$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]]),$$

which will be called a local G-action. Let $W(k)$ denote the ring of Witt vectors of k. The local lifting problem considers the following question: Does there exist an extension $\Lambda/W(k)$, and a representation

$$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]]),$$

such that if t is the reduction of T, then the action of G on $\Lambda[[T]]$ reduces to the action of G on $k[[t]]$? If the answer to the above question is positive, then we say that the G-action lifts to characteristic zero. A group G for which every local G-action on $k[[t]]$ lifts to characteristic zero is called a local Oort group for k.

After studying certain obstructions (the Bertin-obstruction, the KGB-obstruction, the Hurwitz tree obstruction etc.) it is known that the only possible local Oort groups are known to be

(i) Cyclic groups

(ii) Dihedral groups D_{p^h} of order $2p^h$

(iii) The alternating group A_4

The Oort conjecture states that every cyclic group C_q of order $q = p^h$ lifts locally. This conjecture was proved recently by F. Pop [66] using the work of A. Obus and S. Wewers [63]. A. Obus proved that A_4 is local Oort group in [60] and this was also known to F. Pop, I. Bouw and S. Wewers [14]. The case of dihedral groups D_p are known to be local Oort by I. Bouw and S. Wewers for p odd [14] and by G. Pagot [65]. Several cases of dihedral groups D_{p^h} for small p^h have been studied by A. Obus [61] and H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte [23], while the D_4 was studied by B. Weaver [80]. For more details on the lifting problem we refer to [19], [20], [21], [59].

Probably, the most important of the known so far obstructions is the KGB obstruction [20]. It was conjectured that if the p-Sylow subgroup of G is cyclic, then this is the only obstruction for the local lifting problem, see [59], [61]. In particular, the KGB-obstruction for the dihedral group D_q is known to vanish, so the conjecture asserts that the local action of D_q always lifts. We will provide in section 3.5a a counterexample to this conjecture, by proving that the HKG-cover corresponding to D_{125}, with a selection of lower jumps $9, 189, 4689$, which does not lift.

This thesis is splitted into two parts. In the first part, we aim to give a new approach to the deformation theory of curves with automorphisms, which is not based on the deformation theory of representations on the subtle object $\mathrm{Aut}k[[t]]$, but on the deformation theory of the better understood general linear group. In order to do so, we will restrict ourselves to curves that satisfy the mild assumptions of Petri's theorem.

Theorem 1 (Petri's theorem)**.** For a non-singular non-hyperelliptic curve X of genus $g \geqslant 3$ defined over an algebraically closed field with sheaf of differentials Ω_X there is the following short exact sequence:

$$0 \to I_X \to \mathrm{Sym}H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [30], [67]. The ideal I_X is called the canonical ideal and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}^{g-1}$.

For curves that satisfy the assumptions of Petri's theorem and their canonical ideal is generated by quadrics, we prove in section 2.3 the following relative version of Petri's theorem

Proposition 2. Let $f_1, \ldots, f_r \in S := \mathrm{Sym}H^0(X, \Omega_X) = k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal I_X of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathrm{Sym}H^0(\mathscr{X}_A, \Omega_{\mathscr{X}_A/A}) = A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.

This approach allows us to replace several of Grothendieck's equivariant cohomology constructions in terms of linear algebra. Let us mention that in general, it is not so easy to perform explicit computations with equivariant Grothendieck cohomology groups and usually, spectral sequences or a complicated equivariant Chech cohomology is used, see [9], [50, sec.3].

Let $i : X \to \mathbb{P}^{g-1}$ be the canonical embedding. In proposition 2.3.5.1 we prove that elements $[f] \in H^1(X, \mathscr{T}_X)^G = D_{\mathrm{gl}}k[\epsilon]$ correspond to cohomology classes in $H^1(G, M_g(k)/\langle \mathbb{I}_g \rangle)$, where $M_g(k)/\langle \mathbb{I}_g \rangle$ is the space of $g \times g$ matrices with coefficients in k, modulo the vector subspace of scalar multiples of the identity matrix.

Furthermore, in our setting the obstruction to liftings is reduced to an obstruction to the lifting of the linear canonical representation

$$\rho : G \to \mathrm{GL}\big(H^0(X, \Omega_X)\big) \tag{2}$$

and a compatibility criterion involving the defining quadratic equations of our canonically embedded curve, namely in section 2.4 we will prove the following:

Theorem 3. Consider an epimorphism $\Gamma' \to \Gamma \to 0$ of local Artin rings. A deformation $x \in D_{\mathrm{gl}}(\Gamma)$ can be lifted to a deformation $x' \in D_{\mathrm{gl}}(\Gamma')$ if and only if the representation $\rho_\Gamma : G \to \mathrm{GL}_g(\Gamma)$ lifts to a representation $\rho_{\Gamma'} : G \to \mathrm{GL}_g(\Gamma')$ and moreover there is a lifting $X_{\Gamma'}$ of the embedded deformation of X_Γ which is invariant under the lifted action of $\rho_{\Gamma'}$.

Remark 4. The liftability of the representation ρ is a strong condition. In proposition 2.4.0.1 we give an example of a representation $\rho : G \to \mathrm{GL}_2(k)$, for a field k of positive characteristic p, which can not be lifted to a representation $\tilde{\rho} : G \to GL_2(R)$ for $R = W(k)[\zeta_{p^h}]$, meaning that a lifting in some small extension $R/\mathfrak{m}_R^{i+1} \to R/\mathfrak{m}_R^i$ is obstructed. Here R denotes the Witt ring of k with a primitive p^h root of unity added, which has characteristic zero. In our counterexample $G = C_q \rtimes C_m$, $q = p^h$, $(m,p) = 1$.

Remark 5. The invariance of the canonical ideal I_{X_Γ} under the action of G can be checked using Gauss elimination and echelon normal forms, see section 1.2b (or [51, sec 2.2]).

Remark 6. The canonical ideal I_{X_Γ} is determined by r quadratic polynomials which form a $\Gamma[G]$-invariant Γ-submodule V_Γ in the free Γ-module of symmetric $g \times g$ matrices with entries in Γ. When we pass from a deformation $x \in D_{gl}(\Gamma)$ to a deformation $x' \in D_{gl}(\Gamma')$ we require that the canonical ideal $I_{X_{\Gamma'}}$ is invariant under the lifted action, given by the representation $\rho_{\Gamma'} : G \to GL_g(\Gamma')$. In definition 2.2.1.1.1 we introduce an action $T(g)$ on the vector space of symmetric $g \times g$ matrices, and the invariance of the canonical ideal is equivalent to the invariance under the T-action of the Γ'-submodule $V_{\Gamma'}$ generated by the quadratic polynomials generating $I_{X'}$. Therefore, we can write one more representation

$$\rho^{(1)} : G \to GL\big(\mathrm{Tor}_1^S(k, I_X)\big). \tag{3}$$

Set $r = \binom{g-2}{2}$. Liftings of the representations $\rho, \rho^{(1)}$ defined in eq. (2), (3) in $GL_g(\Gamma)$ resp. $GL_r(\Gamma)$ will be denoted by ρ_Γ resp. $\rho_\Gamma^{(1)}$.

Notice that if the representation ρ_Γ lifts to a representation $\rho_{\Gamma'}$ and moreover there is a lifting $X_{\Gamma'}$ of the relative curve X_Γ so that $X_{\Gamma'}$ has an ideal $I_{X_{\Gamma'}}$ which is $\rho_{\Gamma'}$ invariant, then the representation $\rho_\Gamma^{(1)}$ also lifts to a representation $\rho_{\Gamma'}^{(1)}$, see also chapter 1.

The deformation theory of linear representations $\rho, \rho^{(1)}$ gives rise to cocycles D_σ, $D_{\sigma^{-1}}^{(1)}$ in $H^1(G, M_g(k))$, $H^1(G, M_{\binom{g-2}{2}}(k))$, while the deformation theory of curves with automorphisms introduces a cocycle $B_\sigma[f]$ corresponding to $[f] \in H^1(X, \mathscr{T}_X)^G$. We will introduce a compatibility condition in section 2.4b among these cocycles, using the isomorphism

$$\begin{aligned} \psi : M_g(k)/\langle \mathbb{I}_g \rangle &\xrightarrow{\cong} H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \hookrightarrow \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \\ B &\longmapsto \psi_B \end{aligned}$$

defined in In proposition 2.3.4.1.

Proposition 7. The following compatibility condition is satisfied

$$\psi_{D_\sigma} - \psi_{B_\sigma[f]} = D_{\sigma^{-1}}^{(1)}. \tag{4}$$

Our main result, is to give a necessary and sufficient condition for a $C_q \rtimes C_m$-action and in particular for the group D_q to lift. In order to do so, we will employ the Harbater-Katz-Gabber-compactification (HKG for short), which can be used in order to construct complete curves out of local actions. In this way, we have a variety of tools at our disposal and we can transform the local action and its deformations into representations of lineal groups acting on spaces of differentials of the HKG-curve. We will lay the necessary tools in the chapter 2, where we have collected several facts about the relation of liftings of local actions, liftings of curves and liftings of linear representations.

More precisely, the first part consists of three chapters. In the first chapter we study the automorphism group of a curve from the viewpoint of the canonical embedding and Petri's theorem. We give a criterion for identifying the automorphism group as an algebraic subgroup of the general linear group. Furthermore, we extend the action of the automorphism group to a linear action on the generators of the minimal free resolution of the canonical ring of the curve X.

In the second chapter we study the deformation theory of curves by using the canonical ideal. We reduce the problem of lifting curves with automorphisms to a lifting problem of linear representations.

In the last chapter we study the local lifting problem of actions of semidirect products of a cyclic p-group by a cyclic prime to p group, where p is the characteristic of the special fibre. We give a criterion based on Harbater-Katz-Gabber compactification of local actions, which allows us to decide whether a local action lifts or not. In particular for the case of dihedral group we give an example of dihedral local action that cannot lift and in this way we give a stronger obstruction than the KGB-obstruction.

In the second part we give a necessary and sufficient condition for a modular representation of a group $G = C_{p^h} \rtimes C_m$ in a field of characteristic zero to be lifted to a representation over local principal ideal domain of characteristic zero containing the p^h roots of unity. This construction is an essential tool in our study for liftings of the first part.

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Εισαγωγή στα Ελληνικά

Η θεωρία παραμορφώσεων καμπυλών με αυτομορφισμούς, είναι μια σημαντική γενίκευση της κλασσικής θεωρίας παραμορφώσεων καμπυλών. Η θεωρία αυτή συνδέεται με το πρόβλημα ανύψωσης καμπυλών με αυτομορφισμούς, αφού μπορούμε να θεωρήσουμε μια ανύψωση από την θετική χαρακτηριστική στην χαρακτηριστική μηδέν σε μια ακολουθία τοπικών Artin δακτυλίων.

Οι J. Bertin και A. Mézard στο [10], ακολουθώντας την προσέγγιση του Schlessinger [68], εισήγαγαν τον συναρτητή παραμόρφωσης D_{gl} και τον μελέτησαν με την βοήθεια της equivariant συνομολογίας του Grothendieck [31]. Στην προσέγγιση της θεωρίας παραμορφώσεων του Schlessinger, θέλουμε να γνωρίζουμε τον εφαπτόμενο χώρο του συναρτητή παραμόρφωσης $D_{gl}(k[\epsilon])$ και τα πιθανά εμπόδια της ανύψωσης μιας παραμόρφωσης πάνω από έναν τοπικό Artin δακτύλιο Γ σε μια μικρή επέκταση $\Gamma' \to \Gamma$. Ο αναγνώστης που δεν είναι εξοικειωμένος με την θεωρία παραμορφώσεων παραπέμπεται στην ενότητα 2.2a για την ορολογία και παραπομπές στην βιβλιογραφία. Έστω X μια μη ιδιάζουσα, πλήρη, αλγεβρική καμπύλη ορισμένη πάνω από ένα αλγεβρικά κλειστό σώμα k χαρακτηριστικης $p > 0$. Ο εφαπτόμενος χώρος του καθολικού συναρτητή $D_{gl}(k[\epsilon])$ μπορεί να ταυτιστεί με την equivariant ομάδα συνομολογίας του Grothendieck $H^1(G, X, \mathscr{T}_X)$, η οποία γνωρίζουμε πως είναι ίση με των χώρο αναλοιώτων $H^1(X, \mathscr{T}_X)^G$. Επιπλέον έδειξαν ένα τοπικό-καθολικό θεώρημα το οποίο μπορεί να μεταφραστεί στην παρακάτω μικρή ακριβή ακολουθία:

$$\begin{tikzcd} 0 \arrow[r] & H^1(X/G, \pi_*^G(\mathscr{T}_X)) \arrow[r] & H^1(G, X, \mathscr{T}_X) \arrow[r] & H^0(X/G, R^1\pi_*^G(\mathscr{T}_X)) \arrow[r] \arrow[d, "\wr\|"] & 0 \\ & & & \bigoplus_{i=1}^{r} H^1\left(G_{x_i}, \widehat{\mathscr{T}}_{X,x_i}\right) & \end{tikzcd} \tag{5}$$

Το εμπόδιο της ανύψωσης μπορούμε να το δούμε ως ένα στοιχείο μέσα στον χώρο

$$H^2(G, X, \mathscr{T}_X) \cong \bigoplus_{i=1}^{r} H^2\left(G_{x_i}, \widehat{\mathscr{T}}_{X,x_i}\right).$$

όπου στην παραπάνω ισότητα $x_1, \ldots, x_r \in X$ είναι τα σημεία διακλάδωσης, G_{x_i} οι αντίστοιχες ομάδες ισοτροπίας και $\widehat{\mathscr{T}}_{X,x_i}$ η πλήρωση του τοπικού εφαπτόμενου χώρου, $\widehat{\mathscr{T}}_{X,x_i} = k[[t_i]]\frac{d}{dt_i}$, όπου t_i ειναι ο τοπικός uniformizer το x_i. Μπορούμε να δούμε τον χώρο $k[[t_i]]\frac{d}{dt_i}$ ως G_{x_i}-πρότυπο μέσω της δράσης συζυγίας, βλέπε [22, 2.1], [49, 1.5]. Οι Bertin και Mezard μετέτρεψαν τον υπολογισμό του εμποδίου της infinitensimal ανύψωσης της αναπαράστασης της ομάδας ισοτροπίας G_{x_i}, στην δύσκολη ομάδα $\mathrm{Aut}k[[t]]$, όπου με $\mathrm{Aut}k[[t]]$ συμβολίζουμε την ομάδα των συνεχών αυτομορφισμών της ομάδας $k[[t]]$.

Έστω τώρα G μια πεπερασμένη ομάδα, και θεωρούμε τον ομομορφισμο

$$\rho : G \hookrightarrow \mathrm{Aut}(k[[t]]),$$

στο θα αναφερόμαστε ως τοπική G-δράση. Με $W(k)$ συμβολίζουμε τον δακτύλιο των Witt διανυσμάτων του k. Το πρόβλημα της τοπικής ανύψωσης αποτελείται από την επόμενη ερώτηση: Υπάρχει επέκταση $\Lambda/W(k)$ και αναπαράσταση

$$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]]),$$

τέτοια ώστε, αν t είναι η αναγωγή του T, τότε η δράση του G στο $\Lambda[[T]]$ αναγάγετε στην δράση της G στο $k[[t]]$; Αν η απάντηση στην παραπάνω ερώτηση είναι θετική, τότε λέμε ότι η G-δράση ανυψώνεται στην χαρακτηριστική μηδέν. Μια ομάδα για την οποία κάθε τοπική G-δράση στο $k[[t]]$ σηκώνεται στην χαρακτηριστική μηδέν καλείται τοπική Oort ομάδα για το k.

Κατόπιν μελέτης διαφόρων εμποδίων (Bertin-εμπόδιο, KGB-εμπόδιο, Hurwitz tree εμπόδιο, κλπ.) είναι γνώστο ότι οι μόνες πιθανές τοπικές Oort ομάδες μπορούν να είναι οι εξής:

(i) Κυκλικές ομάδες

(ii) Διεδρικές ομάδες D_{p^h} τάξης $2p^h$

(iii) H alternating ομάδα A_4

Η εικασία του Oort λέει ότι κάθε κυκλική ομάδα C_q τάξης $q = p^h$ ανυψώνεται τοπικά. Η εικασία αυτή αποδείχθηκε πρόσφατα από τον F. Pop [66] χρησιμοποιώντας την δουλεία του των A. Obus and S.

Wewers [63]. Ο A. Obus απέδειξε ότι η A_4 είναι τοπική Oort ομάδα στο [60] και ήταν επίσης γνωστό στους F. Pop, I. Bouw and S. Wewers [14]. Η περίπτωση της διεδρικής ομάδας D_p έχει αποδειχθεί ότι είναι επίσης τοπική Oort ομάδα από τους I. Bouw και S. Wewers για p περιττό [14] και από τον G. Pagot [65]. Διάφορες περιπτώσεις της διεδρικής ομάδας D_p^h, για μικρό p^h έχουν μελετηθεί από τον A. Obus [61] και τους H. Dang, S. Das, K. Karagiannis, A. Obus, V. Thatte [23], καθώς και η D_4 από τον B. Weaver [80]. Για περισσότερες λεπτομέρειες στο πρόβλημα ανύψωσης παραπέμπουμε στα [19], [20], [21], [59]. Πιθανότατα το πιο σημαντικό από τα γνωστά εμπόδια, είναι το KGB-εμπόδιο [20]. Ύπήρχε η εικασία πως, αν η p-Sylow υποομάδα της G είναι κυκλική, τότε αυτό ειναι το μόνο εμπόδιο για το πρόβλημα της τοπικής ανύψωσης see [59], [61]. Ειδικότερα, το KGB-εμπόδιο μηδενίζεται για την διεδρική ομάδα D_q, οπότε από την παραπάνω εικασία, πιστευεται πως η τοπική δράση της D_q ανυψώνεται πάντα. Στην ενότητα 3.5a θα κατασκευάσουμε ένα αντιπαράδειγμα για την εικασία αυτή, δείχνοτας ότι το HKG-κάλλυμα που αντιστοιχεί στην D_{125}, με lower jumps $9, 189, 4689$, δεν είναι δυνατό να ανυψωθεί.

Η διατριβή αυτή είναι χωρισμένη σε δύο μέρη. Στο πρώτο μέρος ο στόχος μας είναι να δώσουμε μια νέα προσέγγιση στην θεωρία παραμορφώσεων καμπυλών με αυτομορφισμούς, η οποία δεν βασίζεται στην θεωρία παραμορφώσεων αναπαραστάσεων της περίπλοκης ομάδας $\mathrm{Aut}k[[t]]$, αλλά της πιο κατανοητής γενικής γραμμικής ομάδας. Για να το επιτύχουμε αυτό θα περιοριστούμε σε καμπύλες που πληρούν τις προϋποθέσεις του θεωρήματος του Petri.

Θεώρημα 1 (Θεώρημα του Petri)**.** Για μια μη-ιδιάζουσα και μη-υπερελλειπτική καμπύλη X, γένους $g \geqslant 3$ ορισμένη πάνω από ένα αλγεβρικά κλειστό σώμα με sheaf διαφορικών Ω_X, υπάρχει η εξής μικρή ακριβής ακολουθία:

$$0 \to I_X \to \mathrm{Sym}H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

όπου το I_X παράγεται από στοιχεία τάξης 2 και 3. Ακόμα αν η X δεν είναι μη ιδιάζουσα quintic γένους 6 ή η X δεν είναι trigonal καμπύλη, τότε το I_X παράγεται από στοιχεία τάξης 2.

Για μία απόδειξη αυτού του θεωρήματος παραπέμπουμε στα [30], [67]. Το ιδεώδες I_X καλείται κανονικό ιδεώδες και είναι το ομογενές ιδεώδες της καμπύλης X εμφυτευμένης μέσα στο $\mathbb{P}^{g-1}$. Για καμπύλες που ικανοποιούν τις προϋποθέσεις του θεωρήματος Petri και το ομογενές ιδεώδες τους παράγεται από τετραγωνικά στοιχεία, αποδεικνύουμε στην ενότητα 2.3 την ακόλουθη σχετική μορφή του θεωρήματος Petri.

Πρόταση 2. Έστω $f_1, \dots, f_r \in S := \mathrm{Sym}H^0(X, \Omega_X) = k[\omega_1, \dots, \omega_g]$ τετραγωνικά πολυώνυμα τα οποία παράγουν το κανονικό ιδεώδες I_X της καμπύλης X πάνω απο ένα αλγεβρικά κλειστό σώμα k. Κάθε παραμόρφωση $\mathscr{X}_A$ δίνεται από τετραγωνικά πολυώνυμα $\tilde{f}_1, \dots, \tilde{f}_r \in \mathrm{Sym}H^0(\mathscr{X}_A, \Omega_{\mathscr{X}_A/A}) = A[W_1, \dots, W_g]$, τα οποία πέφτουν στα $f_1, \dots, f_r$ αν πάρουμε πηλίκο με το μέγιστο ιδεώδες $\mathfrak{m}_A$ του A.

Η προσέγγιση αυτή μας επιτρέπει να αντικαταστήσουμε κατασκευές Grothendieck equivariant συνομολογίας με ευκολότερες κατασκευές γραμμικής άλγεβρας. Ας σημειώσουμε ακόμα ότι εν γένει, δεν είναι εύκολοι οι υπολογισμοί με ομάδες equivariant Grothendieck συνομολογίας και συνήθως χρησιμοποιούνται spectral sequences ή equivariant Chech συνομολογία, βλέπε [9], [50, sec.3]. Έστω $i : X \to \mathbb{P}^{g-1}$ η κανονική εμφύτευση. Στην πρόταση 2.3.5.1 αποδεικνύουμε ότι τα στοιχεία $[f] \in H^1(X, \mathscr{T}_X)^G = D_{gl}k[\epsilon]$ αντιστοιχούν σε κλάσεις συνομολογίας στην $H^1(G, M_g(k)/\langle \mathbb{I}_g \rangle)$, όπου $M_g(k)/\langle \mathbb{I}_g \rangle$ είναι ο χώρος $g \times g$ πινάκων με συντελεστές από το k, πηλίκο το διανυσματικό υπόχωρο που παράγεται από τον ταυτοτικό πίνακα. Επιπλέον, στο πλαίσιο αυτό, το εμπόδιο της ανύψωσης μετατρέπεται στο εμπόδιο της ανύψωσης αναπαραστάσεων της γεννικής γραμμικής ομάδας

$$\rho : G \to \mathrm{GL}\big(H^0(X, \Omega_X)\big) \tag{6}$$

και ένα κριτήριο συμβατότητας. Πιο συγκεκριμένα στην ενότητα 2.4 θα αποδείξουμε το ακόλουθο:

Θεώρημα 3. Θεωρούμε επιμορφισμό τοπικών Artin δακτυλίων $\Gamma' \to \Gamma \to 0$. Μία παραμόρφωση $x \in D_{gl}(\Gamma)$ μπορεί να ανυψωθεί σε μια παραμόρφωση $x' \in D_{gl}(\Gamma')$ αν και μόνο αν η αναπαράσταση $\rho_\Gamma : G \to GL_g(\Gamma)$ σηκώνεται σε μια αναπαράσταση $\rho_{\Gamma'} : G \to GL_g(\Gamma')$ και επιπλέον υπάρχει ανύψωση $X_{\Gamma'}$ της εμφυτευμένης παραμόρφωσης X_Γ η οποία είναι αναλλοίωτη από την ανύψωση της δράσης της $\rho_{\Gamma'}$.

Παρατήρηση 4. Η δυνατότητα ανύψωσης της αναπαράστασεις ρ είναι μία ισχυρη συνθήκη. Στην πρόταση 2.4.0.1 δίνουμε ένα παράδειγμα μίας αναπαράστασης $\rho : G \to GL_2(k)$, για ένα σώμα k θετικής χαρακτηριστικής p, η οποία είναι αδύνατο αν ανυψωθεί σε μια αναπαράσταση $\tilde{\rho} : G \to GL_2(R)$ όπου $R = W(k)[\zeta_{p^h}]$, εννοώντας ότι μια ανύψωση σε μία μικρή επέκταση $R/m_R^{i+1} \to R/m_R^i$ εμποδίζεται. Το R συμβολίζει εδώ, τον Witt δακτύλιο του k έχοντας προσθέσειε με μια πρωταρχική p^h ρίζα της μονάδας, χαρακτηριστικής μηδέν. Στο αντιπαράδειγμα μας $G = C_q \rtimes C_m$, $q = p^h$, $(m, p) = 1$.

Παρατήρηση 5. Το αναλλοίωτο του κανονικού ιδεώδους I_{X_Γ} υπό την δράση της G μπορεί να ελεχθεί με απαλοιφή Gauss και echelon κανονικές μορφές, βλέπε ενότητα 1.2b (ή [51, sec 2.2]).

Παρατήρηση 6. Το κανονικό ιδεώδες I_{X_Γ} καθορίζεται από r τετραγωνικά πολυώνυμα, τα οποία σχηματίζουν ένα $\Gamma[G]$-αναλλοίωτο Γ-υποπρότυπο V_Γ του ελεύθερου Γ-προτύπου των συμμετρικών $g \times g$ πινάκων με εγγραφές στο Γ. όταν περνάμε από μία παραμόρφωση $x \in D_{gl}(\Gamma)$ σε μία παραμόρφωση $x' \in D_{gl}(\Gamma')$, απαιτούμε το κανονικό ιδεώδες $I_{X_{\Gamma'}}$ να είναι αναλλοίωτο από την υψωμένη δράση της αναπαράστασης $\rho_{G'} : G \to GL_g(\Gamma')$. Στον ορισμό 2.2.1.1 εισάγουμε την δράση $T(g)$ στον διανυσματικό χώρο των συμμετρικών $g \times g$ πινάκων, και το να είναι αναλλοίωτο το κανονικό ιδεώδες, μεταφράζεται στο να μένει αναλλοίωτο υπό την T-δράση του Γ'-υποπροτύπου $V_{\Gamma'}$, που παράγεται από τα τετραγωνικά πολυώνυμα που παράγουν το $I_{X'}$. Επομένως, μπορούμε να κατασκευάσουμε ακόμα μια αναπαράσταση:

$$\rho^{(1)} : G \to GL\big(\mathrm{Tor}_1^S(k, I_X)\big). \tag{7}$$

Θέτουμε $r = \binom{g-2}{2}$. Οι ανυψώσεις των $\rho, \rho^{(1)}$ που ορίζονται στις προτάσεις (2), (3) στο $GL_g(\Gamma)$ και αντίστοιχα στο $GL_r(\Gamma)$ θα συμβολίζεται με ρ_Γ αντίστοιχα $\rho_\Gamma^{(1)}$.

Αν η αναπαράσταση ρ_G ανυψωθεί σε μια $\rho_{\Gamma'}$ και επιπλέον υπάρχει μια ανύψωση $X_{\Gamma'}$ της σχετικής καμπύλης X_Γ τέτοια ώστε η $X_{\Gamma'}$ να έχει ιδεώδες $I_{X_{\Gamma'}}$ το οποίο είναι $\rho_{\Gamma'}$ αναλλοίωτο, τότε η αναπαράσταση $\rho_\Gamma^{(1)}$ μπορεί επίσης να ανυψωθεί σε μια αναπαράσταση $\rho_{\Gamma'}^{(1)}$, δείτε επίσης το κεφάλαιο 1.

Η θεωρία παραμορφώσεων των αναπαραστάσεων $\rho, \rho^{(1)}$ μας δίνει τους συνκύκλους D_σ, $D_{\sigma^{-1}}^{(1)}$ στο $H^1(G, M_g(k))$, $H^1(G, M_{\binom{g-2}{2}}(k))$, καθώς η θεωρία παραμορφώσεων καμπυλών με αυτομορφισμούς εισάγει συνκύκλους $B_\sigma[f]$ που αντιστοιχούν στο $[f] \in H^1(X, \mathscr{T}_X)^G$. Στην ενότητα 2.4b θα αποδείξουμε μια συνθήκη συμβατότητας μεταξύ αυτών των συνκύκλων, χρησιμοποιώντας τον παρακάτω ισομορφισμο

$$\begin{aligned} \psi : M_g(k)/\langle \mathbb{I}_g \rangle &\xrightarrow{\cong} H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) \hookrightarrow \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \\ B &\longmapsto \psi_B \end{aligned}$$

όπως ορίζεται στην πρόταση 2.3.4.1.

Πρόταση 7. Η παρακάτω σχέση ικανοποιείται

$$\psi_{D_\sigma} - \psi_{B_\sigma[f]} = D_{\sigma^{-1}}^{(1)}. \tag{8}$$

Το κύριο αποτέλεσμα μας είναι να δώσουμε μια ικανή και αναγκαία συνθήκη για την $C_q \rtimes C_m$-δράση και ειδικότερα για την διεδρική ομάδα D_q, για να υψώνεται. Για την επίτευξη του στόχου αυτού θα χρησιμοποιήσουμε την Harbated-Katz-Gabber-compactification (HKG), η οποία μπορεί να χρησιμοποιηθεί για να κατασκευάσει μια πλήρη καμπύλη από μια τοπική δράση. Με αυτό τον τρόπο μπορούμε να μετατρέψουμε μια τοπική δράση και τις παραμορφώσεις της σε αναπαραστάσεις της γενικής γραμμικής ομάδας που δρουν στον χώρο διαφορικών της HKG-καμπύλης. Θα αναπτύξουμε τα κυριότερα εργαλεία στο κεφάλαιο 2.

Πιο συγκεκριμένα το πρώτο κομμάτι χωρίζεται σε τρία κεφάλαια. Στο πρώτο κεφάλαιο μελετάμε την ομάδα αυτομορφισμών μίας καμπύλης από την οπτική της κανονικής εμφύτευσης και του θεωρήματος του Petri. Δίνουμε ένα κριτήριο για να δούμε την ομάδα αυτομορφισμών σαν μια αλγεβρική υποομάδα της γενικής γραμμικής ομάδας. Επιπλέον επεκτείνουμε την δράση των αυτομορφισμών σε μια γραμμική δράση στους γεννήτορες της ελάχιστης ελεύθερης επίλυσης του κανονικού δακτυλίου της καμπύλης.

Στο δεύτερο κεφάλαιο μελετάμε την θεωρία παραμορφώσεων καμπυλών με την χρήση του κανονικού ιδεώδους και μετατρέπουμε το πρόβλημα ανύψωσης καμπυλών με αυτομορφισμούς σε πρόβλημα ανύψωσης γραμμικών αναπαραστάσεων.

Στο τελευταίο κεφάλαιο του πρώτου μέρους κοιτάμε το τοπικό πρόβλημα ανύψωσης ημιευθέων γινομένων μίας κυκλικής p-ομάδας και μίας κυκλικής ομάδας με τάξω πρώτη προς το p, όπου p είναι η χαρακτηριστική του special fibre. Δίνουμε ένα κριτήριο βασισμένο στο HKG-compactification, το οποίο μας επιτρέπει να αποφασίσουμε πότε μια τοπική δράση μπορεί να ανυψωθεί ή όχι. Ειδικότερα στην περίπτωση της διεδρικής ομάδας δίνουμε ένα παράδειγμα τοπικής δράσης το οποίο δεν είναι δυνατό να ανυψωθεί, και με αυτόν τον τρόπο δίνουμε ένα νέο εμπόδιο πιο ισχυρό από το KGB-εμπόδιο.

Στο δεύτερο δίνουμε μια ικανή και αναγκαία συνθήκη για μια modular αναπαράσταση της ομάδας $G = C_{p^h} \rtimes C_m$ ορισμένη σε ένα σώμα θετικής χαρακτηριστικής, για να ανυψωθεί σε μια τοπική περιοχή κυρίων ιδεωδών χαρακτηριστικής μηδέν η οποία περιέχει p^h ρίζες της μονάδας. Αυτό είναι ένα πολύ σημαντικό εργαλείο που χρησιμοποιούμε στην τρίτο κεφάλαιο του πρώτου μέρους και ταυτόχρονα έχει ανεξάρτητο ενδιαφέρον από την θεωρία παραμορφώσεων.

Ευχαριστίες

Θέλω να ξεκινήσω εκφράζοντας τις θερμές μου ευχαριστίες προς τον Καθηγητή Αριστείδη Κοντογεώργη, τον αφοσιωμένο επιβλέποντα της διατριβής μου, ο οποίος είναι ο μοναδικός λόγος που παρέμεινα στην Ελλάδα και επέλεξα να ακολουθήσω το διδακτορικό μου στο Πανεπιστήμιο Αθηνών. Πέρα από τον ρόλο του ως επιβλέποντα, είμαι βαθύτατα ευγνώμων για τον ρόλο του ως μέντορα και φίλο.

Οι ευχαριστίες μου εκτείνονται στους γονείς μου για την υποστήριξή τους, παρέχοντάς μου όχι μόνο τα απαραίτητα, αλλά και ενθαρρύνοντας την αγάπη μου για τα μαθηματικά, την κατάλληλη στιγμή.

Είμαι επίσης υπόχρεος στους συμφοιτητές μου, ιδιαίτερα στον Παναγιώτη Χρήστου, ο οποίος όχι μόνο μοιράστηκε τις μαθηματικές του γνώσεις, αλλά υπήρξε η μοναδική μαθηματική παρέα κατά τη διάρκεια των δύο πρώτων ετών του διδακτορικού. Εκφράζω εξίσου την εκτίμησή μου προς τους Δημήτρη Γαζούλη, Γιώργο Ζαχαρόπουλο, Μίλτο Καρακικέ, Ορέστη Λύγδα, Κωσταντία Μανούσου και Δημήτρη Νούλα, με τους οποίους δημιουργήσαμε μια ομάδα για τα επόμενα δύο χρόνια.

Θέλω να εκφράσω τη θερμή μου εκτίμηση προς τον Κώστα Καραγιάννη για την καθοδήγησή του, προσφέροντας συμβουλές και σκέψεις από την πρώτη παρουσίασή μου. Επιπλέον, η ειλικρινής ευγνωμοσύνη μου πηγαίνει στον Άντριου Όμπους για τα εξαιρετικά χρήσιμα σχόλιά του και την επικοινωνία στην δουλεία μας.

Τέλος, αλλά σίγουρα όχι λιγότερο σημαντικό, θέλω να εκφράσω τις θερμότερες ευχαριστίες μου στην σύζυγό μου Άννα Καλύβα. Η συνεχής κατανόησή της και η αμετακίνητη υποστήριξή της αποτέλεσαν πηγή δύναμης καθόλη τη διάρκεια αυτής της διαδρομής. Τα αδέρφια μου και οι φίλοι που μου στάθηκαν όλο αυτό τον καιρό αξίζουν επίσης ιδιαίτερη μνεία, και ειδικότερα αυτοί που άντεξαν να με ακούνε να μιλάω για μαθηματικά με τις ώρες.

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Ευρωπαϊκή Ένωση
Ευρωπαϊκό Κοινωνικό Ταμείο

Επιχειρησιακό Πρόγραμμα
Ανάπτυξη Ανθρώπινου Δυναμικού,
Εκπαίδευση και Διά Βίου Μάθηση
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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Αθήνα Σεπτέμβριος 2023

Part I

Deformations of curves with Automorhisms

Chapter 1

Automorphisms and the canonical ideal

1.1 Introduction

Let X be a non-singular complete algebraic curve defined over an algebraically closed field of characteristic $p \geqslant 0$. If the genus g of the curve X is $g \geqslant 2$ then the automorphism group $G = \mathrm{Aut}(X)$ of the curve X is finite. The theory of automorphisms of curves is an interesting object of study, see the surveys [2], [16] and the references therein.

On the other hand the theory of syzygies which originates in the work of Hilbert and Sylvester has attracted a lot of researchers and it seems that a lot of geometric information can be found in the minimal free resolution of the ring of functions of an algebraic curve. For an introduction to this fascinating area we refer to [26].

In the first chapter we aim to put together the theory of syzygies of the canonical embedding and the theory of automorphisms of curves. Throughout this chapter X is a non-hyperelliptic, non-trigonal and a non-singular quintic of genus 6 and we also assume $p \neq 2$. These conditions are needed for Petri's theorem to hold, while the $p \neq 2$ condition is needed to ensure the faithful action of the automorphism group on the space of holomorphic differentials $H^0(X, \Omega_X)$.

More precisely, in section 1.2a we use Petri's theorem in order to give a necessary and sufficient condition for an element in $\mathrm{GL}(H^0(X, \Omega_X))$ to act as an automorphism of our curve. In this way we can arrive to

Proposition 1.1.0.1. The automorphism group of a curve X as a finite set can be seen as a subset of the $g^2(g+1)^2-1$-dimensional projective space and can be described by explicit quadratic equations.

In section 1.3 we show that the automorphism group G of the curve acts linearly on a minimal free resolution $\mathbf{F}$ of the ring of regular functions S_X of the curve X canonically embedded in $\mathbb{P}^{g-1}$. Notice that an action of a group G on a graded module M gives rise to a series of linear representations $\rho_d : G \to M_d$ to all linear spaces M_d of degree d for $d \in \mathbb{Z}$. For the case of the free modules F_i of the minimal free resolution $\mathbf{F}$ we relate the actions of the group G in both F_i and in the dual F_{g-2-i} in terms of an inner automorphism of G.

This information is used in order to show that the action of the group G on generators of the modules F_i sends generators of degree d to linear combinations of generators of degree d. Let $S = \mathrm{Sym}(H^0(X, \Omega_X))$ be the symmetric algebra of $H^0(X, \Omega_X)$.

Proposition 1.1.0.2. There is a well defined linear action of the automorphism group G on minimal generators of the free resolution, which sends a minimal generator of degree d of the free module F_i to a linear combination of other generators of degree d.

The degree d-part of $\mathrm{Tor}_i^S(k, S_X)$ will be denoted by $\mathrm{Tor}_i^S(k, S_X)_d$, which is a vector space of dimension $\beta_{i,d}$. We can use our computation in order to show that all $\mathrm{Tor}_i^S(k, S_X)_d$ are acted on by the group

G, but this also follows by Koszul cohomology, see [3]. Indeed, one starts with the vector space $V = H^0(X, \Omega_X)$, $\dim V = g$, $S = \mathrm{Sym}(V)$ and considers the exact Koszul complex

$$0 \to \wedge^g V \otimes S(-g) \to \wedge^{g-1} V \otimes S(-g+1) \to \cdots$$

$$\cdots \to \wedge^2 V \otimes S(-2) \to \nu \otimes S(-1) \to S \to k \to 0.$$

The symmetry property of the Tor functor implies that one can calculate $\mathrm{Tor}_i^S(k, S_X)$ by using the Koszul resolution of k instead of the Koszul resolution of S_X. Since the Koszul resolution of k is a complex of G-modules and all differentials are G-module morphisms the $\mathrm{Tor}_i^S(k, S_X)_d$ are naturally G-modules. On the other hand the passage to the action on generators is not explicit since the isomorphism between the graded components of the terms in the minimal resolution and Koszul cohomology spaces is not explicit, as it comes from the spectral sequence that ensures the symmetry of Tor functor.

Finally, the representations to the d graded space of each F_i, $\rho_{i,d} : G \to \mathrm{GL}(F_{i,d})$ can be expressed as a direct sum of the G-modules $\mathrm{Tor}_i^S(k, S_X)_d$. We conclude by showing that the G-module structure of all F_i is determined by knowledge of the G-module structure of $H^0(X, \Omega_X)$ and the G-module structure of each $\mathrm{Tor}_i^S(k, S_X)$ for all $0 \leqslant i \leqslant g-2$.

1.2 Automorphisms of curves and Petri's theorem

Consider a complete non-singular non-hyperelliptic curve of genus $g \geqslant 3$ over an algebraically closed field K. Let Ω_X denote the sheaf of holomorphic differentials on X.

Theorem 1.2.1 (Noether-Enriques-Petri)**.** There is a short exact sequence

$$0 \to I_X \to \mathrm{Sym} H^0(X, \Omega_X) \to \bigoplus_{n=0}^{\infty} H^0(X, \Omega_X^{\otimes n}) \to 0,$$

where I_X is generated by elements of degree 2 and 3. Also if X is not a non-singular quintic of genus 6 or X is not a trigonal curve, then I_X is generated by elements of degree 2.

For a proof of this theorem we refer to [67], [30]. The ideal I_X is called the canonical ideal and it is the homogeneous ideal of the embedded curve $X \to \mathbb{P}_k^{g-1}$. The automorphism group of the ambient space $\mathbb{P}^{g-1}$ is known to be $\mathrm{PGL}_g(k)$, [36, example 7.1.1 p. 151]. On the other hand every automorphism of X is known to act on $H^0(X, \Omega_X)$ giving rise to a representation

$$\rho : G \to \mathrm{GL}(H^0(X, \Omega_X)),$$

which is known to be faithful, when X is not hyperelliptic and $p \neq 2$, see [46]. The representation ρ in turn gives rise to a series of representations

$$\rho_d : G \to \mathrm{GL}(S_d),$$

where S_d is the vector space of degree d polynomials in the ring $S := k[\omega_1, \ldots, \omega_g]$.

Let $X \subset \mathbb{P}^r$ be a projective algebraic set. Is it true that every automorphism $\sigma : X \to X$ comes as the restriction of an automorphism of the ambient projective space, that is by an element of $\mathrm{PGL}_k(r)$? For instance such a criterion for complete intersections is explained in [48, sec. 2]. In the case of canonically embedded curves $X \subset \mathbb{P}^{g-1}$ it is clear that any automorphism $\sigma \in \mathrm{Aut}(X)$ acts also on $\mathbb{P}^{g-1} = \mathrm{Proj} H^0(X, \Omega_X)$. In this way we arrive at the following:

Lemma 1.2.1.1. Every automorphism $\sigma \in \mathrm{Aut}(X)$ corresponds to an element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ and every element in $\mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$ gives rise to an automorphism of X.

In the next section we will describe the elements $\sigma \in \mathrm{PGL}_g(k)$ such that $\sigma(I_X) \subset I_X$.

1.2a Algebraic equations of automorphisms

For now on we will assume that the canonical ideal I_X is generated by polynomials in $k[\omega_1, \dots, \omega_g] = \mathrm{Sym} H^0(X, \Omega_X)$ of degree 2, that is the requirements for Petri's theorem hold. Consider such a set of quadratic polynomials $\tilde{A}_1, \dots, \tilde{A}_r$ generating I_X.

A polynomial $\tilde{A}_i$ of degree two can be encoded in terms of a symmetric $g \times g$ matrix $A_i = (a_{\nu,\mu})$ as follows. Set $\bar{\omega} = (\omega_1, \dots, \omega_g)^t$. We have

$$\tilde{A}_i(\bar{\omega}) = \bar{\omega}^t A_i \bar{\omega}.$$

The polynomial $\sigma(\tilde{A}_i)$ is still a polynomial of degree two so we write $\sigma(A_i)$ for the symmetric $g \times g$ matrix such that $\sigma(\tilde{A}_i) = \bar{\omega}^t \sigma(A)_i \bar{\omega}$. It is clear that for an element $\sigma \in GL_g(k)$, $\sigma(I_X) \subset I_X$ holds if and only if for all $1 \leqslant i \leqslant r$, $\sigma(A_i) \in \mathrm{span}_k\{A_1, \dots, A_r\}$. This means that

$$(\sigma_{\mu,\nu})^t A_i (\sigma_{\mu,\nu}) = \sum_{j=1}^{r} \lambda(\sigma)_{ji} A_j \qquad \text{for every } 1 \leqslant i \leqslant j. \tag{1.1}$$

1.2b The automorphism group as an algebraic set.

Let $A_1, \dots, A_r$ be a set of linear independent $g \times g$ matrices such that the $w^t A_i w$ $1 \leqslant i \leqslant r$ generate the canonical ideal, and $w^t = (w_1, \dots, w_g)$ is a basis of the space of holomorphic differentials. By choosing an ordered basis of the vector space of symmetric $g \times g$ matrices we can represent any symmetric $g \times g$ matrix A as an element $\bar{A} \in k^{\frac{g(g+1)}{2}}$, that is

$$\begin{aligned} \bar{\cdot} : \text{Symmetric } g \times g \text{ matrices} &\longrightarrow k^{\frac{g(g+1)}{2}} \\ A &\longmapsto \bar{A} \end{aligned}$$

We can now put together the r elements $\bar{A}_i$ as a $g(g+1)/2 \times r$ matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$, which has full rank r, since $\{A_1, \dots, A_r\}$ are assumed to be linear independent.

Proposition 1.2.1.1. An element $\sigma = (\sigma_{ij}) \in GL_g(k)$ induces an action on the curve X, if and only if the $g(g+1)/2 \times 2r$ matrix

$$B(\sigma) = \left[\bar{A}_1, \dots, \bar{A}_r, \overline{\sigma^t A_1 \sigma}, \dots, \overline{\sigma^t A_r \sigma}\right]$$

has rank r.

We have that σ is an automorphism if the $g(g+1)/2 \times 2r$-matrix $B(\sigma)$ has rank r, which means that $(r+1) \times (r+1)$-minors of $B(\sigma)$ are zero. This provides us with a description of the automorphism group as a determinantal variety given by explicit equations of degree $(r+1)^2$.

But we can do better. Using Gauss elimination we can find a $\frac{g(g+1)}{2} \times \frac{g(g+1)}{2}$ invertible matrix Q which puts the matrix $\left(\bar{A}_1 | \cdots | \bar{A}_r\right)$ in echelon form, that is

$$Q \left(\bar{A}_1 | \cdots | \bar{A}_r\right) = \left(\begin{array}{c} \mathbb{I}_r \\ \hline \mathbb{O}_{\left(\frac{g(g+1)}{2} - r\right) \times r} \end{array}\right).$$

But then for each $1 \leqslant i \leqslant r$ eq. (1.1) is satisfied if and only if the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix of the matrix

$$Q \left(\overline{\sigma^t A_1 \sigma} | \cdots | \overline{\sigma^t A_r \sigma}\right) \tag{1.2}$$

is zero, while the top $r \times r$ block matrix gives rise to the representation

$$\rho_1 : G \to GL_r(k),$$

defined by equation (1.1). Assuming that the lower $(\frac{g(g+1)}{2} - r) \times r$ bottom block matrix gives us $r(\frac{g(g+1)}{2} - r)$ equations where the entries $\sigma = (\sigma_{ij})$ are seen as indeterminates. In this way we can write down elements of the automorphism group as a zero dimensional algebraic set, satisfying certain quadratic equations.

1.3 Syzygies

1.3a Extending group actions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module acted on by the group G, generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. We consider the free S-module $F_0 = \bigoplus_{j=1}^r S(-a_j)$ together with the onto map

$$F_0 = \bigoplus_j S(-a_j) \xrightarrow{\pi} M. \tag{1.3}$$

Let us denote by $M_1, \ldots, M_r$ elements of F_0, such that $\pi(M_i) = m_i$, assuming also that $\deg(M_i) = \deg(m_i)$, for $1 \leqslant i \leqslant r$. The action on the generators m_i is given by

$$\sigma(m_i) = \sum_{\nu=1}^{r} a_{\nu,i} m_i, \text{ for some } a_{\nu,i} \in S. \tag{1.4}$$

Remark 1.3.1. We would like to point out here that unlike the theory of vector spaces, an element $x \in F_0$ might admit two different decompositions

$$x = \sum_{i=1}^r a_i m_i = \sum_{i=1}^r b_i m_i, \text{ that is } \sum_{i=1}^r (a_i - b_i) m_i = 0,$$

and if $a_{i_0} - b_{i_0} \neq 0$ we cannot assume that $a_{i_0} - b_{i_0}$ is invertible, so we can't express m_{i_0} as an S-linear combination of the other elements m_i, for $i_0 \neq i, 1 \leqslant i \leqslant r$ in order to contradict minimality. We can only deduce that $\{a_i - b_i\}_{i=1,\ldots,r}$ form a syzygy.

Therefore one might ask if the matrix $(a_{\nu,i})$ given in eq. (1.4) is unique. In proposition 1.3.1.2 we will prove that the elements $a_{\nu,i}$ which appear as coefficients in eq. (1.4) are in the field k and therefore the expression is indeed unique.

The natural action of $\mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$ can be extended to an action on the ring $S = \mathrm{Sym} H^0(X, \Omega_X)$, so that $\sigma(xy) = \sigma(x)\sigma(y)$ for all $x, y \in S$. Therefore if $M = I_X$ then for all $s \in S$, $m \in I_X = M$ we have $\sigma(sm) = \sigma(s)\sigma(m)$. All the actions in the modules we will consider will have this property.

For a free module $F = \bigoplus_{j=1}^s S(-a_j)$, generated by the elements M_i, $1 \leqslant i \leqslant r$, $\deg(M_i) = a_i$ and a map $\pi : F \to M$ we define the action of G by

$$\sigma\left(\sum_{j=1}^r s_j M_j\right) = \sum_{j=1}^r \sigma(s_j) \sum_{\nu=1}^r a_{\nu,j}(\sigma) M_\nu = \sum_{\nu=1}^r \left(\sum_{j=1}^r a_{\nu,j}(\sigma)\sigma(s_j)\right) M_\nu,$$

where $\deg_S a_{\nu,j} + a_\nu = \deg_S m_j$. This means that under the action of $\sigma \in G$ the r-tuple $(s_1, \ldots, s_r)^t$ is sent to

$$\begin{pmatrix} s_1 \\ \vdots \\ s_r \end{pmatrix} \overset{\sigma}{\longmapsto} \begin{pmatrix} a_{1,1}(\sigma) & a_{1,2}(\sigma) & \cdots & a_{1,r}(\sigma) \\ \vdots & \vdots & & \vdots \\ a_{r,1}(\sigma) & a_{r,2}(\sigma) & \cdots & a_{r,r}(\sigma) \end{pmatrix} \begin{pmatrix} \sigma(s_1) \\ \vdots \\ \sigma(s_r) \end{pmatrix}.$$

If $A(\sigma) = \big(a_{i,j}(\sigma)\big)$ is the matrix corresponding to σ then for $\sigma, \tau \in G$ the following cocycle condition holds:

$$A(\sigma\tau) = A(\sigma)A(\tau)^\sigma.$$

If we can assume that G acts trivially on the matrix $A(\tau)$ for every $\tau \in G$ (for instance when $A(\tau)$ is a matrix with entries in k for every $\tau \in G$), then the above cocycle condition becomes a homomorphism condition.

Also if $A(\sigma)$ is a principal derivation, that is there is an $r \times r$ matrix Q, such that

$$A(\sigma) = \sigma(Q) \cdot Q^{-1}$$

then after a basis change of the generators we can show that the action on the coordinates is just given by

$$(s_1, \cdots, s_r)^t \overset{\sigma}{\longmapsto} (\sigma(s_1), \cdots, \sigma(s_r))^t,$$

that is the matrix $A(\sigma)$ is the identity. We will call the action on the free resolution $\mathbf{F}$ obtained by extending the action on M the standard action.

1.3b Group actions on free resolutions

Recall that $S = k[\omega_1, \ldots, \omega_g]$ is the polynomial ring in g variables. Let M be a graded S-module generated by the elements $m_1, \ldots, m_r$ of corresponding degrees $a_1, \ldots, a_r$. Consider the minimal free resolution

$$0 \longrightarrow F_g \xrightarrow{\phi_g} \cdots \longrightarrow F_1 \xrightarrow{\phi_1} F_0 \;, \tag{1.5}$$

where $\mathrm{coker}(\phi_1) = F_0/\mathrm{Im}\phi_1 = F_0/\ker\pi \cong M$. Let $\mathfrak{m}$ be the maximal ideal of S generated by $\langle \omega_1, \ldots, \omega_g \rangle$. Each free module in the resolution can be written as

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}},$$

where the integers $\beta_{i,j}$ are the Betti numbers of the resolution. The Betti numbers satisfy

$$\beta_{i,j} = \beta_{g-2-i,g+1-j}. \tag{1.6}$$

as one can see by using the self duality of the above resolution by twisting by $S(-g)$ see [58, prop. 4.1.1], [26, prop. 9.5] or by using Koszul cohomology, see [27, prop. 4.1].

Assume that M and each F_i is acted on by a group G and that the maps δ_i are G-equivariant. We will now study the action of the group G on the generators of F_i. First of all we have that

$$F_i = \bigoplus_{\nu=1}^{r_i} \bigoplus_{\mu=1}^{\beta_{i,\nu}} e_{i,\nu,\mu} S \cong \bigoplus_{\nu=1}^{r_i} S(-d_{i,\nu})^{\beta_{i,\nu}}.$$

In the above formula we assumed that F_i is generated by elements $e_{i,\nu,\mu}$ such that the degree of $e_{i,\nu,\mu} = d_{i,\nu}$ for all $1 \leqslant \mu \leqslant \beta_{i,\nu}$. We also assume that

$$d_{i,1} < d_{i,2} < \cdots < d_{i,r_i}.$$

The action of σ is respecting the degrees, so an element of minimal degree $d_{i,1}$ is sent to a linear combination of elements of minimal degree $d_{i,1}$. In this way we obtain a representation

$$\rho_{i,1} : G \to \mathrm{GL}(\beta_{i,1}, k).$$

In a similar way an element $e_{i,2,\mu}$ of degree $d_{i,2}$ is sent to an element of degree $d_{i,2}$ and we have that

$$\sigma(e_{i,2,\mu}) = \sum_{j_1=1}^{\beta_{i,2}} \lambda_{i,2,\mu,j_1} e_{i,2,j_1} + \sum_{j_2=1}^{\beta_{i,1}} \lambda'_{i,2,\mu,j_1} e_{i,1,j_2},$$

where all $\lambda_{i,2,\mu,j_1} \in k$ and all $\lambda'_{i,1,\mu,j_2} \in \mathfrak{m}^{d_{i,2}-d_{i,1}}$. In this case we have a representation with entries in an ring instead of a field, which has the form:

$$\rho_{i,2} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,2}-d_{i,1}}),$$

$$\sigma \mapsto \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) \\ 0 & A_2(\sigma) \end{pmatrix},$$

where $A_1(\sigma) \in \mathrm{GL}(\beta_{i,1}, k)$ and $A_2(\sigma) \in \mathfrak{m}^{d_{i,2}-d{i,1}}\mathrm{GL}(\beta_{i,2}, k)$.

By induction the situation in the general setting gives rise to a series of representations:

$$\rho_{i,j} : G \to \mathrm{GL}(\beta_{i,1} + \beta_{i,2}, \mathfrak{m}^{d_{i,j}-d_{i,1}})$$

$$\sigma \mapsto A(\sigma) = \begin{pmatrix} A_1(\sigma) & A_{1,2}(\sigma) & \cdots & A_{1,j}(\sigma) \\ 0 & A_2(\sigma) & & A_{2,j}(\sigma) \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & 0 & A_j(\sigma) \end{pmatrix} \tag{1.7}$$

where $A_\nu(\sigma) \in \mathrm{GL}(\beta_{i,\nu}, k)$ and $A_{\kappa,\lambda}(\sigma)$ is an $\beta_{i,\kappa} \times \beta_{i,\lambda}$ matrix with coefficients in $\mathfrak{m}^{\beta_{i,\lambda}-\beta_{i,\kappa}}$. The representation ρ_{i,r_i} taken modulo $\mathfrak{m}$ reduces to $\mathrm{Tor}_i^S(k, M)$, seen as a $k[G]$-module.

1.3c Unique actions

Let us consider two actions of the automorphisms group G on $H^0(X, \Omega_X)$, which can naturally be extended on the symmetric algebra $\mathrm{Sym} H^0(X, \Omega_X)$. We will denote the first action by $g \star \nu$ and the second action by $g \circ \nu$, where $g \in G$, $\nu \in \mathrm{Sym} H^0(X, \Omega_X)$.

Proposition 1.3.1.1. If the curve X satisfies the conditions of faithful action of $G = \mathrm{Aut}(X)$ on $H^0(X, \Omega_X)$, that is X is not hyperelliptic and $p > 2$, [46, th. 3.2] and moreover both actions $\star, \circ$ restrict to actions on the canonical ideal I_X, then there is an automorphism $i : G \to G$, such that $g \star \nu = i(g) \circ \nu$.

Proof. Both actions of G on $H^0(X, \Omega_X)$ introduce automorphisms of the curve X. That is since $G \star I_X = I_X$ and $G \circ I_X = I_X$, the group G is mapped into $\mathrm{Aut}(X) = G$. This means that for every element $g \in G$ there is an element $g^* \in \mathrm{Aut}(X) = G$ such that $g \star \nu = g^* \nu$, where the action on the right is the standard action of the automorphism group on holomorphic differentials. By the definition of the group action for every $g_1, g_2 \in G$ we have $(g_1 g_2)^* \nu = g_1^* g_2^* \nu$ for all $\nu \in H^0(X, \omega_X)$ and the faithful action of the automorphism group provides us with $(g_1 g_2)^* = g_1^* g_2^*$, i.e. the map $i_* : g \mapsto g^*$ is a homomorphism. Similarly the map corresponding to the $\circ$-action, $i_\circ : g \mapsto g^\circ$ is a homomorphism and the desired homomorphism i is the composition of $i_* i_\circ^{-1}$. □

The map $\mathrm{Hom}_S(F_i, S(-g))$ induces a symmetry of the free resolution $\mathbf{F}$ by sending F_i to F_{g-2-i}. Each free module F_i of the resolution $\mathbf{F}$ is equipped by the extension of the action on holomorphic differentials, according to the construction of section 1.3b. On the other hand since $S(-g)$ is a G-module we have that $F_{g-2-i} \cong \mathrm{Hom}_S(F_i, S(-g))$ is equipped by a second action namely every $\phi : F_i \to S(-g)$ is acted naturally by G in terms of $\phi \mapsto \phi^\sigma = \sigma^{-1} \phi \sigma$. How are the two actions related?

Lemma 1.3.1.1. Denote by $\star$ the action of G on F_i induced by taking the $S(-g)$-dual. The standard and the $\star$-actions are connected in terms of an automorphism ψ_i of G, that is for all $\nu \in F_i$ $g \star \nu = \psi_i(g) \nu$.

Proof. Assume that $i \leqslant g - 2 - i$. Consider the standard action of G on the free resolution $\mathbf{F}$. The module F_{g-2-i} obtains a new action $g \star \nu$ for $g \in G, \nu \in F_i$. By 1.3b this $\star$ action is transferred to an action on all F_j for $j \geqslant g - 2 - i$, including the final term F_{g-2} which is isomorphic to $S(-1)$. This gives us two actions on $H^0(X, \Omega_X)$ which satisfy the requirements of proposition 1.3.1.1. The desired result follows, since the action can be pulled back to all syzygies using either $\mathbf{F}$ or $\mathbf{F}^*$. □

Proposition 1.3.1.2. Under the faithful action requirement we have that all automorphisms $\sigma \in G$ send the direct summand $S(-j)^{\beta_{i,j}}$ of F_i to itself, that is the representation matrix in eq. (1.7) is block diagonal.

Proof. Consider $F_i = \bigoplus_{\nu=1}^{r_i} M_{i,\nu} S$, where $M_{i,1}, \ldots, M_{i,r_i}$ are assumed to be minimal generators of F_i with descending degrees $a_{i,\nu} = \deg(m_{i,\nu})$, $1 \leqslant \nu \leqslant r_i$. The action of an element σ is given in terms of the matrix $A(\sigma)$ given in equation (1.7). The element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is sent to

$$\begin{aligned} h : \mathrm{Hom}_S(F_i, S(-g)) &\xrightarrow{\cong} F_{g-2-i} \\ \phi &\longmapsto \big(\phi(M_{i,1}), \ldots, \phi(M_{i,r_i})\big) \end{aligned} \tag{1.8}$$

Each $\phi(M_{i,\nu})$ can be considered as an element in $S(-g - 1 + \deg(m_{i,\nu}))$ inside F_{g-2-i}. Observe that the element $\phi \in \mathrm{Hom}_S(F_i, S(-g))$ is known if we know all $\phi(M_{i,\nu})$ for $1 \leqslant \nu \leqslant r_i$. From now on we will identify such an element ϕ as a r_i-tuple $\big(\phi(M_{i,\nu})\big)_{1 \leqslant \nu \leqslant r_i}$.

Recall that if A, B are G-modules, then there is an natural action on $\mathrm{Hom}(A, B)$, sending $\phi \in \mathrm{Hom}(A, B)$ to ${}^\sigma\phi$, which is the map

$${}^\sigma\phi : A \ni a \mapsto \sigma\phi(\sigma^{-1} a).$$

We have also a second action on the module F_{g-2-i}. We compute ${}^{\sigma}\phi(M_{i,\nu})$ for all base elements $M_{i,\nu}$ in order to describe ${}^{\sigma}\phi$:

$$\begin{aligned}\sigma\left(\phi(\sigma^{-1}M_{i,\nu})\right)_{1\leqslant\nu\leqslant\kappa} &= \left(\sum_{\mu=1}^{r_i}\sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\sigma\phi(M_{i,\mu})\right)_{1\leqslant\nu\leqslant r_i}\\ &= \left(\sum_{\mu=1}^{r_i}\sigma\big(\alpha_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi(M_{i,\mu})\right)_{1\leqslant\nu\leqslant r_i}\end{aligned}$$

where in the last equation we have used the fact that $\phi(M_i)$ are in the rank one G-module $S(-g) \cong \wedge^{g-1}\Omega^1_X$ hence the action of $\sigma \in G$ is given by multiplication by $\chi(\sigma)$, where $\chi(\sigma)$ is an invertible element is S.

In order to simplify the notation consider i fixed, and denote $M_\nu = M_{i,\nu}$, $r = r_i$, $a_{i,j} = a_j$. We can consider as a basis of $\mathrm{Hom}(F_i, S(-g))$ the morphisms ϕ_μ given by

$$\phi_\mu(M_j) = \delta_{\mu,j}\cdot E, \tag{1.9}$$

where E is a basis element of degree g of the rank 1 module $S(-g) \cong S\cdot E$. This is a different basis than the basis $M_{g-2-i,\nu}$, $1 \leqslant n \leqslant r_{g-2-i}$ of F_{g-2-i} we have already introduced.

According to eq. (1.6) if M_j has degree a_j then the element ϕ_j has degree $g+1-a_j$. Assume that M_r has maximal degree a_r. Then, ϕ_r has minimal degree. Moreover, in order to describe ${}^{\sigma}\phi_r$ we have to consider the tuple $({}^{\sigma}\phi_r(M_1),\ldots,{}^{\sigma}\phi_r(M_r))$. We have

$$\begin{aligned}\left({}^{\sigma}\phi_r(M_\nu)\right)_{1\leqslant\nu\leqslant r} &= \left(\sum_{\mu=1}^{r}\sigma\big(\alpha^{(i)}_{\mu,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_r(M_\mu)\right)_{1\leqslant\nu\leqslant r}\\ &\overset{(1.9)}{=\!=} \left(\sigma\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)E\right)_{1\leqslant\nu\leqslant r}\end{aligned}$$

and we finally conclude that

$${}^{\sigma}\phi_r = \sum_{\nu=1}^{r}\sigma^{-1}\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)\phi_\nu.$$

In this way every element $x \in F_{g-2-i}$ is acted on by σ in terms of the action

$$\sigma \star x = h\big({}^{\sigma}h^{-1}(x)\big),$$

where h is the map given in eq. (1.8). On the other hand the elements $h(\phi_r)$ are in F_{g-2-i} and by lemma 1.3.1.1 there is an element $\sigma' \in G$ such that

$$\sigma' h(\phi_r) = \sum_{\nu=1}^{r}\alpha^{(g-2-i)}_{\nu,r}(\sigma')h(\phi_\nu).$$

Since the element ϕ_ν has maximal degree among generators of F_i the element $h(\phi_r)$ has minimal degree. This means that all coefficients

$$\alpha^{(g-2-i)}_{\nu,r}(\sigma') = \sigma\big(\alpha^{(i)}_{r,\nu}(\sigma^{-1})\big)\chi(\sigma)$$

are zero for all ν such that $\deg m_\nu < \deg \mu_r$. Therefore all coefficients $a^{(i)}_{\nu,r}(\sigma)$ for ν such that $\deg m_\nu < \deg m_r$ are zero. This holds for all $\sigma \in G$. By considering in this way all elements $\phi_{r-1}, \phi_{r-2}, \ldots, \phi_1$, which might have greater degree than the degree of ϕ_r the result follows. □

1.4 Representations on the free resolution

Each S-module F_i in the minimal free resolution can be seen as a series of representations of the group G. Indeed, the modules F_i are graded and there is an action of G on each graded part $F_{i,d}$, given by representations

$$\rho_{i,d} : G \to GL\big(F_{i,d}\big),$$

where $F_{i,d}$ is the degree d part of the S-module F_i. The space $\mathrm{Tor}_i^S(k, S_X))$ is clearly a G-module, and by proposition 1.3.1.2 there is a decomposition of G-modules

$$\mathrm{Tor}_i^S(k, S_X) = \bigoplus_{j\in\mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_j,$$

where $\mathrm{Tor}_i^S(k, S_X)_j$ is the k-vector space generated by generators of F_i that have degree j. This is a vector space of dimension $\beta_{i,j}$.

Denote by $\mathrm{Ind}(G)$ the set of isomorphism classes of indecomposable $k[G]$-modules. If k is of characteristic $p > 0$ and G has no-cyclic p-Sylow subgroup then the set $\mathrm{Ind}(G)$ is infinite, see [7, p.26]. Suppose that each $\mathrm{Tor}_i^S(k, S_X)_j$ admits the following decomposition in terms of $U \in \mathrm{Ind}(G)$:

$$\mathrm{Tor}_i^S(k, S_X)_j = \bigoplus_{U\in\mathrm{Ind}(G)} a_{i,j,U} U \text{ where } a_{i,j,U} \in \mathbb{Z}.$$

We obviously have that

$$\beta_{i,j} = \sum_{U\in\mathrm{Ind}(G)} a_{i,j,U} \dim_k U.$$

The G-structure of F_i is given by

$$\mathrm{Tor}_i^S(k, S_X) \otimes S,$$

that is the G-module structure of $F_{i,d}$ is given by

$$F_{i,d} = \bigoplus_{d\in\mathbb{Z}} \bigoplus_{j\in\mathbb{Z}} \mathrm{Tor}_i^S(k, S_X)_{d-j} \otimes S_j.$$

1.5 An example: the Fermat curve

Consider the projective non singular curve given by equation

$$F_n : x_1^n + x_2^n + x_0^n = 0$$

This curve has genus $g = \frac{(n-2)(n-1)}{2}$. Set $x = x_1/x_0$, $y = x_2/x_0$. For $\omega = \frac{dx}{y^{n-1}} = -\frac{dy}{x^{n-1}}$ we have that the set

$$x^i y^j \omega \text{ for } 0 \leqslant i + j \leqslant n - 3 \tag{1.10}$$

forms a basis for holomorphic differentials, [47], [75], [76]. These g differentials are ordered lexicographically according to (i, j), that is

$$\omega_{0,0} < \omega_{0,1} < \cdots < \omega_{0,n-3} < \omega_{1,0} < \omega_{1,1} < \cdots < \omega_{1,n-4} < \cdots < \omega_{n-3,0}.$$

The case $n = 2$ is a rational curve, the case $n = 3$ is an elliptic curve, the case $n = 4$ has genus 3 and gonality 3, the case $n = 5$ has genus 6 and is quintic so the first Fermat curve which has canonical ideal generated by quadratic polynomial is the case $n = 6$ which has genus 10.

Proposition 1.5.0.1. The canonical ideal of the Fermat curve F_n for $n \geqslant 6$ consists of two sets of relations

$$G_1 = \{\omega_{i_1,j_1}\omega_{i_2,j_2} - \omega_{i_3,j_3}\omega_{i_4,j_4} : i_1 + i_2 = i_3 + i_4, j_1 + j_2 = j_3 + j_4\}, \tag{1.11}$$

and

$$G_2 = \left\{ \omega_{i_1,j_1}\omega_{i_2,j_2} + \omega_{i_3,j_3}\omega_{i_4,j_4} + \omega_{i_5,j_5}\omega_{i_6,j_6} = 0 : \begin{smallmatrix} i_1+i_2=n+a, & j_1+j_2=b \\ i_3+i_4=a, & j_3+j_4=n+b \\ i_5+i_6=a, & j_5+j_6=b \end{smallmatrix} \right\} \tag{1.12}$$

where $0 \leqslant a, b$ are selected such that $0 \leqslant a + b \leqslant n - 3$.

We will now prove proposition 1.5.0.1 for $n \geqslant 6$, following the method developed in [18]. Observe that the holomorphic differentials given in eq. (1.10) are in 1-1 correspondence with the elements of the set $\mathbf{A} = \{(i, j) : 0 \leqslant i + j \leqslant n - 3\} \subset \mathbb{N}^2$. First we introduce the following term order on the polynomial algebra $S := \mathrm{Sym} H^0(X, \Omega_X)$.

Definition 1.5.0.1. Choose any term order $\prec_t$ for the variables $\{\omega_{N,\mu} : (N,\mu) \in A\}$ and define the term order $\prec$ on the monomials of S as follows:

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s} \text{ if and only if} \tag{1.13}$$

- $d < s$ or
- $d = s$ and $\sum \mu_i > \sum \mu'_i$ or
- $d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i < \sum N'_i$
- $d = s$ and $\sum \mu_i = \sum \mu'_i$ and $\sum N_i = \sum N'_i$ and

$$\omega_{N_1,\mu_1}\omega_{N_2,\mu_2}\cdots\omega_{N_d,\mu_d} \prec_t \omega_{N'_1,\mu'_1}\omega_{N'_2,\mu'_2}\cdots\omega_{N'_s,\mu'_s}.$$

By evaluating $\sum_{i=0}^{E}\sum_{j=0}^{E-i} 1$ we can see that

$$\#\{(i,j) \in \mathbb{N}^2 : 0 \leqslant i+j \leqslant E\} = (E+1)(E+2)/2 \tag{1.14}$$

We will use the following lemma, for a proof see [18].

Lemma 1.5.0.1. Let J be the ideal generated by the elements G_1, G_2 and let I be the canonical ideal. Assume that the cannonical ideal is generated by elements of degree 2. If $\dim_L (S/\mathrm{in}_\prec(J))_2 \leqslant 3(g-1)$, then $I = J$.

We extend the correspondence between the variables $\omega_{i,j}$ and the points of $\mathbf{A}$ to a correspondence between monomials in S of standard degree 2 and points of the Minkowski sum of $\mathbf{A}$ with itself, defined as

$$\mathbf{A} + \mathbf{A} = \{(i+i', j+j') \mid (i,j), (i',j') \in \mathbf{A}\} \subseteq \mathbb{N}^2. \tag{1.15}$$

Proposition 1.5.0.2. Let $\mathbf{A}$ be the set of exponents of the basis of holomorphic differentials, and let $\mathbf{A}+\mathbf{A}$ denote the Minkowski sum of $\mathbf{A}$ with itself, as defined in (1.15). Then

$$(\rho, T) \in \mathbf{A}+\mathbf{A} \Leftrightarrow \exists\ \omega_{i,j}\omega_{i',j'} \in S \text{ such that } \mathrm{mdeg}(\omega_{i,j}\omega_{i',j'}) = (2,\rho,T).$$

For each $n \in \mathbb{N}$ we write $\mathbb{T}^n$ for the set of monomials of degree n in S and proceed with the characterization of monomials that do not appear as leading terms of binomials in $G_1 \subseteq J$.

Proposition 1.5.0.3. Let σ be the map of sets

$$\begin{array}{rcl} \sigma : \mathbf{A}+\mathbf{A} & \to & \mathbb{T}^2 \\ (\rho, T) & \mapsto & \min_\prec\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho,T) = (i+i', j+j')\} \end{array}$$

Then

$$\sigma(\mathbf{A}+\mathbf{A}) = \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \omega_{i,j}\cdot\omega_{i',j'} \neq \mathrm{in}_\prec(f),\ \forall\, f \in G_1\}$$

The above proposition gives a characterization of the monomials that do not appear as initial terms of elements of G_1, therefore they survive in the quotient $(S/\mathrm{in}_\prec(J))_2$. Indeed, the minimal of the set $\{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid (\rho,T) = (i+i', j+j')\}$ will never appear as the initial term of an element in G_1. Therefore $\mathbf{A}+\mathbf{A}$ is bijective with a basis of the vector space $(S/\mathrm{in}_\prec G_1)_2$. However, some of these monomials appear as initial terms of polynomials in G_2 and these have to be subtracted in order to compute $\dim_L (S/\mathrm{in}_\prec(J))_2$

Proposition 1.5.0.4. Let

$$C = \{(\rho, b) \in \mathbf{A} + \mathbf{A} \mid \rho = n + a, 0 \leqslant a + b \leqslant n - 6, a, b \in \mathbb{N}\}$$

Then

$$\sigma(C) \subseteq \{\omega_{i,j}\omega_{i',j'} \in \mathbb{T}^2 \mid \exists\, g \in G_2 \text{ such that } \omega_{i,j}\omega_{i',j'} = \mathrm{in}_{\prec}(g)\}$$

Moreover $\#C = \#\sigma(C) = (n-5)(n-4)/2$.

Proof. Observe that elements in G_2 are mapped into elements of the form $x^a y^b (x^n + y^n + 1)\omega^2 \in H^0(X, \Omega_X^{\otimes 2})$. By the form of the initial term of such an element of G_2 we have for $i_1 + i_2 = n + a = \rho, j_1 + j_2 = b$. Therefore

$$i_3 + i_4 = a = \rho - n, j_3 + j_4 = n + b, i_5 + i_6 = a = \rho - n, j_5 + j_6 = b = T$$

We should have $0 \leqslant a + b \leqslant n - 6$ and by eq. (1.14) we have that the cardinality of C equals $(n-5)(n-4)/2$. □

We now observe that

$$\mathbf{A} + \mathbf{A} \subset \{i, j \in \mathbb{N} : i + j \leqslant 2n - 6\}$$

so $\#(\mathbf{A} + \mathbf{A}) \leqslant (2n-5)(2n-4)/2$ and

$$\begin{aligned} \dim_L (S/\mathrm{in}_{\prec}(J))_2 &= \#\big((\mathbf{A} + \mathbf{A}) \setminus C\big) = \#(\mathbf{A} + \mathbf{A}) - \#C \\ &\leqslant \frac{(2n-5)(2n-4)}{2} - \frac{(n-5)(n-4)}{2} = 3(g-1). \end{aligned}$$

so by lemma 1.5.0.1 we have that $I = J$.

1.5a Automorphisms of the Fermat curve

The group of automorphisms of the Fermat curve is given by [77], [54]

$$G = \begin{cases} \mathrm{PGU}(3, p^h), & \text{if } n = 1 + p^h \\ (\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes S_3, & \text{otherwise} \end{cases}$$

The action of the automorphism group is given in terms of a 3×3 matrix A sending

$$x = (x_1/x_0) \mapsto \frac{\sum_{i=0}^2 a_{1,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i} \qquad y = (x_2/x_0) \mapsto \frac{\sum_{i=0}^2 a_{2,i}x_i}{\sum_{i=0}^2 a_{0,i}x_i},$$

In characteristic 0, the matrix A is a monomial matrix, that is, it has only one non-zero element in each row and column and this element is an n-th root of unity. Two matrices A_1, A_2 give rise to the same automorphism if and only if they differ by an element in the group $\{\lambda\mathbb{I}_3 : \lambda \in k\}$. In any case the group G is naturally a subgroup of $\mathrm{PGL}_3(k)$. Finding the representation matrix of G as an element in $\mathrm{PGL}_{g-1}(k)$ is easy when $n \neq 1 + p^h$ and more complicated in $n = 1 + p^h$ case. We have two different embeddings of the Fermat curve F_n in projective space

$$\mathbb{P}_k^{g-1} \longleftarrow F_n \longrightarrow \mathbb{P}_k^2.$$

In both cases the automorphism group is given as restriction of the automorphism group of the ambient space.

The computation of the automorphism group in terms of the vanishing of the polynomials given in equation (1.2) is quite complicated.

We have performed this computation in magma [12], and it turns out the automorphism group for the $n = 6$ case is described as an algebraic set described by $g^2 = 100$ variables and 756 equations.

Chapter 2

The canonical ideal and the deformation theory of curves with automorphisms

2.1 Introduction

The structure of the section is as follows. In section 2.2b we will present together the deformation theory of linear representations $\rho : G \to GL(V)$ and the deformation theory of representations of the form $\rho : G \to \mathrm{Aut}k[[t]]$. The deformation theory of linear representations is a better-understood object of study, see [56], which played an important role in topology [41] and also in the proof of Fermat's last theorem, see [57]. The deformation theory of representations in $\mathrm{Aut}k[[t]]$ comes out from the study of local fields and it is related to the deformation problem of curves with automorphisms after the local global theory of Bertin Mézard. There is also an increased interest related to the study of Nottingham groups and $\mathrm{Aut}k[[t]]$, see [17], [25],[52].

It seems that the similarities between these two deformation problems are known to the expert, see for example [64, prop. 3.13]. For the convenience of the reader and in order to fix the notation, we also give a detailed explanation and comparison of these two deformation problems.

In section 2.3 we revise the theory of relative canonical ideals and the work of H. Charalambous, K. Karagiannis and A. Kontogeorgis [18] aiming at the deformation problem of curves with automorphisms. More precisely a relative version of Petri's theorem is proved, which implies that the relative canonical ideal is generated by quadratic polynomials.

In section 2.4 we study both the obstruction and the tangent space problem of the deformation theory of curves with automorphisms using the relative canonical ideal point of view. In this section theorem 3 is proved.

2.2 Deformation theory of curves with automorphisms

2.2a Global deformation functor

Let Λ be a complete local Noetherian ring with residue field k, where k is an algebraically closed field of characteristic $p \geqslant 0$. Let $\mathscr{C}$ be the category of local Artin Λ-algebras with residue field k and homomorphisms the local Λ-algebra homomorphisms $\phi : \Gamma' \to \Gamma$ between them, that is $\phi^{-1}(\mathfrak{m}_\Gamma) = \mathfrak{m}_{\Gamma'}$. The deformation functor of curves with automorphisms is a functor D_{gl} from the category $\mathscr{C}$ to the category of sets

$$D_{gl} : \mathscr{C} \to \mathrm{Sets}, \Gamma \mapsto \left\{ \begin{array}{l} \text{Equivalence classes} \\ \text{of deformations of} \\ \text{couples } (X, G) \text{ over } \Gamma \end{array} \right\}$$

defined as follows. For a subgroup G of the group $\mathrm{Aut}(X)$, a deformation of the couple (X, G) over the local Artin ring Γ is a proper, smooth family of curves

$$X_\Gamma \to \mathrm{Spec}(\Gamma)$$

parametrized by the base scheme $\mathrm{Spec}(\Gamma)$, together with a group homomorphism $G \to \mathrm{Aut}_\Gamma(X_\Gamma)$, such that there is a G-equivariant isomorphism ϕ from the fibre over the closed point of Γ to the original curve X:

$$\phi : X_\Gamma \otimes_{\mathrm{Spec}(\Gamma)} \mathrm{Spec}(k) \to X.$$

Two deformations X^1_Γ, X^2_Γ are considered to be equivalent if there is a G-equivariant isomorphism ψ that reduces to the identity in the special fibre and making the following diagram commutative:

$$\begin{tikzcd} X^1_\Gamma \arrow[rr, "\psi"] \arrow[dr] & & X^2_\Gamma \arrow[dl] \\ & \mathrm{Spec}\Gamma & \end{tikzcd}$$

Given a small extension of Artin local rings

$$0 \to E \cdot k \to \Gamma' \to \Gamma \to 0 \tag{2.1}$$

and an element $x \in D_{gl}(\Gamma)$ we have that the set of lifts $x' \in D_{gl}(\Gamma')$ extending x is a principal homogeneous space under the action of $D_{gl}(k[\epsilon])$ and such an extension x' exists if certain obstruction vanishes. It is well known, see section 2.2b, that similar behavior have the deformation functors of representations.

2.2b Lifting of representations

Let $\mathscr{G} : \mathscr{C} \to \mathrm{Groups}$ be a group functor, see [24, ch. 2]. We will be mainly interested in two group functors. The first one, GL_g, will be represented by the by the group scheme $G_g = \Lambda[x_{11}, \ldots, x_{gg}, \det(x_{ij})^{-1}]$, that is $\mathrm{GL}_g(\Gamma) = \mathrm{Hom}_\Lambda(G_g, \Gamma)$. The second one is the group functor from the category of rings to the category of groups $\mathscr{N} : \Gamma \mapsto \mathrm{Aut}\Gamma[[t]]$.

We also assume that each group $\mathscr{G}(\Gamma)$ is embedded in the group of units of some ring $\mathscr{R}(\Gamma)$ depending functorially on Γ. This condition is asked since our argument requires us to be able to add certain group elements. We also assume that the additive group of the ring $\mathscr{R}(\Gamma)$ has the structure of direct product Γ^I, while $\mathscr{R}(\Gamma) = \mathscr{R}(\Lambda) \otimes_\Lambda \Gamma$. Notice, that I might be an infinite set, but since all rings involved are Noetherian Γ^I is flat, see [53, 4F].

A representation of the finite group G in $\mathscr{G}(\Gamma)$ is a group homomorphism

$$\rho : G \to \mathscr{G}(\Gamma),$$

where Γ is a commutative ring.

Remark 2.2.1. Consider two sets X, Y acted on by the group G. Then every function $f : X \to Y$ is acted on by G, by defining ${}^\sigma f : X \to Y$, sending $x \mapsto \sigma f \sigma^{-1}(x)$.

More precisely we will use the following actions

Definition 2.2.1.1. (i) Let $M_g(\Gamma)$ denote the set of $g \times g$ matrices with entries in ring Γ. An element $A \in M_g(\Gamma)$ will be acted on by $g \in G$ in terms of the action

$$T(g)A = \rho(g^{-1})^t A \rho(g^{-1}).$$

This is the natural action coming from the action of G on $H^0(X, \Omega_{X/k})$ and on the quadratic forms $\omega^t A \omega$. We raise the group element in -1 in order to have a left action, that is $T(gh)A = T(g)T(h)A$. Notice also that $T(g)$ restricts to an action on the space $\mathscr{S}_g(\Gamma)$ of symmetric $g \times g$ matrices with entries in Γ.

(ii) The adjoint action on elements $A \in M_g(\Gamma)$, comes from the action to the tangent space of the general linear group.

$$\mathrm{Ad}(g)A = \rho(g)A\rho(g^{-1}).$$

(iii) Actions on elements which can be seen as functions between G-spaces as in remark 2.2.1. This action will be denoted as $f \mapsto {}^\sigma f$.

Examples

1. Consider the groups $\mathrm{GL}_g(\Gamma)$ consisted of all invertible $g \times g$ matrices with coefficients in Γ. The group functor

$$\Gamma \mapsto \mathrm{GL}_g(\Gamma) = \mathrm{Hom}(R, \Gamma),$$

is representable by the affine Λ-algebra $R = k[x_{11}, \ldots, x_{gg}, \det\left((x_{ij})\right)^{-1}]$, see [73, 2.5]. In this case the ring $\mathscr{R}(\Gamma)$ is equal to $\mathrm{End}(\Gamma^g)$, while $I = \{i, j \in \mathbb{N} : 1 \leqslant i, j, \leqslant g\}$.

We can consider the subfunctor $\mathrm{GL}_{g,\mathbb{I}_g}$ consisted of all elements $f \in \mathrm{GL}_g(\Gamma)$, which reduce to the identity modulo the maximal ideal $\mathfrak{m}_\Gamma$. The tangent space $T_{\mathbb{I}_g}\mathrm{GL}_g$ of GL_g at the identity element $\mathbb{I}_g$, that is the space $\mathrm{Hom}(\mathrm{Spec}k[\epsilon], \mathrm{Spec}R)$ or equivalently the set $\mathrm{GL}_{g,\mathbb{I}_g}(k[\epsilon])$ consisted of $f \in \mathrm{Hom}(R, k[\epsilon])$, so that $f \equiv \mathbb{I}_g \bmod \langle \epsilon \rangle$. This set is a vector space according to the functorial construction given in [57, p. b 272] and can be identified to the space of $\mathrm{End}(k^g) = M_g(k)$, by identifying

$$\mathrm{Hom}(R, k[\epsilon]) \ni f \mapsto \mathbb{I}_g + \epsilon M, M \in M_g(k).$$

The later space is usually considered as the tangent space of the algebraic group $\mathrm{GL}_g(k)$ at the identity element or equivalently as the Lie algebra corresponding to $\mathrm{GL}_g(k)$.

The representation $\rho : G \to \mathrm{GL}_g(\Gamma)$ equips the space $T_{\mathbb{I}_g}\mathrm{GL}_g = M_g(k)$ with the adjoint action, which is the action described in remark 2.2.1, when the endomorphism M is seen as an operator $V \to V$, where V is a G-module in terms of the representation ρ:

$$\begin{aligned} G \times M_g(k) &\longrightarrow M_g(k) \\ (g, M) &\longmapsto \mathrm{Ad}(g)(M) = gMg^{-1}. \end{aligned}$$

In order to make clear the relation with the local case below, where the main object of study is the automorphism group of a completely local ring we might consider the completion $\hat{R}_{\mathbb{I}}$ of the localization of $R = k[x_{11}, \ldots, x_{gg}, \det\left((x_{ij})\right)^{-1}]$ at the identity element. We can now form the group $\mathrm{Aut}\hat{R}_{\mathbb{I}}$ of automorphisms of the ring $\hat{R}_{\mathbb{I}}$ which reduce to the identity modulo $\mathfrak{m}_{\hat{R}_{\mathbb{I}}}$. The later automorphism group is huge but it certainly contains the group G acting on $\hat{R}_{\mathbb{I}}$ in terms of the adjoint representation. We have that elements $\sigma \in \mathrm{Aut}\hat{R}_{\mathbb{I}} \otimes k[\epsilon]$ are of the form

$$\sigma(x_{ij}) = x_{ij} + \epsilon\beta(x_{ij}), \text{ where } \beta(x_{ij}) \in \hat{R}_{\mathbb{I}}.$$

Moreover, the relation

$$\sigma(f \cdot g) = fg + \epsilon\beta(fg) = (f + \epsilon\beta(f))(g + \epsilon\beta(f)),$$

implies that the map β is a derivation and

$$\beta(fg) = f\beta(g) + \beta(f)g.$$

Therefore, β is a linear combination of $\frac{\partial}{\partial x_{ij}}$, with coefficients in $\hat{R}_{\mathbb{I}}$, that is

$$\beta = \sum_{0 \leqq i,j \leqslant g} a_{i,j} \frac{\partial}{\partial x_{ij}}$$

Remark 2.2.2. In the literature of Lie groups and algebras, the matrix notation $M_g(k)$ for the tangent space is frequently used for the Lie algebra-tangent space at identity, instead of the later vector field-differential operator approach, while in the next example the differential operator notation for the tangent space is usually used.

2. Consider now the group functor $\Gamma \mapsto \mathscr{N}(\Gamma) = \mathrm{Aut}\Gamma[[t]]$. An element $\sigma \in \mathrm{Aut}\Gamma[[t]]$ is fully described by its action on t, which can be expressed as an element in $\Gamma[[t]]$. When Γ is an Artin local algebra then an automorphism is given by

$$\sigma(t) = \sum_{\nu=0}^{\infty} a_\nu t^\nu, \text{ where } a_i \in \Gamma, a_0 \in \mathfrak{m}_\Gamma \text{ and } a_1 \text{ is a unit in } \Gamma.$$

If a_1 is not a unit in Γ or $a_0 \notin \mathfrak{m}_\Gamma$ then σ is an endomorphism of $\Gamma[[t]]$. In this way $\mathrm{Aut}\Gamma[[t]]$ can be seen as the group of invertible elements in $\Gamma[[t]] = \mathrm{End}\Gamma[[t]] = \mathscr{R}(\Gamma)$. In this case, the set I is equal to the set of natural numbers, where Γ^I can be identified as the set of coefficients of each powerseries.

$$\mathrm{Aut}(k[\epsilon][[t]]) = \left\{ t \mapsto \sigma(t) = \sum_{\nu=1}^{\infty} a_i t^{\nu} : a_i = \alpha_i + \epsilon\beta_i,\ \alpha_i, \beta_i \in k, \alpha_1 \neq 0 \right\}$$

Exactly as we did in the general linear group case let as consider the subfunctor $\Gamma \mapsto \mathscr{N}_{\mathbb{I}}(\Gamma)$, where $\mathscr{N}_{\mathbb{I}}(\Gamma)$ consists of all elements in $\mathrm{Aut}\Gamma[[t]]$ which reduce to the identity mod $\mathfrak{m}_{\Gamma}$.

Such an element $\sigma \in \mathscr{N}_{\mathbb{I}}(k[\epsilon])$ transforms $f \in k[[t]]$ to a formal powerseries of the form

$$\sigma(f) = f + \epsilon F_{\sigma}(f),$$

where $F_{\sigma}(f)$ is fully determined by the value of $\sigma(t)$. The multiplication condition $\sigma(f_1 f_2) = \sigma(f_1)\sigma(f_2)$ implies that

$$F_{\sigma}(f_1 f_2) = f_1 F_{\sigma}(f_2) + F_{\sigma}(f_1) f_2,$$

that is F_{σ} is a $k[[t]]$-derivation, hence an element in $k[[t]]\frac{d}{dt}$.

The local tangent space of $\Gamma[[t]]$ is defined to be the space of differential operators $f(t)\frac{d}{dt}$, see [10], [22], [49]. The G action on the element $\frac{d}{dt}$ is given by the adjoint action, which is given as a composition of operators, and is again compatible with the action given in remark 2.2.1:

$$\begin{array}{ccccccc}
\Gamma[[t]] & \xrightarrow{\rho(\sigma^{-1})} & \Gamma[[t]] & \xrightarrow{\frac{d}{dt}} & \Gamma[[t]] & \xrightarrow{\rho(\sigma)} & \Gamma[[t]] \\
t & \longmapsto & \rho(\sigma^{-1})(t) & \longmapsto & \frac{d\rho(\sigma^{-1})(t)}{dt} & \longmapsto & \rho(\sigma)\left(\frac{d\rho(\sigma^{-1})(t)}{dt}\right)
\end{array}$$

So the G-action on the local tangent space $k[[t]]\frac{d}{dt}$ is given by

$$f(t)\frac{d}{dt} \longmapsto \mathrm{Ad}(\sigma)\left(f(t)\frac{d}{dt}\right) = \rho(\sigma)(f(t)) \cdot \rho(\sigma)\left(\frac{d\rho(\sigma^{-1})(t)}{dt}\right)\frac{d}{dt},$$

see also [49, lemma 1.10], for a special case.

$\mathscr{G}(\Gamma)$	$\mathscr{R}(\Gamma)$	tangent space	action
$\mathrm{GL}_g(\Gamma)$	$\mathrm{End}_g(\Gamma)$	$\mathrm{End}_g(k) = M_g(k)$	$M \mapsto \mathrm{Ad}(\sigma)(M)$
$\mathrm{Aut}\Gamma[[t]]$	$\mathrm{End}(\Gamma[[t]])$	$k[[t]]\frac{d}{dt}$	$f(t)\frac{d}{dt} \longmapsto \mathrm{Ad}(\sigma)\left(f(t)\frac{d}{dt}\right)$

Table 2.1: Comparing the two group functors

Motivated by the above two examples we can define

Definition 2.2.2.1. Let $\mathscr{G}_{\mathbb{I}}$ be the subfunctor of $\mathscr{G}$, defined by

$$\mathscr{G}_{\mathbb{I}}(\Gamma) = \{f \in \mathscr{G}(\Gamma) : f = \mathbb{I} \bmod \mathfrak{m}_{\Gamma}\}.$$

The tangent space to the functor $\mathscr{G}$ at the identity element is defined as $\mathscr{G}_{\mathbb{I}}(k[\epsilon])$, see [57]. Notice, that $\mathscr{G}_{\mathbb{I}}(k[\epsilon]) \cong \mathscr{R}(k)$, is k-vector space, acted on in terms of the adjoint representation, given by

$$\begin{aligned} G \times \mathscr{G}_{\mathbb{I}}(\Gamma) &\longrightarrow \mathscr{G}_{\mathbb{I}}(\Gamma) \\ (\sigma, f) &\longmapsto \rho(\sigma) \cdot f \cdot \rho(\sigma)^{-1}. \end{aligned}$$

If $\mathscr{R}(\Gamma)$ can be interpreted as an endomorphism ring, then the above action can be interpreted in terms of the action on functions as described in remark 2.2.1.

We will define the tangent space in our setting as $\mathscr{T} = \mathscr{R}(k)$, which is equipped with the adjoint action.

2.2c Deforming representations

We can now define the deformation functor F_ρ for any local Artin algebra Γ with maximal ideal $\mathfrak{m}_\Gamma$ in $\mathscr{C}$ to the category of sets:

$$F_\rho : \Gamma \in \mathrm{Ob}(\mathscr{C}) \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathscr{G}(k) \\ \text{to } \rho_\Gamma : G \to \mathscr{G}(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathscr{G}(\Gamma) \to \mathscr{G}(k)) \end{array} \right\} \tag{2.2}$$

Let

$$0 \longrightarrow \langle E \rangle = E \cdot \Gamma' = E \cdot k \xrightarrow[\phi']{} \Gamma' \underset{\phi}{\overset{i}{\rightleftarrows}} \Gamma \longrightarrow 0 \tag{2.3}$$

be a small extension in $\mathscr{C}$, that is the kernel of the natural onto map ϕ is a principal ideal, generated by E and $E \cdot \mathfrak{m}_{\Gamma'} = 0$. In the above diagram $i : \Gamma \to \Gamma'$ is a section, which is not necessarily a homomorphism. Since the kernel of ϕ is a principal ideal $E \cdot \Gamma'$ annihilated by $\mathfrak{m}_{\Gamma'}$ it is naturally a $k = \Gamma'/\mathfrak{m}_{\Gamma'}$-vector space, which is one dimensional.

Lemma 2.2.2.1. For a small extension as given in eq. (2.3) consider two liftings $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ of the representation ρ_Γ. The map

$$\begin{aligned} d : G &\longrightarrow \mathscr{T} := \mathscr{R}(k) \\ \sigma &\longmapsto d(\sigma) = \frac{\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_{\Gamma'}}{E} \end{aligned}$$

is a cocycle.

Proof. We begin by observing that $\phi\left(\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_{\Gamma'}\right) = 0$, hence

$$\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} = \mathbb{I}_{\Gamma'} + E \cdot d(\sigma), \text{ where } d(\sigma) \in \mathscr{T}.$$

Also, we compute that

$$\begin{aligned} \mathbb{I}_{\Gamma'} + E \cdot d(\sigma\tau) &= \rho^1_{\Gamma'}(\sigma\tau)\rho^2_{\Gamma'}(\sigma\tau)^{-1} \\ &= \rho^1_{\Gamma'}(\sigma)\rho^1_{\Gamma'}(\tau)\rho^2_{\Gamma'}(\tau)^{-1}\rho^2_{\Gamma'}(\sigma)^{-1} \\ &= \rho^1_{\Gamma'}(\tau)\big(\mathbb{I}_{\Gamma'} + Ed(\sigma)\big)\rho^2_{\Gamma'}\tau)\big)^{-1} \\ &= \rho^1_{\Gamma'}(\tau)\rho^2_{\Gamma'}(\tau)^{-1} + E \cdot \rho^1_{\Gamma'}(\tau)d(\sigma)\rho^2_{\Gamma'}(\tau)^{-1} \\ &= \mathbb{I}_{\Gamma'} + E \cdot d(\tau) + E \cdot \rho_k(\tau)d(\sigma)\rho_k(\tau)^{-1}, \end{aligned}$$

since E annihilates $\mathfrak{m}_{\Gamma'}$, so the values of both $\rho^1_{\Gamma'}(\tau))$ and $\rho^2_{\Gamma'}(\tau)$ when multiplied by E are reduced modulo the maximal ideal $\mathfrak{m}_{\Gamma'}$. We, therefore, conclude that

$$d(\sigma\tau) = d(\tau) + \rho_k(\tau)d(\sigma)\rho_k(\tau)^{-1} = d(\tau) + \mathrm{Ad}(\tau)d(\sigma).$$

□

Similarly if $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ are equivalent extensions of ρ_Γ, that is

$$\rho^1_{\Gamma'}(\sigma) = \big(\mathbb{I}_{\Gamma'} + EQ\big)\rho^2_{\Gamma'}(\sigma)\big(\mathbb{I}_{\Gamma'} + EQ\big)^{-1},$$

then

$$d(\sigma) = Q - \mathrm{Ad}(\sigma)Q,$$

that is $d(\sigma)$ is a coboundary. This proves that the set of liftings $\rho_{\Gamma'}$ of a representation $\rho_{\Gamma'}$ is a principal homogeneous space, provided it is non-empty.

The obstruction to the lifting can be computed by considering a naive lift $\rho_{\Gamma'}$ of ρ_Γ (that is we don't assume that $\rho_{\Gamma'}$ is a representation) and by considering the element

$$\phi(\sigma, \tau) = \rho_{\Gamma'}(\sigma) \circ \rho_{\Gamma'}(\tau) \circ \rho_{\Gamma'}(\sigma\tau)^{-1}, \qquad \text{for } \sigma, \tau \in G$$

which defines a cohomology class as an element in $H^2(G, \mathscr{T})$. Two naive liftings $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ give rise to cohomologous elements ϕ^1, ϕ^2 if their difference $\rho^1_{\Gamma'} - \rho^2_{\Gamma'}$ reduce to zero in Γ'. If this class is zero, then the representation ρ_Γ can be lifted to Γ'.

Examples Notice that in the theory of deformations of representations of the general linear group, this is a classical result, see [57, prop. 1], [56, p.30] while for deformations of representations in $\mathrm{Aut}\Gamma[[t]]$, this is in [22],[10].

The functors in these cases are given by

$$F : \mathrm{Ob}(\mathscr{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{liftings of } \rho : G \to \mathrm{GL}_n(k) \\ \text{to } \rho_\Gamma : G \to \mathrm{GL}_n(\Gamma) \text{ modulo} \\ \text{conjugation by an element} \\ \text{of } \ker(\mathrm{GL}_n(\Gamma) \to \mathrm{GL}_n(k)) \end{array} \right\} \tag{2.4}$$

$$D_P : \mathrm{Ob}(\mathscr{C}) \ni \Gamma \mapsto \left\{ \begin{array}{l} \text{lifts } G \to \mathrm{Aut}(\Gamma[[t]]) \text{ of } \rho \text{ mod-} \\ \text{ulo conjugation with an element} \\ \text{of } \ker(\mathrm{Aut}\Gamma[[t]] \to \mathrm{Aut}k[[t]]) \end{array} \right\} \tag{2.5}$$

Let V be the n-dimensional vector space k, and let $\mathrm{End}_A(V)$ be the Lie algebra corresponding to the algebraic group $\mathrm{GL}(V)$. The space $\mathrm{End}_A(V)$ is equipped with the adjoint action of G given by:

$$\begin{aligned} \mathrm{End}_A(V) &\to \mathrm{End}_A(V) \\ e &\mapsto (g \cdot e)(v) = \rho(g)(e(\rho(g)^{-1})(v)) \end{aligned}$$

The tangent space of this deformation functor equals to

$$F(k[\epsilon]) = H^1(G, \mathrm{End}_A(V)),$$

where the later cohomology group is the group cohomology group and $\mathrm{End}_A(V)$ is considered as a G-module with the adjoint action.

More precisely, if

$$0 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 0$$

is a small extension of local Artin algebras then we consider the diagram of small extensions

$$\begin{tikzcd} & \mathrm{GL}_n(\Gamma') \arrow[d, "\phi"] \\ G \arrow[ur, "{\rho^1_{\Gamma'}, \rho^2_{\Gamma'}}"] \arrow[r, "\rho_\Gamma"'] & \mathrm{GL}_n(\Gamma) \end{tikzcd}$$

where $\rho^1_{\Gamma'}, \rho^2_{\Gamma'}$ are two liftings of ρ_Γ in Γ'.

We have the element

$$d(\sigma) := \frac{1}{E}\left(\rho^1_{\Gamma'}(\sigma)\rho^2_{\Gamma'}(\sigma)^{-1} - \mathbb{I}_n\right) \in H^1(G, \mathrm{End}_n(k)).$$

To a naive lift $\rho_{\Gamma'}$ of ρ_Γ we can attach the 2-cocycle $\alpha(\sigma,\tau) = \rho_{\Gamma'}(\sigma)\rho_{\Gamma'}(\tau)\rho_{\Gamma'}(\sigma\tau)^{-1}$ defining a cohomology class in $H^2(G, \mathrm{End}_n(k))$.

The following proposition shows us that a lifting is not always possible.

Proposition 2.2.2.1. Let k be an algebraically closed field of positive characteristic $p > 0$, and let $R = W(k)[\zeta_q]$ be the Witt ring of k with a primitive $q = p^h$ root adjoined. Consider the group $G = C_q \rtimes C_m$, where C_m and C_q are cyclic groups of orders m and q respectively and $(m,p) = 1$. Assume that σ and τ are generators for C_m and C_q respectively and moreover

$$\sigma\tau\sigma^{-1} = \tau^a$$

for some integer a (which should satisfy $a^m \equiv 1 \bmod q$.) There is a linear representation $\rho : G \to \mathrm{GL}_2(k)$, which can not be lifted to a representation $\rho_R : G \to \mathrm{GL}_2(R)$.

Proof. Consider the field $\mathbb{F}_p \subset k$ and let λ be a generator of the cyclic group $\mathbb{F}_p^*$. The matrices

$$\sigma = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfy

$$\sigma^{p-1} = 1, \tau^q = 1, \sigma\tau\sigma^{-1} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \sigma^a$$

and generate a subgroup of $\mathrm{GL}_2(k)$, isomorphic to $C_q \rtimes C_m$ for $m = p-1$, giving a natural representation $\rho : G \to \mathrm{GL}_2(\bar{\mathbb{F}}_p) \subset \mathrm{GL}_2(k)$.

Suppose that there is a faithful representation $\tilde{\rho} : G \to \mathrm{GL}_n(R)$ which gives a faithful representation of $\tilde{\rho} : G \to \mathrm{GL}_n(\mathrm{Quot}(R))$. Since $\tilde{\rho}(\tau)$ is of finite order after a $\mathrm{Quot}(R)$ linear change of basis we might assume that $\tilde{\rho}(\tau)$ is diagonal with q-roots of unity in the diagonal (we have considered $R = W(k)[\zeta]$ so that the necessary diagonal elements exist in $\mathrm{Quot}(R)$). We have

$$\tilde{\rho}(\tau) = \mathrm{diag}(\lambda_1, \ldots, \lambda_n).$$

At least one of the diagonal elements say $\lambda = \lambda_{i_0}$ in the above expression is a primitive q-th root of unity. Let E be an eigenvector, that is

$$\tilde{\rho}(\tau)E = \lambda E.$$

The equality $\tau\sigma = \sigma\tau^a$ implies that σE is an eigenvector of the eigenvalue λ^a. This means that n should be greater than the order of $a \bmod q$ since we have at least as many different (and linearly independent) eigenvectors as the different values $\lambda, \lambda^a, \lambda^{a^2}, \ldots$.

Since, for large prime ($p > 3$) we have $2 = n < p - 1$ the representation ρ can not be lifted to R. $\square$

Local Actions By the local-global theorems of J.Bertin and A. Mézard [10] and the formal patching theorems of D. Harbater, K. Stevenson [33], [34], the study of the functor D_{gl} can be reduced to the study of the deformation functors D_P attached to each wild ramification point P of the cover $X \to X/G$, as defined in eq. (2.5). The theory of automorphisms of formal powerseries rings is not as well understood as is the theory of automorphisms of finite dimensional vector spaces, i.e. the theory of general linear groups.

As in the theory of liftings for the general linear group, we consider small extensions

$$1 \to \langle E \rangle \to \Gamma' \xrightarrow{\phi} \Gamma \to 1$$

An automorphism $\rho^\Gamma(\sigma) \in \mathrm{Aut}\Gamma[[t]]$ is completely described by a powerseries

$$\rho^\Gamma(\sigma)(t) = f_\sigma = \sum_{\nu=1}^{\infty} a_\nu^\Gamma(\sigma) t^\nu,$$

where $a_\nu^\Gamma(\sigma) \in \Gamma$. Given a naive lift

$$\rho^{\Gamma'}(\sigma)(t) = \sum_{\nu=1}^{\infty} a_\nu^{\Gamma'}(\sigma) t^\nu,$$

where $a_\nu^{\Gamma'}(\sigma) \in \Gamma'$ we can again form a two cocycle

$$\alpha(\sigma, \tau) = \rho^{\Gamma'}(\sigma) \circ \rho^{\Gamma'}(\tau) \circ \rho^{\Gamma'}(\sigma\tau)^{-1}(t),$$

defining a cohomology class in $H^2(G, \mathscr{T}_{k[[t]]})$. The naive lift $\rho^{\Gamma'}(\sigma)$ is an element of $\mathrm{Aut}\Gamma'[[t]]$ if and only if α is cohomologous to zero.

Suppose now that $\rho_1^{\Gamma'}, \rho_2^{\Gamma'}$ are two lifts in $\mathrm{Aut}\Gamma'[[t]]$. We can now define

$$d(\sigma) := \frac{1}{t}\left(\rho_1^{\Gamma'}(\sigma)\rho_2^{\Gamma'}(\sigma)^{-1} - \mathrm{Id}\right) \in H^1(G, \mathscr{T}_{k[[t]]}).$$

2.3 Relative Petri's theorem.

Recall that a functor $F : \mathscr{C} \to \mathrm{Sets}$ can be extended to a functor $\hat{F} : \hat{\mathscr{C}} \to \mathrm{Sets}$ by letting for every $R \in \mathrm{Ob}(\hat{\mathscr{C}})$, $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}_R^{n+1})$. An element $\hat{u} \in \hat{F}(R)$ is called a formal element, and by definition it can be represented as a system of elements $\{u_n \in F(R/\mathfrak{m}_R^{n+1})\}_{n \geqslant 0}$, such that for each $n \geqslant 1$, the map $F(R/\mathfrak{m}_R^{n+1}) \to F(R/\mathfrak{m}_R^n)$ induced by $R/\mathfrak{m}_R^{n+1} \to R/\mathfrak{m}_R^n$ sends $u_n \mapsto u_{n-1}$. For $R \in \mathrm{Ob}(\hat{\mathscr{C}})$ and a formal element $\hat{u} \in \hat{F}(R)$, the couple $(R, \hat{u})$ is called a formal couple. It is known that there is a 1-1 correspondence between $\hat{F}(R)$ and the set of morphisms of functors $h_R := \mathrm{Hom}_{\hat{\mathscr{C}}}(R, -) \to F$, see [70, lemma 2.2.2.]. The formal element $\hat{u} \in \hat{F}(R)$ will be called versal if the corresponding morphism $h_R \to F$ is smooth. For the definition of a smooth map between functors, see [70, def. 2.2.4]. The ring R will be called versal deformation ring.

Schlessinger [68, 3.7] proved that the deformation functor D for curves without automorphisms, admits a ring R as versal deformation ring. Schlessinger calls the versal deformation ring the hull of the deformation functor. Indeed, since there are no obstructions to liftings in small extensions for curves, see [68, rem. 2.10] the hull R of D_{gl} is a powerseries ring over Λ, which can be taken as an algebraic extension of $W(k)$. Moreover $R = \Lambda[[x_1, \ldots, x_{3g-3}]]$, as we can see by applying [9, cor. 3.3.5], when G is the trivial subgroup of the automorphism group. In this case the quotient map $f : X \to \Sigma = X/\{\mathrm{Id}\} = X$ is the identity. Indeed, for the equivariant deformation functor, in the case of the trivial group, there are no ramified points and the short exact sequence in eq. (1) reduces to an isomorphism of the first two spaces. We have $\dim_k H^1(X/G, \pi_*^G(\mathscr{T}_X)) = \dim_k H^1(X, \mathscr{T}_X) = 3g - 3$. The deformation $\mathscr{X} \to \mathrm{Specf} R$ can be extended to a deformation $\mathscr{X} \to \mathrm{Spec} R$ by Grothendieck's effectivity theorem, see [70, th. 2.5.13], [32].

The versal element $\hat{u}$ corresponds to a deformation $\mathscr{X} \to \mathrm{Spec}R$, with generic fibre $\mathscr{X}_\eta$ and special fibre $\mathscr{X}_0$. The couple $(R, \hat{u})$ is called the versal [70, def. 2.2.6] element of the deformation functor D of curves (without automorphisms). Moreover, the element u defines a map $h_{R/\Lambda} \to D$, which by definition of the hull is smooth, so every deformation $X_A \to \mathrm{Spec} A$ defines a homomorphism $R \to A$, which allows us to see A as an R-algebra.

Indeed, for the Artin algebra $A \to A/\mathfrak{m}_A = k$ we consider the diagram

$$h_{R/\Lambda} = \mathrm{Hom}_{\hat{\mathscr{C}}}(R, A) \to h_{R/\Lambda}(k) \times_{D(k)} D(A)$$

This section aims to prove the following

Proposition 2.3.0.1. Let $f_1, \ldots, f_r \in k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.

For $n \geqslant 1$, we write $\Omega_{\mathscr{X}/R}^{\otimes n}$ for the sheaf of holomorphic polydifferentials on $\mathscr{X}$. By [36, lemma II.8.9] the R–modules $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ are free of rank $d_{n,g}$ for all $n \geqslant 1$, with $d_{n,g}$ given by eq. (2.6)

$$d_{n,g} = \begin{cases} g, & \text{if } n = 1 \\ (2n-1)(g-1), & \text{if } n > 1. \end{cases} \tag{2.6}$$

Indeed, by a standard argument using Nakayama's lemma, see [36, lemma II.8.9],[43] we have that the R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ is free. Notice that to use Nakayama's lemma we need the deformation over R to have both a special and generic fibre and this was the reason we needed to consider a deformation over the spectrum of R instead of the formal spectrum.

Lemma 2.3.0.1. For every Artin algebra A the A-module $H^0(X_A, \Omega_{X_A/A}^{\otimes n})$ is free.

Proof. This follows since $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ is a free R-module and [36, prop. II.8.10], which asserts that

$\Omega_{X_A/A} \cong g'^*(\Omega_{\mathscr{X}/R})$, where g' is shown in the next commutative diagram:

$$\begin{tikzcd}
X_A = \mathscr{X} \times_{\mathrm{Spec}R} \mathrm{Spec}A \arrow[r, "g'"] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}A \arrow[r] & \mathrm{Spec}R
\end{tikzcd}$$

We have by definition of the pullback

$$g'^*(\Omega_{\mathscr{X}/R})(X_A) = (g')^{-1}\Omega_{\mathscr{X}/R}(X_A) \otimes_{(g')^{-1}\mathscr{O}_{\mathscr{X}}(X_A)} \mathscr{O}_{X_A}(X_A) \tag{2.7}$$

and by definition of the fiber product $\mathscr{O}_{X_A} = \mathscr{O}_{\mathscr{X}} \otimes_R A$. Observe also that since A is a local Artin algebra the schemes X_A and $\mathscr{X}$ share the same underlying topological space so

$$g'^{-1}(\Omega_{\mathscr{X}/R}(X_A)) = \Omega_{\mathscr{X}/R}(\mathscr{X})$$

and $g'^{-1}\mathscr{O}_{\mathscr{X}}(X_A) = \mathscr{O}_{\mathscr{X}}(\mathscr{X})$. So eq. (2.7) becomes

$$\begin{aligned}
H^0(X_A, \Omega_{X_A/A}) &= \Omega_{X_A/A}(X_A) = g'^*(\Omega_{\mathscr{X}/R})(X_A)) = \\
&= \Omega_{\mathscr{X}/R}(\mathscr{X}) \otimes_{\mathscr{O}_{\mathscr{X}}(\mathscr{X})} \otimes \mathscr{O}_{\mathscr{X}}(\mathscr{X}) \otimes_{R_{gl}} A \\
&= H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}) \otimes_R A.
\end{aligned}$$

So $H^0(X_A, \Omega_{X_A/A})$ is a free A-module of the same rank as $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$.

The proof for $H^0(X_A, \Omega_{X_A/A}^{\otimes n})$ follows in the same way. □

We select generators $W_1, \dots, W_g$ for the symmetric algebra

$$\mathrm{Sym}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})) = R[W_1, \dots, W_g].$$

Similarly, we write

$$\mathrm{Sym}(H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L})) = L[\omega_1, \dots, \omega_g] \text{ and } \mathrm{Sym}(H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k})) = k[w_1, \dots, w_g],$$

where

$$\omega_i = W_i \otimes_R L \qquad w_i = W_i \otimes_R k \text{ for all } 1 \leqslant i \leqslant g.$$

We have the following diagram relating special and generic fibres.

$$\begin{tikzcd}
\mathrm{Spec}(k) \times_{\mathrm{Spec}(R)} \mathscr{X} = \mathscr{X}_0 \arrow[r, hook] \arrow[d] & \mathscr{X} \arrow[d] & \mathscr{X}_\eta = \mathrm{Spec}(L) \times_{\mathrm{Spec}(R)} \mathscr{X} \arrow[l, hook'] \arrow[d] \\
\mathrm{Spec}(k) \arrow[r, hook] & \mathrm{Spec}(R) & \mathrm{Spec}(L) \arrow[l, hook']
\end{tikzcd} \tag{2.8}$$

Our work is based on the following relative version of Petri's theorem.

Theorem 2.3.1. Diagram (2.8) induces a deformation-theoretic diagram of canonical embeddings

$$\begin{tikzcd}
0 \arrow[r] & I_{\mathscr{X}_\eta} \arrow[r, hook] & S_L := L[\omega_1, \dots, \omega_g] \arrow[r, two heads, "\phi_\eta"] & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_\eta, \Omega_{\mathscr{X}_\eta/L}^{\otimes n}) \arrow[r] & 0 \\
0 \arrow[r] & I_{\mathscr{X}} \arrow[r, hook] \arrow[u, hook, "\otimes_R L"'] \arrow[d, "\otimes_R R/m"] & S_R := R[W_1, \dots, W_g] \arrow[r, two heads, "\phi"] \arrow[u, hook, "\otimes_R L"'] \arrow[d, "\otimes_R R/m"] & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n}) \arrow[r] \arrow[u, hook, "\otimes_R L"'] \arrow[d, "\otimes_R R/m"] & 0 \\
0 \arrow[r] & I_{\mathscr{X}_0} \arrow[r, hook] & S_k := k[w_1, \dots, w_g] \arrow[r, two heads, "\phi_0"] & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n}) \arrow[r] & 0
\end{tikzcd} \tag{2.9}$$

where $I_{\mathscr{X}} = \ker \phi_\eta$, $I_{\mathscr{X}} = \ker \phi$, $I_{\mathscr{X}_0} = \ker \phi_0$, each row is exact and each square is commutative.

Moreover, the ideal $I_{\mathscr{X}}$ can be generated by elements of degree 2 as an ideal of S_R.

The commutativity of the above diagram was proved in [18] by H. Charalambous, K. Karagiannis and A. Kontogeorgis. For proving that $I_{\mathscr{X}}$ is generated by elements of degree 2 as in the special and generic fibers we argue as follows: Since L is a field it follows by Petri's Theorem, that there are elements $\tilde{f_1}, \ldots, \tilde{f_r} \in S_L$ of degree 2 such that

$$I_{\mathscr{X}_\eta} = \langle \tilde{f_1}, \ldots, \tilde{f_r} \rangle.$$

Now we choose an element $c \in R$ such that $f_i := c\tilde{f_i} \in S_R$ for all i and notice that $\deg(f_i) = \deg(\tilde{f_i}) = 2$.
• Assume first that the element $c \in R$ is invertible in R. Consider the ideal $I = \langle f_1, \ldots, f_r \rangle$ of S_R. We will prove that $I = I_{\mathscr{X}}$. Consider the multiplicative system R^*. We will prove first $I \subset I_{\mathscr{X}} = \ker\phi$. Indeed, using the commuting upper square every element $a = \sum_{\nu=1}^r a_i f_i \in I$ maps to $\sum_{\nu=1}^r a_i f_i \otimes_R 1$ which in turn maps to 0 by ϕ_η. The same element maps to $\phi(a)$ and $\phi(a) \otimes_R 1$ should be zero. Since all modules $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes n})$ are free $\phi(a) = 0$ and $a \in I_{\mathscr{X}}$.

Since the family $\mathscr{X} \to \mathrm{Spec} R$ is flat we have that $I_{\mathscr{X}} \otimes_R L = I_{\mathscr{X}_\eta}$, that is we apply the $\otimes_R L$ functor on the middle short exact sequence of eq. (2.9). The ideal $I = I_{\mathscr{X}_\eta} \cap S_R = (I_{\mathscr{X}} \otimes_R L) \cap S_R$. By [6, prop. 3.11ii] this gives that

$$I = \cup_{s \in R^*} (I_{\mathscr{X}} : s) \supset I_{\mathscr{X}},$$

so $I_{\mathscr{X}} = I$. In the above formula $(I_{\mathscr{X}} : s) = \{x \in S_R : xs \in I_{\mathscr{X}}\}$.
• From now on we don't assume that the element c is an invertible element of R.

Let $\bar{g}$ be an element of degree 2 in $I_{\mathscr{X}_0}$, we will prove that we can select an element $g \in I_{\mathscr{X}}$ such that $g \otimes 1_k = \bar{g}$, so that g has degree 2.

Let us choose a lift $\tilde{g} \in S_R$ of degree 2 by lifting each coefficient of g from k to R. This element is not necessarily in $I_{\mathscr{X}}$. We have $\phi(g) \otimes 1_k = \phi_0(g \otimes 1_k) = \phi_0(\bar{g}) = 0$. Let $\bar{e}_1, \ldots, \bar{e}_{3g-3}$ be generators of the free R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}^{\otimes 2})$ and choose $e_1, \ldots, e_{3g-3} \in S_R$ such that $\phi(e_i) = \bar{e}_i$. Let us write $\phi(\tilde{g}) = \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i$, with $\lambda_i \in R$. Since $\phi_0(\bar{g}) = 0$ we have that all $\lambda_i \in \mathfrak{m}_R$ for all $1 \leqslant i \leqslant 3g-3$. This means that the element $g = \tilde{g} - \sum_{i=1}^{3g-3} \lambda_i e_i \in S_R$ reduces to $\bar{g}$ modulo $\mathfrak{m}_R$ and also $\phi(g) = \phi(\tilde{g}) - \sum_{i=1}^{3g-3} \lambda_i \bar{e}_i = 0$, so $g \in I_{\mathscr{X}}$.

Let $\bar{g}_1, \ldots, \bar{g}_s \in I_{\mathscr{X}_0}$ be elements of degree 2 such that

$$I_{\mathscr{X}_0} = \langle \bar{g}_1, \ldots, \bar{g}_s \rangle$$

and, using the previous construction, we take g_i lifts in $I_{\mathscr{X}} \lhd S_R$, i.e. such that $g_i \otimes 1_k = \bar{g}_i$ and also assume that the elements g_i have also degree 2.

We will now prove that the elements $g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L \in S_L$ generate the ideal $I_{\mathscr{X}_\eta}$. By the commutativity of the diagram in eq. (2.9) we have $\langle g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L \rangle \subset I_{\mathscr{X}_\eta} = \ker \phi_\eta$. Observe that any linear relation

$$\sum_{\nu=1}^{s} (a_\nu g_\nu \otimes_{S_R} 1_L) = 0, \text{ with } a_\nu \in L$$

gives rise to a relation for some $c \in R$

$$\sum_{\nu=1}^{s} c \cdot a_\nu g_\nu = 0, \qquad c \cdot a_\nu \in S_R,$$

which implies that $c \cdot a_\nu \in \mathfrak{m}_R$.

We will prove that the elements $g_i \otimes_{S_R} 1_L$ are linear independent.

Lemma 2.3.1.1. Let $\bar{v}_1, \ldots, \bar{v}_n \in k^m$ be linear independent elements and $v_1, \ldots, v_n$ be lifts in R^m. Then

$$\sum_{\nu=1}^{n} a_\nu v_\nu = 0 \qquad a_\nu \in R,$$

implies that $a_1 = \cdots = a_n = 0$.

Proof. We have $n \leqslant m$. We write the elements $v_1, \ldots, v_n$ (resp. $\bar{v}_1, \ldots, \bar{v}_n$) as columns and in this way we obtain an $m \times n$ matrix J (resp. $\bar{J}$). Since the elements are linear independent in k^m there is an $n \times n$ minor matrix with an invertible determinant. Without loss of generality, we assume that there is an $n \times n$ invertible matrix $\bar{Q}$ with coefficients in k such that $\bar{Q} \cdot \bar{J}^t = \left(\begin{array}{c|c} \mathbb{I}_n & \bar{A} \end{array} \right)$, where $\bar{A}$ is an $(m-n) \times n$ matrix. We now get lifts Q, J and A of $\bar{Q}, \bar{J}$ and $\bar{A}$ respectively, with coefficients in R, i.e.

$$Q \cdot J^t \equiv \left(\begin{array}{c|c} \mathbb{I}_n & A \end{array} \right) \mathrm{mod} \mathfrak{m}_R.$$

The columns of J are lifts of the elements $\bar{v}_1, \ldots, \bar{v}_n$. It follows that $Q \cdot J^t = \left(\begin{array}{c|c} \mathbb{I}_n & A \end{array} \right) + \left(\begin{array}{c|c} C & D \end{array} \right)$, where C, D are matrices with entries in $\mathfrak{m}_R$. The determinant of $\mathbb{I}_n + C$ is $1 + m$, for some element $m \in \mathfrak{m}_R$, and this is an invertible element in the local ring R. Similarly, the matrix Q is invertible. Therefore,

$$J^t = \left(\begin{array}{c|c} Q^{-1}(\mathbb{I}_n + C) & Q^{-1}(A + D) \end{array} \right)$$

has the first $n \times n$ block matrix invertible and the desired result follows.

□

Remark 2.3.2. It is clear that over a ring where 2 is invertible, there is an 1-1 correspondence between symmetric $g \times g$ matrices and quadratic polynomials. Indeed, a quadratic polynomial can be written as

$$f(w_1, \ldots, w_g) = \sum_{1 \leqslant i,j \leqslant g} a_{ij} w_i w_j = w^t A w,$$

where $A = (a_{ij})$. Even if the matrix A is not symmetric, the matrix $(A + A^t)/2$ is and generates the same quadratic polynomial

$$w^t A w = w^t \left(\frac{A + A^t}{2} \right) w.$$

Notice that the map

$$A \mapsto \frac{A + A^t}{2}$$

is onto the space of symmetric matrices and has as kernel the space of antisymmetric matrices.

A minimal set of quadratic generators is given by a set of polynomials $f_1, \ldots, f_r$, with $f_i = w^t A_i w$, where the symmetric polynomials are linearly independent.

By the general theory of Betti tables we know that in the cases the canonical ideal is generated by quadratic polynomials, the dimension of this set of matrices equals $\binom{g-2}{2}$, see [26, prop. 9.5]. Therefore we begin on the special fibre with the $s = \binom{g-2}{2}$ generators $\bar{g}_1, \ldots, \bar{g}_s$ elements. As we have proved in theorem 2.3.1 we can lift them to elements $g_1, \ldots, g_s \in I_{\mathscr{X}}$ so that for $J := \langle g_1, \ldots, g_s \rangle$ we have

(i) $J \otimes_R L = I_{\mathscr{X}_\eta}$.

(ii) $J \otimes_R k = I_{\mathscr{X}_0}$.

In this way we obtain the linear independent elements $g_1 \otimes_{S_R} 1_L, \ldots, g_s \otimes_{S_R} 1_L$ in I_{X_η}. We have seen that the $s = \binom{g-2}{2}$ linear independent quadratic elements generate also $I_{\mathscr{X}_\eta}$.

By following Lemma 5 (ii) of [18] we have the next lemma.

Lemma 2.3.2.1. Let G be a set of polynomials in S_R such that $\langle G \rangle \otimes_R L = I_{\mathscr{X}_\eta}$ and $\langle G \rangle \otimes_R k = I_{\mathscr{X}_0}$. Then $I_{\mathscr{X}} = \langle G \rangle$.

Essential for the proof of lemma 2.3.2.1 was that the ring R has a generic fibre. The deformation theory is concerned with deformations over local Artin algebras which do not have generic fibres. But by tensoring with A in the middle sequence of eq. (2.9) we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & I_{X_A} \arrow[r, hook] \arrow[d, "\otimes_A A/\mathfrak{m}_A"] & S_A := A[W_1, \ldots, W_g] \arrow[r, two heads, "\phi"] \arrow[d, "\otimes_A A/\mathfrak{m}_A"] & \bigoplus_{n=0}^{\infty} H^0(X_A, \Omega_{X_A/A}^{\otimes n}) \arrow[r] \arrow[d, "\otimes_A A/\mathfrak{m}_A"] & 0 \\
0 \arrow[r] & I_{\mathscr{X}_0} \arrow[r, hook] & S_k := k[w_1, \ldots, w_g] \arrow[r, two heads, "\phi_0"] & \bigoplus_{n=0}^{\infty} H^0(\mathscr{X}_0, \Omega_{\mathscr{X}_0/k}^{\otimes n}) \arrow[r] & 0
\end{tikzcd}$$

Indeed, since $H^0(\mathscr{X},\Omega^{\otimes n}_{\mathscr{X}/A})$ is free the left top arrow in the above diagram is injective. Moreover the relative canonical ideal I_{X_A} is still generated by quadratic polynomials in S_A.

2.3a Embedded deformations

Let Z be a scheme over k and let X be a closed subscheme of Z. An embedded deformation $X' \to \mathrm{Spec} k[\epsilon]$ of X over $\mathrm{Spec} k[\epsilon]$ is a closed subscheme $X' \subset Z' = Z \times \mathrm{Spec} k[\epsilon]$ fitting in the diagram:

$$\begin{tikzcd}
 & Z \arrow[rr] \arrow[dd] & & Z \times \mathrm{Spec} k[\epsilon] \arrow[dd] \\
X \arrow[ur, hook] \arrow[rr] \arrow[dr] & & X' \arrow[ur, hook] \arrow[dr] & \\
 & \mathrm{Spec} k \arrow[rr] & & \mathrm{Spec} k[\epsilon]
\end{tikzcd}$$

Let $\mathscr{I}$ be the ideal sheaf describing X as a closed subscheme of Z and

$$\mathscr{N}_{X/Z} = \mathscr{H}om_Z(\mathscr{I},\mathscr{O}_X) = \mathscr{H}om_X(\mathscr{I}/\mathscr{I}^2,\mathscr{O}_X), \tag{2.10}$$

be the normal sheaf. In particular for an affine open set U of X we set $B' = \mathscr{O}_{Z'}(U) = B \oplus \epsilon B$, where $B = \mathscr{O}_Z(U)$ and we observe that describing the sheaf of ideals $\mathscr{I}'(U) \subset \mathscr{B}'$ is equivalent to giving an element

$$\phi_U \in \mathrm{Hom}_{\mathscr{O}_Z(U)}\big(\mathscr{I}(U), \mathscr{O}_Z(U)/\mathscr{I}(U)\big),$$

see [37, prop. 2.3].

We will take $Z = \mathbb{P}^{g-1}$ and consider the canonical embedding $f : X \to \mathbb{P}^{g-1}$. We will denote by N_f the sheaf $\mathscr{N}_{X/\mathbb{P}^{g-1}}$. Let $\mathscr{I}_X$ be the sheaf of ideals of the curve X seen as a subscheme of $\mathbb{P}^{g-1}$. Since the curve X satisfies the conditions of Petri's theorem it is fully described by certain quadratic polynomials $f_1 = \tilde{A}_1,\ldots,f_r = \tilde{A}_r$ which correspond to a set $g \times g$ matrices $A_1,\ldots,A_r$, as we described in chapter 1. The elements $f_1,\ldots,f_r$ generate the ideal I_X corresponding to the projective cone $C(X)$ of X, $C(X) \subset \mathbb{A}^g$.

We have

$$H^0(X,N_f) = \mathrm{Hom}_S(I_X,\mathscr{O}_X).$$

Assume that X is deformed to a curve $X_\Gamma \to \mathrm{Spec}\Gamma$, where Γ is a local Artin algebra, $X_\Gamma \subset \mathbb{P}^{g-1}_\Gamma = \mathbb{P}^{g-1} \times \mathrm{Spec}\Gamma$. Our initial curve X is described in terms of the homogeneous canonical ideal I_X, generated by the elements $\{w^t A_1 w,\ldots,w^t A_r w\}$. For a local Artin algebra Γ let $\mathscr{S}_g(\Gamma)$ denote the space of symmetric $g \times g$ matrices with coefficients in Γ. The deformations X_Γ are expressed in terms of the ideals I_{X_Γ}, which by the relative Petri's theorem are also generated by elements $w^t A_1^\Gamma w,\ldots,w^t A_r^\Gamma w$, where A_i^Γ is in $\mathscr{S}_g(\Gamma)$. This essentially fits with Schlessinger's observation in [69], where the deformations of the projective variety are related to the deformations of the affine cone, notice that in our case all relative projective curves are smooth and the assumptions of [69, th. 2] are satisfied. We can thus replace the sheaf theoretic description of eq. (2.10) and work with the affine cone instead.

Remark 2.3.3. A set of quadratic generators $\{w^t A_1 w,\ldots,w^t A_r w\}$ is a minimal set of generators if and only if the elements $A_1,\ldots,A_r$ are linear independent in the free Γ-module $\mathscr{S}_g(\Gamma)$ of rank $(g+1)g/2$.

Embedded deformations and small extensions

Let

$$0 \to \langle E \rangle \to \Gamma' \xrightarrow{\pi} \Gamma \to 0$$

be a small extension and a curve $\mathbb{P}^{g-1}_{\Gamma'} \supset X_{\Gamma'} \to \mathrm{Spec}\Gamma'$ be a deformation of X_Γ and X. The curve $X_{\Gamma'}$ is described in terms of quadratic polynomials $w^t A_i^{\Gamma'} w$, where $A_i^{\Gamma'} \in \mathscr{S}_g(\Gamma')$, which reduce to A_i^Γ modulo $\langle E \rangle$. This means that

$$A_i^{\Gamma'} \equiv A_i^\Gamma \bmod \ker(\pi) \text{ for all } 1 \leqslant i \leqslant r \tag{2.11}$$

and if we select a naive lift $i(A_i^\Gamma)$ of A_i^Γ, then we can write

$$A_i^{\Gamma'} = i(A_i^\Gamma) + E \cdot B_i, \text{ where } B_i \in \mathscr{S}_g(k).$$

The set of liftings of elements $A_i^{\Gamma'}$ of elements A_i^{Γ}, for $1 \leqslant i \leqslant r$ is a principal homogeneous space, under the action of $H^0(X, N_f)$, since two such liftings $\{A_i^{(1)}(\Gamma'), 1 \leqslant i \leqslant r\}$, $\{A_i^{(2)}(\Gamma'), 1 \leqslant i \leqslant r\}$ differ by a set of matrices in $\{B_i(\Gamma') = A_i^{(1)}(\Gamma') - A_i^{(2)}(\Gamma'), 1 \leqslant i \leqslant r\}$ with entries in $\langle E \rangle \cong k$, see also [37, thm. 6.2].

Define a map $\phi : \langle A_1, \ldots, A_r \rangle \to \mathscr{S}_g(k)$ by $\phi(A_i) = B_i(\Gamma')$ and we also define the a corresponding map on polynomials $\tilde{\phi}(\tilde{A}_i) = w^t \phi(A_i) w$. we obtain a map $\tilde{\phi} \in \mathrm{Hom}_S(I_X, \mathscr{O}_X) = H^0(X, N_f)$, see also [37, th. 6.2], where $S = S_k$. Obstructions to such liftings are known to reside in $H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}} \otimes_k \ker \pi)$, which we will prove it is zero, see remark 2.3.4.

Embedded deformations and tangent spaces

Let us consider the $k[\epsilon]/k$ case. Since $i : X \hookrightarrow \mathbb{P}^{g-1}$ is non-singular we have the following exact sequence

$$0 \to \mathscr{T}_X \to i^* \mathscr{T}_{\mathbb{P}^{g-1}} \to \mathscr{N}_{X/\mathbb{P}^{g-1}} \to 0$$

which gives rise to

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, \mathscr{T}_X) \arrow[r] & H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \arrow[r] & H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \arrow[dll, "\delta"] \\ & H^1(X, \mathscr{T}_X) \arrow[r] & H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \arrow[r] & H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \arrow[r] & 0 \end{tikzcd}$$

Remark 2.3.4. In the above diagram, the last entry in the bottom row is zero since it corresponds to a second cohomology group on a curve. By Riemann-Roch theorem we have that $H^0(X, \mathscr{T}_X) = 0$ for $g \geqslant 2$. Also, the relative Petri theorem implies that the map δ is onto. We will give an alternative proof that δ is onto by proving that $H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) = 0$. This proves that $H^1(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) = 0$ as well, so there is no obstruction in lifting the embedded deformations.

Each of the above spaces has a deformation theoretic interpretation, see [35, p.96]:

- The space $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ is the space of deformations of the map $i : X \hookrightarrow \mathbb{P}^{g-1}$, that is both $X, \mathbb{P}^{g-1}$ are trivially deformed, see [70, p. 158, prop. 3.4.2.(ii)]
- The space $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ is the space of embedded deformations, where $\mathbb{P}^{g-1}$ is trivially deformed see [37, p. 13, Th. 2.4)].
- The space $H^1(X, \mathscr{T}_X)$ is the space of all deformations of X.

The dimension of the space $H^1(X, \mathscr{T}_X)$ can be computed using Riemann-Roch theorem on the dual space $H^0(X, \Omega_X^{\otimes 2})$ and equals $3g - 3$. In next section we will give a linear algebra interpretation for the spaces $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$, $H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}})$ allowing us to compute its dimensions.

2.3b Some matrix computations

We begin with the Euler exact sequence (see. [36, II.8.13], [78, p. 581] and [40] MO)

$$0 \to \mathscr{O}_{\mathbb{P}^{g-1}} \to \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g} \to \mathscr{T}_{\mathbb{P}^{g-1}} \to 0.$$

We restrict this sequence to the curve X:

$$0 \to \mathscr{O}_X \to i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g} = \omega_X^{\oplus g} \to i^* \mathscr{T}_{\mathbb{P}^{g-1}} \to 0.$$

We now take the long exact sequence in cohomology

$$\begin{tikzcd} 0 \arrow[r] & k = H^0(X, \mathscr{O}_X) \arrow[r, "f_1"] & H^0(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \arrow[r, "f_2"] & H^0(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \arrow[dlll, "f_3"] \\ H^1(X, \mathscr{O}_X) \arrow[r, "f_4"] & H^1(X, i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \arrow[r, "f_5"] & H^1(X, i^* \mathscr{T}_{\mathbb{P}^{g-1}}) \arrow[r] & H^2(X, \mathscr{O}_X) = 0 \end{tikzcd} \tag{2.12}$$

The spaces involved above have the following dimensions:

- $i^* \mathscr{O}_{\mathbb{P}^{g-1}}(1) = \Omega_X$ (canonical bundle)

- $\dim H^0(X, i^*\mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) = g \cdot \dim H^0(X, \Omega_X) = g^2$
- $\dim H^1(X, \mathscr{O}_X) = \dim H^1(X, \Omega_X) = g$
- $\dim H^1(X, i^*\mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) = g \cdot \dim H^0(X, \mathscr{O}_X) = g$

We will return to the exact sequence given in eq. (2.12) and the above dimension computations in the next section.

Study of $H^0(X, N_f)$

By relative Petri theorem the elements $\phi(A_i)$ are quadratic polynomials not in I_X, that is elements in a vector space of dimension $(g+1)g/2 - \binom{g-2}{2} = 3g-3$, where $(g+1)g/2$ is the dimension of the symmetric $g \times g$ matrices and $\binom{g-2}{2}$ is the dimension of the space generated by the generators of the canonical ideal, see [26, prop. 9.5].

The set of matrices $\{A_1, \ldots, A_r\}$ can be assumed to be linear independent but this does not mean that an arbitrary selection of quadratic elements $\omega^t B_i \omega \in \mathscr{O}_X$ will lead to a homomorphism of rings. Indeed, the linear independent elements A_i might satisfy some syzygies, see the following example where the linear independent elements

$$x^2 = \begin{pmatrix} x & y \end{pmatrix}^t \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad xy = \begin{pmatrix} x & y \end{pmatrix}^t \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

satisfy the syzygy

$$y \cdot x^2 - x \cdot xy = 0.$$

Therefore, a map of modules ϕ, should be compatible with the syzygy and satisfy the same syzygy. This is known as the fundamental Grothendieck flatness criterion, see [69, 1.1] and also [5, lem. 5.1, p. 28].

Proposition 2.3.4.1. The map

$$\begin{aligned} \psi : M_g(k) &\longrightarrow \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \\ B &\longmapsto \psi_B : \omega^t A_i \omega \mapsto \omega^t (A_i B + B^t A_i)\omega \bmod I_X \end{aligned}$$

identifies the vector space $M_g(k)/\langle \mathbb{I}_g \rangle$ to $H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) \subset H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. The map ψ is equivariant, where $M_g(k)$ is equipped with the adjoint action

$$B \mapsto \rho(g) B \rho(g^{-1}) = \mathrm{Ad}(g) B,$$

that is

$${}^g\psi_B = \psi_{\mathrm{Ad}(g)B}.$$

Proof. Recall that the space $H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$ can be identified to the space of deformations of the map f, where X, $\mathbb{P}^{g-1}$ are both trivially deformed. By [69] a map $\phi \in \mathrm{Hom}_S(I_X, S/I_X) = \mathrm{Hom}_S(I_X, \mathscr{O}_X)$ gives rise to a trivial deformation if there is a map

$$w_j \mapsto w_j + \epsilon \delta_j(w),$$

where $\delta_j(w) = \sum_{\nu=1}^{g} b_{j,\nu} w_\nu$. The map can be defined in terms of the matrix $B = (b_{j,\nu})$,

$$w \mapsto w + \epsilon B w$$

so that for all $\tilde{A}_i$, $1 \leqslant i \leqslant r$

$$\nabla \tilde{A}_i \cdot B w = \phi(\tilde{A}_i) = \phi(w^t A_i w) \bmod I_X. \tag{2.13}$$

But for $\tilde{A}_i = w^t A_i w$ we compute $\nabla \tilde{A}_i = w^t A_i$, therefore eq. (2.13) is transformed to

$$w^t A_i B w = w^t B_i w \bmod I_X, \tag{2.14}$$

for a symmetric $g \times g$ matrix B_i in $\mathscr{S}_g(k[\epsilon])$. Therefore if 2 is invertible according to remark 2.3.2 we replace the matrix $A_i B$ appearing in eq. (2.14) by the symmetric matrix $A_i B + B^t A_i$. Since we are interested in the projective algebraic set defined by homogeneous polynomials the $1/2$ factor of remark 2.3.2 can be omitted.

For every $B \in M_g(k)$ we define the map $\psi_B \in \mathrm{Hom}_S(I_X, S/I_X) = \mathrm{Hom}_S(I_X, \mathscr{O}_X)$ given by

$$\tilde{A}_i = \omega^t A_i \omega \mapsto \omega^t(A_i B + B^t A_i)\omega \bmod I_X,$$

and we have just proved that the functions ψ_B are all elements in $H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$. The kernel of the map $\psi : B \mapsto \psi_B$ consists of all matrices B satisfying:

$$A_i B = -B^t A_i \bmod I_X \text{ for all } 1 \leqslant i \leqslant \binom{g-2}{2}. \tag{2.15}$$

This kernel seems to depend on the selection of the elements A_i. This is not the case. We will prove that the kernel consists of all multiples of the identity matrix. Indeed,

$$\dim H^0(X, i^*\mathscr{T}_X) = g^2 - \ker\psi.$$

We now rewrite the spaces in eq. (2.12) by their dimensions we get

$$\begin{tikzcd}
(0) \arrow[r] & (1) \arrow[r, "f_1"] & (g^2) \arrow[r, "f_2"] & (g^2 - \ker\psi) \arrow[dlll, "f_3"'] \\
(g) \arrow[r] & (g) \arrow[r] & (?) \arrow[r] & (0)
\end{tikzcd}$$

So

- $\dim\ker f_2 = \dim \mathrm{Im}\, f_1 = 1$
- $\dim\ker f_3 = \dim \mathrm{Im}\, f_2 = g^2 - 1$
- $\dim \mathrm{Im}\, f_3 = (g^2 - \dim\ker\psi) - (g^2 - 1) = 1 - \dim\ker\psi$

It is immediate that $\dim\ker\psi = 0$ or 1. But obviously $\mathbb{I}_g \in \ker\psi$, and hence

$$\dim\ker\psi = 1.$$

Finally $\dim \mathrm{Im}\, f_3 = 0$, i.e. f_3 is the zero map and we get the small exact sequence,

$$0 \longrightarrow k = H^0(X, \mathscr{O}_X) \longrightarrow H^0(X, i^*\mathscr{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g}) \longrightarrow H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) \longrightarrow 0$$

It follows that

$$\dim H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) = g^2 - 1.$$

We have proved that $\psi : M_g(k)/\langle \mathbb{I}_g \rangle \to H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$ is an isomorphism of vector spaces. We will now prove it is equivariant.

Using remark 2.2.1 we have that the action of the group G on the function

$$\psi_B : A_i \mapsto A_i B + B^t A_i,$$

seen as an element in $H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$ is given:

$$\begin{aligned}
A_i \mapsto T(\sigma^{-1})A_i &\overset{\psi_B}{\longmapsto} T(\sigma)\left(\rho(\sigma)^t A_i \rho(\sigma) B + B^t \rho(\sigma)^t A_i \rho(\sigma)\right) \\
&= \left(A_i \rho(\sigma) B \rho(\sigma^{-1}) + (\rho(\sigma) B \rho(\sigma^{-1}))^t A_i\right)
\end{aligned}$$

□

Corolarry 2.3.4.1. The space $H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})^G$ is generated by the elements $B \neq \{\lambda \mathbb{I}_g : \lambda \in k\}$ such that

$$\rho(\sigma) B \rho(\sigma^{-1}) B^{-1} = [\rho(\sigma), B] \in \langle A_1, \ldots, A_r \rangle \text{ for all } \sigma \in \mathrm{Aut}(X).$$

Remark 2.3.5. This construction allows us to compute the space $H^1(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$. Indeed, we know that f_4 is isomorphism and hence f_5 is the zero map, on the other hand f_5 is surjective, it follows that $H^1(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) = 0$. This provides us with another proof of the exactness of the sequence

$$0 \longrightarrow H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) \longrightarrow H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}}) \xrightarrow{\delta} H^1(X, \mathscr{T}_X) \longrightarrow 0 \tag{2.16}$$

2.3c Invariant spaces

Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence of G-modules. We have the following sequence of G-invariant spaces

$$0 \to A^G \to B^G \to C^G \xrightarrow{\delta_G} H^1(G, A) \to \cdots$$

where the map δ_G is computed as follows: an element c is given as a class $b \bmod A$ and it is invariant if and only if $gb - b = a_g \in A$. The map $G \ni g \mapsto a_g$ is the cocycle defining $\delta_G(c) \in H^1(G, A)$.

Using this construction on the short exact sequence of eq. (2.16) we arrive at

$$\begin{tikzcd} 0 \arrow[r] & H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})^G \arrow[r] & H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})^G \arrow[r, "\delta"] & H^1(X, \mathscr{T}_X)^G \arrow[dll, "\delta_G"] \\ & H^1\big(G, H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})\big) \arrow[r] & \cdots \end{tikzcd}$$

We will use eq. (2.16) in order to represent elements in $H^1(X, \mathscr{T}_X)$ as elements $[f] \in H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})/H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})/\mathrm{Im}\psi$.

Proposition 2.3.5.1. Let $[f] \in H^1(X, \mathscr{T}_X)^G$ be a class of a map $f : I_X \to S/I_X$ modulo $\mathrm{Im}\psi$. For each element $\sigma \in G$ there is a matrix $B_\sigma[f]$, depending on f, which defines a class in $M_g(k)/\langle \mathbb{I}_g \rangle$ satisfying the cocycle condition in eq. (2.18), such that

$$\delta_G(f)(\sigma) : A_i \mapsto A_i\,(B_\sigma[f]) + \big(B^t_\sigma[f]\big)\,A_i \bmod \langle A_1, \ldots, A_g \rangle.$$

Proof. Let $[f] \in H^1(X, \mathscr{T}_X)^G$, where $f : I_X \to S/I_X$ that is $f \in H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. The $\delta_G(f)$ is represented by an 1-cocycle given by $\delta_G(f)(\sigma) = {}^\sigma f - f$. Using the equivariant isomorphism of $\psi : M_g(k)/\langle \mathbb{I}_g \rangle \to H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}})$ of proposition 2.3.4.1 we arrive at the diagram:

$$\begin{tikzcd} G \arrow[r] & H^0(X, i^*\mathscr{T}_{\mathbb{P}^{g-1}}) \arrow[r, "\psi^{-1}"] & M_g(k)/\langle \mathbb{I}_g \rangle \\ \sigma \arrow[r, mapsto] & \delta_G(f)(\sigma) \arrow[r] & B[f]_\sigma := \psi^{-1}(\delta_G(f)(\sigma)) \end{tikzcd}$$

We will now compute

$${}^\sigma f : A_i \xrightarrow{T(\sigma^{-1})} T(\sigma^{-1})A_i \xrightarrow{f} f(T(\sigma^{-1})A_i) \xrightarrow{T(\sigma)} T(\sigma)f(T(\sigma^{-1})A_i).$$

We set

$$T(\sigma^{-1})(A_i) = \rho(\sigma)^t A_i \rho(\sigma) = \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma) A_i$$

so

$$\begin{aligned} \delta_G(f)(\sigma)(A_i) &= \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma) \cdot \rho(\sigma^{-1})^t f(A_\nu)\rho(\sigma^{-1}) - f(A_i) \\ &= A_i B_\sigma[f] + B_\sigma[f]^t A_i \bmod I_X \end{aligned} \tag{2.17}$$

for some matrix $B_\sigma[f] \in M_g(k)$ such that for all $\sigma, \tau \in G$ we have

$$\begin{aligned} B_{\sigma\tau}[f] &= B_\sigma[f] + \sigma B_\tau[f]\sigma^{-1} + \lambda(\sigma,\tau)\mathbb{I}_g \\ &= B_\sigma[f] + \mathrm{Ad}(\sigma)B_\tau[f] + \lambda(\sigma,\tau)\mathbb{I}_g. \end{aligned} \tag{2.18}$$

In the above equation we have used the fact that $\sigma \mapsto B_\sigma[f]$ is a 1-cocycle in the quotient space $M_g(k)/\mathbb{I}_g$, therefore the cocycle condition holds up to an element of the form $\lambda(\sigma,\tau)\mathbb{I}_g$. □

Remark 2.3.6. Let

$$\lambda(\sigma,\tau)\mathbb{I}_g = B_{\sigma\tau}[f] - B_{\sigma}[f] - \mathrm{Ad}(œ)B_{ø}[f].$$

The map $G \times G \to k$, $(\sigma,\tau) \mapsto \lambda(\sigma,\tau)$ is a normalized 2-cocycle (see [81, p. 184]), that is

$$\begin{aligned}
0 &= \lambda(\sigma,1) = \lambda(1,\sigma) && \text{for all } \sigma \in G\\
0 &= \mathrm{Ad}(œ_1)\lambda(\sigma_2,\sigma_3) - \lambda(\sigma_1\sigma_2,\sigma_3) + \lambda(\sigma_1,\sigma_2\sigma_3) - \lambda(\sigma_1,\sigma_2) && \text{for all } \sigma_1,\sigma_2,\sigma_3 \in G\\
&= \lambda(\sigma_2,\sigma_3) - \lambda(\sigma_1\sigma_2,\sigma_3) + \lambda(\sigma_1,\sigma_2\sigma_3) - \lambda(\sigma_1,\sigma_2) && \text{for all } \sigma_1,\sigma_2,\sigma_3 \in G
\end{aligned}$$

For the last equality notice that the Ad-action is trivial on scalar multiples of the identity.

Proof. The first equation is clear. For the second one,

$$\lambda(\sigma_1\sigma_2,\sigma_3)\mathbb{I}_g = B_{\sigma_1\sigma_2\sigma_3}[f] - B_{\sigma_1\sigma_2}[f] - \mathrm{Ad}(œ_1œ_2)B_{œ_3}[f]$$

and

$$\lambda(\sigma_1,\sigma_2)\mathbb{I}_g = B_{\sigma_1\sigma_2}[f] - B_{\sigma_1}[f] - \mathrm{Ad}(œ_1)B_{œ_2}[f].$$

Hence

$$\begin{aligned}
\lambda(\sigma_1\sigma_2,\sigma_3)\mathbb{I}_g + \lambda(\sigma_1,\sigma_2)\mathbb{I}_g =& B_{\sigma_1\sigma_2\sigma_3}[f] - \mathrm{Ad}(œ_1œ_2)B_{œ_3}[f] - B_{œ_1}[f] - \mathrm{Ad}(œ_1)B_{œ_2}[f]\\
=& B_{\sigma_1\sigma_2\sigma_3}[f] - B_{\sigma_1}[f] - \mathrm{Ad}(œ_1)B_{œ_2œ_3}[f] +\\
& + \mathrm{Ad}(œ_1)B_{œ_2,œ_3}[f] - \mathrm{Ad}(œ_1)B_{œ_2}[f] - \mathrm{Ad}(œ_1œ_2)B_{œ_3}[f]\\
=& \lambda(\sigma_1,\sigma_2\sigma_3)\mathbb{I}_g + \mathrm{Ad}(œ_1)\big(B_{œ_2,œ_3}[f] - B_{œ_2}[f] - \mathrm{Ad}(œ_1)B_{œ_3}[f]\big)\\
=& \mathrm{Ad}(œ_1)\,\check{}\,(œ_2,œ_3)\mathbb{I}_g + \check{}\,(œ_1,œ_2œ_3)\mathbb{I}_g.
\end{aligned}$$

□

Corolarry 2.3.6.1. If $f(\omega^t A_i \omega) = \omega^t B_i \omega$, where $B_i \in M_g(k)$ are the images of the elements defining the canonical ideal in the small extension $\Gamma' \to \Gamma$, then the symmetric matrices defining the canonical ideal $I_X(\Gamma')$ are given by $A_i + E \cdot B_i$. Using proposition 2.3.5.1 we have

$$\begin{aligned}
({}^{\sigma}f - f)(A_i) &= \sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma)T(\sigma)(B_\nu) - B_i \\
&= \left(A_i B_\sigma[f] + B_\sigma^t[f]A_i\right) \mod\langle A_1,\ldots,A_r\rangle \\
&= \psi_{B_\sigma[f]}A_i.
\end{aligned} \tag{2.19}$$

Therefore, using also eq. (2.17)

$$\sum_{\nu=1}^{r} \lambda_{i,\nu}(\sigma)(B_\nu) - T(\sigma^{-1})B_i = T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i). \tag{2.20}$$

2.4 On the deformation theory of curves with automorphisms

Let $1 \to \langle E\rangle \to \Gamma' \to \Gamma \to 0$ be a small extension of Artin local algebras and consider the diagram

$$\begin{tikzcd}
X_\Gamma \arrow[r] \arrow[d] & X_{\Gamma'} \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}(\Gamma) \arrow[r] & \mathrm{Spec}(\Gamma') \arrow[r] & \mathrm{Spec}(R)
\end{tikzcd}$$

Suppose that G acts on X_Γ, that is every automorphism $\sigma \in G$ satisfies $\sigma(I_{X_\Gamma}) = I_{X_\Gamma}$. If the action of the group G is lifted to $X_{\Gamma'}$ then we should have a lift of the representations $\rho, \rho^{(1)}$ defined in eq. (2), (3) to Γ' as well. The set of all such liftings is a principal homogeneous space parametrized by the spaces $H^1(G, M_g(k)), H^1(G, M_r(k))$, provided that the corresponding lifting obstructions in $H^2(G, M_g(k)), H^2(G, M_r(k))$ both vanish.

Assume that there is a lifting of the representation

$$\begin{tikzcd} & \mathrm{GL}_g(\Gamma') \arrow[d, "\mathrm{mod}\langle E\rangle"] \\ G \arrow[ur, "\rho_{\Gamma'}"] \arrow[r, "\rho_\Gamma"'] & \mathrm{GL}_g(\Gamma) \end{tikzcd} \tag{2.21}$$

This lift gives rise to a lifting of the corresponding automorphism group to the curve $X_{\Gamma'}$ if

$$\rho_{\Gamma'}(\sigma) I_{X_{\Gamma'}} = I_{X_{\Gamma'}} \quad \text{for all } \sigma \in G,$$

that is if the relative canonical ideal is invariant under the action of the lifted representation $\rho_{\Gamma'}$. In this case the free Γ'-modules $V_{\Gamma'}$, defined in remark 6, are G-invariant and the T-action, as defined in definition 2.2.1.1.1 restricts to a lift of the representation

$$\begin{tikzcd} & \mathrm{GL}_r(\Gamma') \arrow[d, "\mathrm{mod}\langle E\rangle"] \\ G \arrow[ur, "\rho^{(1)}_{\Gamma'}"] \arrow[r, "\rho^{(1)}_\Gamma"'] & \mathrm{GL}_r(\Gamma) \end{tikzcd} \tag{2.22}$$

In section 1.2b (or [51, sec. 2.2]) we gave an efficient way to check this compatibility in terms of linear algebra:

Consider an ordered basis Σ of the free Γ-module $\mathscr{S}_g(\Gamma)$ generated by the matrices $\Sigma(ij) = (\sigma(ij))_{\nu,\mu}$, $1 \leqslant i \leqslant j \leqslant g$ ordered lexicographically, with elements

$$\sigma(ij)_{\nu,\mu} = \begin{cases} \delta_{i,\nu}\delta_{j,\mu} + \delta_{i,\mu}\delta_{j,\nu}, & \text{if } i \neq j \\ \delta_{i,\nu}\delta_{i,\mu} & \text{if } i = j. \end{cases}$$

For example, for $g = 2$ we have the elements

$$\sigma(11) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \sigma(12) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma(22) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For every symmetric matrix A, let $F(A)$ be the column vector consisted of the coordinates of A in the basis Σ. Consider the symmetric matrices $A_1^{\Gamma'}, \ldots, A_r^{\Gamma'}$, which exist since at the level of curves there is no obstruction of the embedded deformation. For each $\sigma \in G$ the $(g+1)g/2 \times 2r$ matrix

$$F_{\Gamma'}(\sigma) = \left[F\left(A_1^{\Gamma'}\right), \ldots, F\left(A_r^{\Gamma'}\right), F\left(\rho_{\Gamma'}(\sigma)^t A_1^{\Gamma'} \rho_{\Gamma'}(\sigma)\right), \ldots, F\left(\rho_{\Gamma'}(\sigma)^t A_r^{\Gamma'} \rho_{\Gamma'}(\sigma)\right)\right]. \tag{2.23}$$

The automorphism σ acting on the relative curve X_Γ is lifted to an automorphism σ of $X_{\Gamma'}$ if and only if the matrix given in eq. (2.23) has rank r.

Proposition 2.4.0.1. The obstruction to lifting an automorphism of X_Γ to $X_{\Gamma'}$ has a global obstruction given by vanishing the class of

$$A(\sigma, \tau) = \rho_{\Gamma'}(\sigma)\rho_{\Gamma'}(\tau)\rho_{\Gamma'}(\sigma\tau)^{-1}$$

in $H^2(G, M_g(k))$ and a compatibility rank condition given by requiring that the matrix $F_{\Gamma'}(\sigma)$ equals r for all elements $\sigma \in G$.

2.4a An example

Let k be an algebraically closed field of positive characteristic $p > 0$. Consider the Hermitian curve, defined over k, given by the equation

$$H : y^p - y = \frac{1}{x^{p+1}}, \tag{2.24}$$

which has the group $\mathrm{PGU}(3,p^2)$ as an automorphism group, [79, th. 7]. As an Artin-Schreier extension of the projective line, this curve fits within the Bertin-Mézard model of curves, and the deformation functor with respect to the subgroup $\mathbb{Z}/p\mathbb{Z} \cong \mathrm{Gal}(H/\mathbb{P}^1) = \{y \mapsto y+1\}$ has versal deformation ring $W(k)[\zeta][[x_1]]$, where ζ is a primitive p root of unity which resides in an algebraic extension of $\mathrm{Quot}(W(k))$ [10], [43]. Indeed, $m = p+1 = 2p-(p-1) = qp - l$, so in the notation of [10] $q=2$ and $l = p-1$.

The reduction of the universal curve in the Bertin-Mezard model modulo $\mathfrak{m}_{W(k)[\zeta]}$ is given by the Artin-Schrein equation:

$$X^p - X = \frac{x^{p-1}}{(x^2+x_1x)^p} \tag{2.25}$$

which has special fibre at the specialization $x_1 = 0$ the original Hermitian curve given in eq. (2.24).

The initial Hermitian curve admits the automorphism $\sigma : y \mapsto y, x \mapsto \zeta_{p+1}x$, where ζ_{p+1} is a primitive $p+1$ root of unity. We will use the tools developed in this chapter in order to show that the automorphism σ does not lift even in positive characteristic.

We set $a(x) = x^2 + x_1x$ and $\lambda = \zeta - 1 \in W(k)[\zeta]$. In [43] S. Karanikolopoulos and A. Kontogeorgis proved that the free R-module $H^0(\mathscr{X}, \Omega_{\mathscr{X}/R})$ has basis

$$\mathbf{c} = \left\{ W_{N,\mu} = \frac{x^N a(x)^{p-1-\mu} X^{p-1-\mu}}{a(x)^{p-1}(\lambda X+1)^{p-1}} dx : \left\lfloor \frac{\mu\ell}{p} \right\rfloor \leqslant N \leqslant \mu q - 2,\ 1 \leqslant \mu \leqslant p-1 \right\}.$$

From the form of the holomorphic differentials it is clear that the representation of $\langle\sigma\rangle$ on $H^0(H, \Omega_{H/k})$ is diagonal, since $a(x) = x^2 + x_1x$ reduces to x^2 for $x_1 = 0$. In our example, we have $q = \deg a(x) = 2$ so in the special fibre we have

$$w_{N,\mu} = x^{N-2\mu} X^{p-1-\mu} dx$$
$$\sigma(w_{N,\mu}) = \zeta_{p+1}^{N-2\mu+1} w_{N,\mu}$$

and

$$\sigma(w_{N,\mu} w_{N',\mu'}) = \zeta_{p+1}^{N+N'-2(\mu+\mu')+2} w_{N,\mu} w_{N',\mu'}. \tag{2.26}$$

Thus, the action of σ on holomorphic differentials on the special fibre is given by a diagonal matrix.

To decide, whether the action lifts to the Artin local ring $k[\epsilon]$, we have to see first whether the diagonal representation can be lifted, that is whether we have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{GL}_g(k[\epsilon]) \arrow[d] \\ \langle\sigma\rangle \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"'] & \mathrm{GL}_g(k) \end{tikzcd}$$

Since $\rho(\sigma) = \mathrm{diag}(\delta_1, \ldots, \delta_g) =: \Delta$ a possible lift will be given by $\tilde{\rho}(\sigma) = \Delta + \epsilon B$, for some $g \times g$ matrix B with entries in k. The later element should have order $p+1$, that is

$$\mathbb{I}_g = (\Delta + \epsilon B)^{p+1} = \Delta^{p+1} + \epsilon \Delta^p B,$$

which in turn implies that $\Delta^p B = 0$ and since Δ is invertible $B = 0$. This means that the representation of the cyclic group generated by σ is trivially deformed to a representation into $\mathrm{GL}_g(k[\epsilon])$.

The next step is to investigate whether the canonical ideal is kept invariant under the action of σ for $x_1 \neq 0$. The canonical ideal for Bertin-Mézard curves was recently studied by H. Haralampous K. Karagiannis and A. Kontogeorgis, [18]. Namely, using the notation of [18] we have

$$\begin{aligned} a(x)^{p-i} = (x^2 + x_1x)^{p-i} &= \sum_{j=j_{\min}}^{2(p-1)} c_{j,p-i} x^j \\ &= \sum_{j=0}^{p-i} \binom{p-i}{j} x_1^{p-i-j} x^{j+p-i} \end{aligned}$$

so by setting $J = j + p - i$, $p - i \leqslant J \leqslant 2(p-i)$ we have

$$c_{J,p-i} = \begin{cases} \binom{p-i}{J-(p-i)} x_1^{2(p-i)-J} & \text{if } J \geqslant p-i \\ 0 & \text{if } J < p-i \end{cases}$$

This means that $c_{2(p-i),p-i}=1$, $c_{2(p-i)-1,p-i}=(p-i)x_1$ and for all other values of J, the quantity $c_{J,p-i}$ is either zero or a monomial in x_1 of degree $\geqslant 2$.

It is proved in [18] that the canonical ideal is generated by two sets of generators G_1 and G_2 given by:

$$G_1^c=\{W_{N_1,\mu_1}W_{N_1',\mu_1'}-W_{N_2,\mu_2}W_{N_2',\mu_2'}\in S\ :\ W_{N_1,\mu_1}W_{N_1',\mu_1'},W_{N_2,\mu_2}W_{N_2',\mu_2'}\in\mathbb{T}^2 \text{ and } N_1+N_1'=N_2+N_2',\quad \mu_1+\mu_1'=\mu_2+\mu_2'\}.$$

$$\begin{aligned}G_2^c=\Bigg\{&W_{N,\mu}W_{N',\mu'}-W_{N'',\mu''}W_{N''',\mu'''}\\ &+\sum_{i=1}^{p-1}\sum_{j=j_{\min}(i)}^{(p-i)q}\lambda^{i-p}\binom{p}{i}c_{j,p-i}W_{N_j,\mu_i}W_{N_j',\mu_i'}\in S\ :\\ &N''+N'''=N+N'+p-1,\quad \mu''+\mu'''=\mu+\mu'+p,\\ &N_j+N_j'=N+N'+j,\quad \mu_i+\mu_i'=\mu+\mu'+p-i\\ &\text{for } 0\leqslant i\leqslant p,\ j_{\min}(i)\leqslant j\leqslant(p-i)q\Bigg\}.\end{aligned}$$

The reduction modulo $\mathfrak{m}_{W(k)[\zeta]}$, of the set G_1^c is given by simply replacing each $W_{n,\mu}$ by $w_{N,\mu}$ and does not depend on x_1. Therefore it does not give us any condition to deform σ.

The reduction of the set G_2^c modulo $\mathfrak{m}_{W(k)[\zeta]}$ is given by

$$\begin{aligned}G_2^c\otimes_R k=\Bigg\{&w_{N,\mu}w_{N',\mu'}-w_{N'',\mu''}w_{N''',\mu'''}-\sum_{j=j_{\min}(1)}^{(p-1)q}c_{j,p-1}w_{N_j,\mu_j}w_{N_j',\mu_j'}\in S\ :\\ &N''+N'''=N+N'+p-1,\quad \mu''+\mu'''=\mu+\mu'+p,\\ &N_j+N_j'=N+N'+j,\quad \mu_i+\mu_i'=\mu+\mu'+p-i\\ &\text{for } j_{\min}(1)\leqslant j\leqslant(p-1)q\Bigg\}.\end{aligned}$$

If we further consider this set modulo $\langle x_1^2\rangle$, that is if we consider the canonical curve as a family over first-order infinitesimals then, only the terms $c_{2(p-1),p-1}=1$, $c_{2(p-1)-1,p-1}=(p-1)x_1$ survive.

Using eq. (2.26) and the definition of G_2^c we have that for

$$W=w_{N,\mu}w_{N',\mu'}-w_{N'',\mu''}w_{N''',\mu'''}-w_{N_{2(p-1)},\mu_{p-1}}w_{N'_{2(p-1)},\mu'_{p-1}}$$

$$\sigma(W)=\zeta_{p+1}^{N+N'-2(\mu+\mu')+2}W$$

Set

$$W''=w_{N_{2(p-1)-1},\mu_{p-1}}w_{N'_{2(p-1)-1},\mu'_{p-1}}.$$

The automorphism lifts if and only if the element

$$W'=W+x_1W''$$

we have

$$\sigma(W')=\chi(\sigma)\big(W'\big).$$

But this is not possible since for

$$\sigma(W'')=\zeta_{p+1}^{N_{2(p-1)-1}+N_{2(p-1)-1}-2(\mu_{p-1}+\mu'_{p-1})+2}W''$$

and

$$N_{2(p-1)-1}+N_{2(p-1)-1}-2(\mu_{p-1}+\mu'_{p-1})+2=N+N'-2(\mu+\mu')+2-1.$$

2.4b A tangent space condition

All lifts of X_Γ to $X_{\Gamma'}$ form a principal homogeneous space under the action of $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$. This paragraph aims to provide the next compatibility relation given in eq. (4) by selecting the deformations of the curve and the representations.

Let $\{A_1^\Gamma, \ldots, A_r^\Gamma\}$ be a basis of the canonical Ideal I_{X_Γ}, where X_Γ is a canonical curve. Assume also that the special fibre is acted on by the group G, and we assume that the action of the group G is lifted to the relative curve X_Γ. Since X_Γ is assumed to be acted on by G, we have the action

$$T(\sigma^{-1})(A_i^\Gamma) = \rho_\Gamma(\sigma)^t A_i^\Gamma \rho_\Gamma(\sigma) = \sum_j \lambda_{i,j}^\Gamma(\sigma) A_j(\Gamma) \text{ for each } i = 1, \ldots, r, \tag{2.27}$$

where ρ_Γ is a lift of the representation ρ induced by the action of G on $H^0(X_\Gamma, \Omega_{X/\Gamma})$, and $\lambda_{i,j}^\Gamma(\sigma)$ are the entries of the matrix of the lifted representation $\rho_\Gamma^{(1)}$ induced by the action of G on $A_1^\Gamma, \ldots, A_r^\Gamma$. Notice that the matrix $\rho_\Gamma(\sigma) \in \mathrm{GL}_g(\Gamma)$. We will denote by $A_1^{\Gamma'}, \ldots, A_r^{\Gamma'} \in \mathscr{S}_g(\Gamma')$ a set of liftings of the matrices $A_1^\Gamma, \ldots, A_r^\Gamma$. Since the couple (X_Γ, G) is lifted to $(X_{\Gamma'}, G)$, there is an action

$$T(\sigma^{-1})(A_i^{\Gamma'}) = \rho_{\Gamma'}(\sigma)^t A_i^{\Gamma'} \rho_{\Gamma'}(\sigma) = \sum_j \lambda_{i,j}^{\Gamma'}(\sigma) A_j^{\Gamma'} \text{ for each } i = 1, \ldots, r,$$

where $\lambda_{ij}^{\Gamma'}(\sigma) \in \Gamma'$. All other liftings extending X_Γ form a principal homogeneous space under the action of $H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ that is we can find matrices $B_1, \ldots, B_r \in \mathscr{S}_g(k)$, such that the set

$$\{A_1^{\Gamma'} + E \cdot B_1, \ldots, A_r^{\Gamma'} + E \cdot B_r\}$$

forms a basis for another lift $I_{X^1_{\Gamma'}}$ of the canonical ideal of I_{X_Γ}. That is all lifts of the canonical curve I_{X_Γ} differ by an element $f \in \mathrm{Hom}_S(I_X, S/I_X) = H^0(X, \mathscr{N}_{X/\mathbb{P}^{g-1}})$ so that $f(A_i) = B_i$.

In the same manner, if $\rho_{\Gamma'}$ is a lift of the representation ρ_Γ every other lift is given by

$$\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma),$$

where $\tau(\sigma) \in M_g(k)$.

We have to find out when $\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)$ is an automorphism of the relative curve $X_{\Gamma'}$, i.e. when

$$T(\rho_{\Gamma'}(\sigma^{-1}) + E \cdot \tau(\sigma^{-1}))(A_i^{\Gamma'} + E \cdot B_i) \in \mathrm{span}_{\Gamma'}\{A_1^{\Gamma'} + E \cdot B_1, \ldots, A_r^{\Gamma'} + E \cdot B_r\}, \tag{2.28}$$

that is

$$(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left(A_i^{\Gamma'} + E \cdot B_i\right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) = \sum_{j=1}^r \tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \left(A_j^{\Gamma'} + E \cdot B_j\right), \tag{2.29}$$

for some $\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \in \Gamma'$. Since

$$T_{\Gamma'}(\sigma^{-1}) A_i^{\Gamma'} = \rho_\Gamma(\sigma)^t A_i^\Gamma \rho_\Gamma(\sigma) \bmod \langle E \rangle$$

we have that $\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) = \lambda_{i,j}^\Gamma(\sigma) \bmod E$, therefore we can write

$$\tilde{\lambda}_{ij}^{\Gamma'}(\sigma) = \lambda_{ij}^{\Gamma'}(\sigma) + E \cdot \mu_{ij}(\sigma), \tag{2.30}$$

for some $\mu_{ij}(\sigma) \in k$. We expand first the right-hand side of eq. (2.29) using eq. (2.30). We have

$$\sum_{j=1}^r \tilde{\lambda}_{ij}^{\Gamma'}(\sigma) \left(A_j^{\Gamma'} + E \cdot B_j\right) = \sum_{j=1}^r \left(\lambda_{ij}^{\Gamma'}(\sigma) + E \cdot \mu_{ij}(\sigma)\right) \left(A_j^{\Gamma'} + E \cdot B_j\right) \tag{2.31}$$

$$= \sum_{j=1}^r \lambda_{ij}^{\Gamma'}(\sigma) A_j^{\Gamma'} + E\big(\mu_{ij}(\sigma) A_j + \lambda_{ij}(\sigma) B_j\big). \tag{2.32}$$

Here we have used the fact that $E\mathfrak{m}_\Gamma = E\mathfrak{m}_{\Gamma'}$ so $E \cdot x = E \cdot (x \bmod \mathfrak{m}_{\Gamma'})$ for every $x \in \Gamma'$.

We now expand the left-hand side of eq. (2.29).

$$\begin{aligned}(\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma))^t \left(A_i^{\Gamma'} + E \cdot B_i\right) (\rho_{\Gamma'}(\sigma) + E \cdot \tau(\sigma)) &= \rho_{\Gamma'}(\sigma)^t A_i^{\Gamma'} \rho_{\Gamma'}(\sigma) \\ &+ E \cdot \left(\rho(\sigma)^t B_i \rho(\sigma) + \tau^t(\sigma) A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma)\right).\end{aligned}$$

Set $D_\sigma = \tau(\sigma)\rho(\sigma)^{-1} = d(\sigma)$ according to the notation of lemma 2.2.2.1, we can write

$$\begin{aligned}\tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma) \\ &= \rho(\sigma)^t \rho(\sigma^{-1})^t \tau(\sigma)^t A_i \rho(\sigma) + \rho(\sigma)^t A_i \tau(\sigma)\rho(\sigma)^{-1}\rho(\sigma) \\ &= \rho(\sigma)^t (D_\sigma^t A_i)\rho(\sigma) + \rho(\sigma)^t (A_i D_\sigma)\rho(\sigma) \\ &= T(\sigma^{-1})\psi_{D_\sigma}(A_i).\end{aligned} \tag{2.33}$$

while eq. (2.20) implies that

$$\rho(\sigma)^t B_i \rho(\sigma) - \sum_{j=1}^{r} \lambda_{ij}(\sigma^{-1}) B_j = -T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i). \tag{2.34}$$

For the above computations recall that for a $g \times g$ matrix B, the map ψ_B is defined by

$$\psi_B(A_i) = A_i B + B^t A_i.$$

Combining now eq. (2.33) and (2.34) we have that eq. (2.29) is equivalent to

$$\begin{aligned} T(\sigma^{-1})\big(\psi_{D_\sigma}(A_i)\big) - T(\sigma^{-1})\psi_{B_\sigma[f]}(A_i) &= \sum_{j=1}^{r} \mu_{ij}(\sigma) A_j \\ \big(\psi_{D_\sigma}(A_i)\big) - \psi_{B_\sigma[f]}(A_i) &= \sum_{j=1}^{r} T(\sigma)\mu_{ij}(\sigma) A_j. \\ &= \sum_{j=1}^{r}\sum_{\nu=1}^{r} \mu_{ij}(\sigma)\lambda_{j\nu}(\sigma^{-1}) A_\nu. \end{aligned} \tag{2.35}$$

On the other hand the action T on $A_1, \ldots, A_r$ is given in terms of the matrix $(\lambda_{i,j})$ while the right hand side of eq. (2.35) $\big(\mu_{i,j}(\sigma^{-1})\big)\big(\lambda_{ij}(\sigma)\big)$ corresponds to the derivation $D^{(1)}(\sigma^{-1})$ of the ρ_1-representation. Equation (4) is now proved.

Chapter 3

A new obstruction to the local lifting problem

3.1 Introduction

Consider a local action $\rho : G \to \mathrm{Aut}k[[t]]$ of the group $G = C_q \rtimes C_m$. The Harbater-Katz-Gabber compactification theorem asserts that there is a Galois cover $X \to \mathbb{P}^1$ ramified wildly and completely only at one point P of X with Galois group $G = \mathrm{Gal}(X/\mathbb{P}^1)$ and tamely on a different point P' with ramification group C_m, so that the action of G on the completed local ring $\mathscr{O}_{X,P}$ coincides with the original action of G on $k[[t]]$. Moreover, it is known that the local action lifts if and only if the corresponding HKG-cover lifts.

In particular, we have proved that in order to lift a subgroup $G \subset \mathrm{Aut}(X)$, the representation $\rho : G \to \mathrm{GL}H^0(X, \Omega_X)$ should be lifted to characteristic zero and also the lifting should be compatible with the deformation of the curve. More precisely, in chapter 2 we have proved the following relative version of Petri's theorem

Proposition 3.1.0.1. Let $f_1, \ldots, f_r \in S := \mathrm{Sym}H^0(X, \Omega_X) = k[\omega_1, \ldots, \omega_g]$ be quadratic polynomials which generate the canonical ideal I_X of a curve X defined over an algebraic closed field k. Any deformation $\mathscr{X}_A$ is given by quadratic polynomials $\tilde{f}_1, \ldots, \tilde{f}_r \in \mathrm{Sym}H^0(\mathscr{X}_A, \Omega_{\mathscr{X}_A/A}) = A[W_1, \ldots, W_g]$, which reduce to $f_1, \ldots, f_r$ modulo the maximal ideal $\mathfrak{m}_A$ of A.

And we also gave the following liftability criterion:

Theorem 3.1.1. Consider an epimorphism $R \to k \to 0$ of local Artin rings. Let X be a curve which is is canonically embedded in $\mathbb{P}^g_k$ and the canonical ideal is generated by quadratic polynomials, and acted on by the group G. The curve $X \to \mathrm{Spec}(k)$ can be lifted to a family $\mathscr{X} \to \mathrm{Spec}(R) \in D_{gl}(R)$ along with the G-action, if and only if the representation $\rho_k : G \to \mathrm{GL}_g(k) = \mathrm{GL}(H^0(X, \Omega_X))$ lifts to a representation $\rho_R : G \to \mathrm{GL}_g(R) = \mathrm{GL}(H^0(\mathscr{X}, \Omega_{\mathscr{X}/R}))$ and moreover the lift of the canonical ideal is left invariant by the action of $\rho_R(G)$.

In section 3.2 we collect results concerning deformations of HKG covers, Artin representations and orbit actions and also provide a geometric explanation of the KGB-obstruction in remark 3.2.2. In section 3.3 we prove that the HKG-cover is canonically generated by quadratic polynomials, therefore theorem 3 can be applied.

In order to decide whether a linear representation of $G = C_q \rtimes C_m$ can be lifted we will we use the following criterion for the lifting of the linear representation, based on the decomposition of a $k[G]$-module into intecomposable summands. We begin by describing the indecomposable $k[G]$-modules for the group $G = C_q \rtimes C_m$:

Proposition 3.1.1.1. Suppose that the group $G = C_q \rtimes C_m$ is represented in terms of generators

σ, τ and relations as follows:

$$G = \langle \sigma, \tau | \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^\alpha \rangle,$$

for some $\alpha \in \mathbb{N}, 1 \leqslant \alpha \leqslant p^h - 1, (\alpha, p) = 1$. Every indecomposable $k[G]$-module has dimension $1 \leqslant \kappa \leqslant q$ and is of the form $V_\alpha(\lambda, \kappa)$, where the underlying space of $V_\alpha(\lambda, \kappa)$ has the set of elements $\{(\tau - 1)^\nu e, \nu = 0, \dots, \kappa - 1\}$ as a basis for some $e \in V_\alpha(\lambda, \kappa)$, and the action of σ on e is given by $\sigma e = \zeta_m^\lambda e$, for a fixed primitive m-th root of unity.

Proof. We will prove this in the secon part, in section 4.3. Notice also that $(\tau - 1)^\kappa e = 0$. □

Remark 3.1.2. In the chapter 2 $V_\alpha(\lambda, \kappa)$ notation is used. In this chapter we will need the Galois module structure of the space of homolomorphic differentials of a curve and we will employ the results of [11], where the $U_{\ell,\mu}$ notation is used. These modules will be defined in section 3.5, notice that $V_\alpha(\lambda, \kappa) = U_{(\lambda + a_0(\kappa-1) \bmod m, \kappa}$, see lemma 3.5.0.1.

Notice that in section 3.5 we will give an alternative description of the indecomposable $k[G]$-modules, which is compatible with the results of [11].

Theorem 3.1.3. Consider a $k[G]$-module M which is decomposed as a direct sum

$$M = V_\alpha(\epsilon_1, \kappa_1) \oplus \cdots \oplus V_\alpha(\epsilon_s, \kappa_s).$$

The module lifts to an $R[G]$-module if and only if the set $\{1, \dots, s\}$ can be written as a disjoint union of sets I_ν, $1 \leqslant \nu \leqslant t$ so that

a. $\sum_{\mu \in I_\nu} \kappa_\mu \leqslant q$, for all $1 \leqslant \nu \leqslant t$.

b. $\sum_{\mu \in I_\nu} \kappa_\mu \equiv a \bmod m$ for all $1 \leqslant \nu \leqslant t$, where $a \in \{0, 1\}$.

c. For each ν, $1 \leqslant \nu \leqslant t$ there is an enumeration $\sigma : \{1, \dots, \#I_\nu\} \to I_\nu \subset \{1, .., s\}$, such that

$$\epsilon_{\sigma(2)} = \epsilon_{\sigma(1)} \alpha^{\kappa_{\sigma(1)}}, \epsilon_{\sigma(3)} = \epsilon_{\sigma(3)} \alpha^{\kappa_{\sigma(3)}}, \dots, \epsilon_{\sigma(s)} = \epsilon_{\sigma(s-1)} \alpha^{\kappa_{\sigma(s-1)}}.$$

Condition b., with $a = 1$ happens only if the lifted C_q-action in the generic fibre has an eigenvalue equal to 1 for the generator τ of C_q.

Proof. The above theorem is actually the proposition 4.1.0.1 and we prove it in part 4. □

The idea of this, is that indecomposable $k[G]$-modules in the decomposition of $H^0(X, \Omega_X)$ of the special fibre, should be combined together in order to give indecomposable modules in the decomposition of holomorphic differentials of the relative curve.

We will have the following strategy. We will consider a HKG-cover

$$\begin{tikzcd} X \arrow[r, "C_q"'] \arrow[rr, bend left, "G"] & \mathbb{P}^1 \arrow[r, "C_m"'] & \mathbb{P}^1 \end{tikzcd}$$

of the G-action. This has a cyclic subcover $X \to \mathbb{P}^1$ with Galois group C_q. We lift this cover using Oort's conjecture for C_q-groups to a cover $\mathscr{X} \to \operatorname{Spec}\Lambda$. This gives rise to a representation

$$\rho : G \longrightarrow \mathrm{GL} H^0(X, \Omega_X), \tag{3.1}$$

together with a lifting

$$\begin{tikzcd} & \mathrm{GL}H^0(\mathscr{X}, \Omega_{\mathscr{X}/\Lambda}) = \mathrm{GL}_g(\Lambda) \arrow[d, "\bmod m_\Lambda"] \\ C_q \arrow[ur] \arrow[r] & \mathrm{GL}H^0(X, \Omega_X) = \mathrm{GL}_g(k) \end{tikzcd} \tag{3.2}$$

of the representation of the cyclic part C_q of G. We then lift, checking the conditions of theorem 3.1.3 the linear action of eq. (3.1) in characteristic zero in a such a way that the restriction to the C_q

group is our initial lifting of the representation of the C_q subgroup coming from the lifting assured by Oort's conjecture given in eq. (3.2). Notice that the lifting of the cyclic group acting on a curve of characteristic zero in the generic fibre has the additional property that every eigenvalue of a generator of C_q is different than one, see eq. 3.4.0.1. Then using theorem 3 we will modify the initial lifting $\mathscr{X}$ to a lifting $\mathscr{X}'$ so that $\mathscr{X}'$ is acted on by G.

Notice that $m = 2$, that is for the case of dihedral groups D_q of order $2q$, there is no need to pair two indecomposable $k[D_q]$-modules together in order to lift them into an indecomposable $R[D_q]$-module. The sets I_ν can be singletons and the conditions of theorem 3.1.3 are trivially satisfied. For example, condition 3.1.3.b. does not give any information since every integer is either odd or even. This means that the linear representations always lift.

In our geometric setting on the other hand, we know that in the generic fibre cyclic actions do not have identity eigenvalues, see proposition 3.4.0.1. This means that we have to consider lifts that satisfy 3.1.3.b. with $a = 0$. Therefore, indecomposable modules for $G = C_q \rtimes C_2 = D_q$ of odd dimmension d_1 should find an other indecomposable module of odd dimension d_2 in order lift to an $R[G]$-indecomposable module of even dimension $d_1 + d_2$. Moreover this dimension should satisfy $d_1 + d_2 \leqslant q$. If we also take care of the condition 3.1.3.c. we arrive at the following

Criterion 3.1.4. The HKG-curve with action of D_q lifts in characteristic zero if and only if all indecomposable summands $V_\alpha(\epsilon, d)$, where $\epsilon \in \{0,1\}$ and $1 \leqslant d \leqslant q^h$ with d odd have a pair $V_\alpha(\epsilon', d')$, with $\epsilon' \in \{0,1\} - \{\epsilon\}$ and d' odd and $d + d' \leqslant q^h$. Notice that since, d, d' are both odd we have

$$V_\alpha(\epsilon, d) = U_{\epsilon+d-1 \bmod 2, d} = U_{\epsilon, d}, \quad V_\alpha(\epsilon', d') = U_{\epsilon'+d'-1 \bmod 2, d'} = U_{\epsilon', d'}.$$

The indecomposable modules given above will be called complementary. We will apply this criterion for complementary modules in the $U_{\epsilon,d}$-notation.

In section 3.4 we will show that given a lifting $\mathscr{X}$ of the C_q action using Oort conjecture, and a lifting of the linear representation satisfying criterion 3.1.4 the lift $\mathscr{X}$ can be modified to a lift $\mathscr{X}'$, which lifts the action of D_q. In order to apply this idea we need a detailed study of the direct $k[G]$-summands of $H^0(X, \Omega_X)$, for $G = C_q \rtimes C_m$. This is considered in section 3.5, where we employ the joint work of F. Bleher and T. Chinburg and A. Kontogeorgis [11], in order to compute the decomposition of $H^0(X, \Omega_X)$ into indecomposable kG-modules, in terms of the ramification filtration of the local action.

Then the lifting criterion of theorem 3.1.3 is applied. Our method gives rise to an algorithm which takes as input a group $C_q \rtimes C_m$, with a given sequence of lower jumps and decides whether the action lifts to characteristic zero.

In section 3.5a we give an example of an $C_{125} \rtimes C_4$ HKG-curve which does not lift and then we restrict ourselves to the case of dihedral groups. The possible ramification filtrations for local actions of the group $C_q \rtimes C_m$ were computed in the work of A. Obus and R. Pries in [62]. We focus on the case of dihedral groups D_q with lower jumps

$$b_\ell = w_0 \frac{p^{2\ell} + 1}{p + 1}, 0 \leqslant \ell \leqslant h - 1. \tag{3.3}$$

For the values $w_0 = 9$ we show in this section that the local action does not lift, providing a counterexample to the conjecture that the KGB-obstruction is the only obstruction to the local lifting problem.

Finally, in section 3.5b we prove that the jumps of eq. (3.3) for the value $w_0 = 1$ lift in characteristic zero. This result is a special case of the result of A. Obus in [61, Th. 8.7] proved by completely different methods.

We also have developed a program in sage [74] in order to compute the decomposition of $H^0(X, \Omega_X)$ into intecomposable summands, which is freely available[1].

In the last chapter of the first part we will study metacyclic groups $G = C_q \rtimes C_m$, where $q = p^h$ is a power of the characteristic and $m \in \mathbb{N}, (m, p) = 1$. Let τ be a generator of the cyclic group C_q and σ be a generator of the cyclic group C_m.

The group G is given in terms of generators and relations as follows:

$$G = \langle \sigma, \tau | \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^\alpha \rangle, \tag{3.4}$$

for some $\alpha \in \mathbb{N}, 1 \leqslant \alpha \leqslant p^h - 1, (\alpha, p) = 1$. The integer α satisfies the following congruence:

$$\alpha^m \equiv 1 \bmod q \tag{3.5}$$

[1] https://www.dropbox.com/sh/uo0dg91l0vuqulr/AACarhRxsru_zuIp5ogLvy6va?dl=0

as one sees by computing $\tau = \sigma^m \tau \sigma^{-m} = \tau^{\alpha^m}$. Also the α can be seen as an element in the finite field $\mathbb{F}_p$, and it is a $(p-1)$-th root of unity, not necessarily primitive. In particular the following holds:

Lemma 3.1.4.1. Let ζ_m be a fixed primitive m-th root of unity. There is a natural number a_0, $0 \leqslant a_0 < m-1$ such that $\alpha = \zeta_m^{a_0}$.

Proof. The integer α if we see it as an element in k is an element in the finite field $\mathbb{F}_p \subset k$, therefore $\alpha^{p-1} = 1$ as an element in $\mathbb{F}_p$. Let $\mathrm{ord}_p(\alpha)$ be the order of α in $\mathbb{F}_p^*$. By eq. 3.5 we have that $\mathrm{ord}_p(\alpha) \mid p-1$ and $\mathrm{ord}_p(\alpha) \mid m$, that is $\mathrm{ord}_p(\alpha) \mid (p-1, m)$.

The primitive m-th root of unity ζ_m generates a finite field $\mathbb{F}_p(\zeta_m) = \mathbb{F}_{p^\nu}$ for some integer ν, which has cyclic multiplicative group $\mathbb{F}_{p^\nu} \backslash \{0\}$ containing both the cyclic groups $\langle \zeta_m \rangle$ and $\langle \alpha \rangle$. Since for every divisor δ of the order of a cyclic group C there is a unique subgroup $C' < C$ of order δ we have that $\alpha \in \langle \zeta_m \rangle$, and the result follows. □

Remark 3.1.5. For the case $C_q \rtimes C_m$ the KGB-obstruction vanishes if and only if the first lower jump h satisfies $h \equiv -1 \bmod m$. For this to happen the conjugation action of C_m on C_q has to be faithful, see [59, prop. 5.9]. Also notice that by [62, th. 1.1], that if $u_0, u_1, \ldots, u_{h-1}$ is the sequence upper ramification jumps for the C_q subgroup, then the condition $h \equiv -1 \bmod m$, then all upper jumps $u_i \equiv -1 \bmod m$. In remark 3.2.2 we will explain the necessity of the KGB-obstruction in terms of the action of C_m, on the fixed horizontal divisor of the C_q group.

3.2 Deformation of covers

3.2a Splitting the branch locus

Consider a deformation $\mathscr{X} \to \mathrm{Spec} A$ of the curve X together with the action of G. Denote by $\tilde{\tau} = \tilde{\rho}(\tau)$ a lift of the action of the element $\tau \in \mathrm{Aut}(X)$. Weierstrass preparation theorem [13, prop. VII.6] implies that:

$$\tilde{\tau}(T) - T = g_{\tilde{\tau}}(T) u_{\tilde{\tau}}(T),$$

where $g_{\tilde{\tau}}(T)$ is a distinguished Weierstrass polynomial of degree $m+1$ and $u_{\tilde{\tau}}(T)$ is a unit in $R[[T]]$.

The polynomial $g_{\tilde{\tau}}(T)$ gives rise to a horizontal divisor that corresponds to the fixed points of $\tilde{\tau}$. This horizontal divisor might not be irreducible. The branch divisor corresponds to the union of the fixed points of any element in $G_1(P)$. Next lemma gives an alternative definition of a horizontal branch divisor for the relative curves $\mathscr{X} \to \mathscr{X}^G$, that works even when G is not a cyclic group.

Lemma 3.2.0.1. Let $\mathscr{X} \to \mathrm{Spec} A$ be an A-curve, admitting a fibrewise action of the finite group G, where A is a Noetherian local ring. Let $S = \mathrm{Spec} A$, and $\Omega_{\mathscr{X}/S}$, $\Omega_{\mathscr{Y}/S}$ be the sheaves of relative differentials of $\mathscr{X}$ over S and $\mathscr{Y}$ over S, respectively. Let $\pi : \mathscr{X} \to \mathscr{Y}$ be the quotient map. The sheaf

$$\mathscr{L}(-D_{\mathscr{X}/\mathscr{Y}}) = \Omega^{-1}_{\mathscr{X}/S} \otimes_S \pi^* \Omega_{\mathscr{Y}/S}$$

is the ideal sheaf of the horizontal Cartier divisor $D_{\mathscr{X}/\mathscr{Y}}$. The intersection of $D_{\mathscr{X}/\mathscr{Y}}$ with the special and generic fibre of $\mathscr{X}$ gives the ordinary branch divisors for curves.

Proof. We will first prove that the above defined divisor $D_{\mathscr{X}/\mathscr{Y}}$ is indeed an effective Cartier divisor. According to [45, Cor. 1.1.5.2] it is enough to prove that

- $D_{\mathscr{X}/\mathscr{Y}}$ is a closed subscheme which is flat over S.
- for all geometric points $\mathrm{Spec} k \to S$ of S, the closed subscheme $D_{\mathscr{X}/\mathscr{Y}} \otimes_S k$ of $\mathscr{X} \otimes_S k$ is a Cartier divisor in $\mathscr{X} \otimes_S k/k$.

In our case the special fibre is a nonsingular curve. Since the base is a local ring and the special fibre is nonsingular, the deformation $\mathscr{X} \to \mathrm{Spec} A$ is smooth. (See the remark after the definition 3.35 p.142 in [55]). The smoothness of the curves $\mathscr{X} \to S$, and $\mathscr{Y} \to S$, implies that the sheaves $\Omega_{\mathscr{X}/S}$ and $\Omega_{\mathscr{X}/S}$ are S-flat, [55, cor. 2.6 p.222].

On the other hand the sheaf $\Omega_{\mathscr{Y},\mathrm{Spec} A}$ is by [45, Prop. 1.1.5.1] $\mathscr{O}_{\mathscr{Y}}$-flat. Therefore, $\pi^*(\Omega_{\mathscr{Y},\mathrm{Spec} A})$ is $\mathscr{O}_{\mathscr{X}}$-flat and $\mathrm{Spec} A$-flat [36, Prop. 9.2]. Finally, observe that the intersection with the special and generic fibre is the ordinary branch divisor for curves according to [36, IV p.301]. □

For a curve X and a branch point P of X we will denote by $i_{G,P}$ the order function of the filtration of G at P. The Artin representation of the group G is defined by $\mathrm{ar}_P(\sigma) = -f_P i_{G,P}(\sigma)$ for $\sigma \neq 1$ and $\mathrm{ar}_P(1) = f_P \sum_{\sigma\neq 1} i_{G,P}(\sigma)$ [72, VI.2]. We are going to use the Artin representation at both the special and generic fibre. In the special fibre we always have $f_P = 1$ since the field k is algebraically closed. The field of quotients of A should not be algebraically closed therefore a fixed point there might have $f_P \geqslant 1$. The integer $i_{G,P}(\sigma)$ is equal to the multiplicity of $P \times P$ in the intersection of $\Delta.\Gamma_\sigma$ in the relative A-surface $\mathscr{X} \times_{\mathrm{Spec}A} \mathscr{X}$, where Δ is the diagonal and Γ_σ is the graph of σ [72, p. 105].

Since the diagonals Δ_0, Δ_η and the graphs of σ in the special and generic fibres respectively of $\mathscr{X} \times_{\mathrm{Spec}A} \mathscr{X}$ are algebraically equivalent divisors we have:

Proposition 3.2.0.1. Assume that A is an integral domain, and let $\mathscr{X} \to \mathrm{Spec}A$ be a deformation of X. Let $\bar{P}_i$, $i = 1, \cdots, s$ be the horizontal branch divisors that intersect at the special fibre, at point P, and let P_i be the corresponding points on the generic fibre. For the Artin representations attached to the points P, P_i we have:

$$\mathrm{ar}_P(\sigma) = \sum_{i=1}^{s} \mathrm{ar}_{P_i}(\sigma). \tag{3.6}$$

This generalizes a result of J. Bertin [8]. Moreover if we set $\sigma = 1$ to the above formula we obtain a relation for the valuations of the differents in the special and the generic fibre, since the value of the Artin's representation at 1 is the valuation of the different [72, prop. 4.IV,prop. 4.VI]. This observetion is equivalent to claim 3.2 in [29] and is one direction of a local criterion for good reduction theorem proved in [29, 3.4], [44, sec. 5].

3.2b The Artin representation on the generic fibre

We can assume that after a base change of the family $\mathscr{X} \to \mathrm{Spec}(A)$ the points P_i at the generic fibre have degree 1. Observe also that at the generic fibre the Artin representation can be computed as follows:

$$\mathrm{ar}_Q(\sigma) = \begin{cases} 1 \text{ if } \sigma(Q) = Q, \\ 0 \text{ if } \sigma(Q) \neq Q. \end{cases}$$

The set of points $S := \{P_1, \ldots, P_s\}$ that are the intersections of the ramification divisor and the generic fibre are acted on by the group G.

We will now restrict our attention to the case of a cyclic group $H = C_q$ of order q. Let S_k be the subset of S fixed by $C_{p^{h-k}}$, i.e.

$$P \in S_k \text{ if and only if } H(P) = C_{p^{h-k}}.$$

Let s_k be the order of S_k. Observe that since for a point Q in the generic fibre $\sigma(Q)$ and Q have the same stabilizers (in general they are conjugate, but here H is abelian) the sets S_k are acted on by H. Therefore $\#S_k =: s_k = p^k i_k$ where i_k is the number of orbits of the action of H on S_k.

Let $b_0, b_1, \ldots, b_{h-1}$ be the jumps in the lower ramification filtration. Observe that

$$H_{j_k} = \begin{cases} C_{p^{h-k}} & \text{for } 0 \leqslant k \leqslant h-1 \\ \{1\} & \text{for } k \geqslant h. \end{cases}$$

An element in H_{b_k} fixes only elements in S with stabilizers that contain H_{b_k}. So H_{b_0} fixes only S_0, H_{b_1} fixes both S_0 and S_1 and H_{b_k} fixes all elements in $S_0, S_1, \ldots, S_k$. By definition of the Artin representation an element σ in $H_{b_k} - G_{b_{k+1}}$ satisfies $\mathrm{ar}_P(\sigma) = b_k + 1$ and by using equation (3.6) we arive at

$$b_k + 1 = i_0 + p i_1 + \cdots + p^k i_k.$$

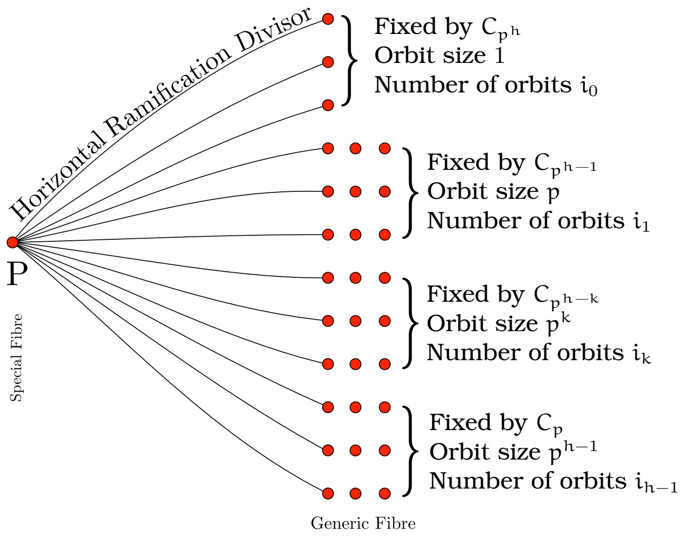
Remark 3.2.1. This gives us a geometric interpretation of the Hasse-Arf theorem, which states that for the cyclic p-group of order $q = p^h$, the lower ramification filtration is given by

$$H_0 = H_1 = \cdots = H_{b_0} \gneqq H_{b_0+1} = \cdots = H_{b_1} \gneqq H_{b_1+1} = \cdots = H_{b_{h-1}} \gneqq \{1\},$$

i.e. the jumps of the ramification filtration appear at the integers $b_0, \ldots, b_{h-1}$. Then

$$b_k + 1 = i_0 + i_1 p + i_2 p^2 + \cdots + i_k p^k. \tag{3.7}$$

Figure 3.1: The horizontal Ramification divisor



The set of horizontal branch divisors is illustrated in figure 3.1. Notice that the group C_m acts on the set of ramification points of $H = C_q$ on the special fibre but it can't fix any of them since there are already fixed by a subgroup of C_q and if a branch point P of C_q was also fixed by an element of C_m, then the isotropy subgroup of P could not be cyclic. This proves that m divides the numbers of all orbits $i_0, \dots, i_{n-1}$.

Remark 3.2.2. In this way we can recover the necessity of the KGB-obstruction since by eq. (3.7) the upper ramification jumps are $i_0 - 1, i_0 + i_1 - 1, \dots, i_0 + \cdots + i_{n-1} - 1$.

The Galois cover $X \to X/G$ breaks into two covers $X \to X^{C_q}$ and $X^{C_q} \to C^G$. The genus of C^G is zero by assumption and in the cover $X^{C_q} \to C^G$ there are exactly two ramified points with ramification indices m. An application of the Riemann-Hurwitz formula shows that the genus of X^{C_q} is zero as well.

The genus of the curve X can be computed either by the Riemann-Hurwitz formula in the special fibre

$$\begin{aligned} g &= 1 - p^n + \frac{1}{2}\sum_{i=0}^{\infty}(|G_i| - 1) \\ &= 1 - p^n + \frac{1}{2}\big((b_0+1)(p^n-1) + (b_1 - b_0)(p^{n-1}-1) \\ &+ (b_2 - b_1)(p^{n-2}-1) + \cdots + (b_n - b_{n-1})(p-1)\big) \end{aligned}$$

or by the Riemann-Hurwitz formula on the generic fibre:

$$g = 1 - p^n + \frac{1}{2}\left(i_0(p^n-1) + i_1 p(p^{n-1}-1) + \cdots i_{n-1}p^{n-1}(p-1)\right). \tag{3.8}$$

Using eq. (3.7) we see that the two formulas for g give the same result as expected.

3.3 HKG-covers and their canonical ideal

Lemma 3.3.0.1. Consider the Harbater-Katz-Gabber curve corresponding to the local group action $C_q \rtimes C_m$, where $q = p^h$ that is a power of the characteristic p. If one of the following conditions holds:

- $h \geqslant 3$ or $h = 2, p > 3$
- $h = 1$ and the first jump i_0 in the ramification filtration for the cyclic group satisfies $i_0 \neq 1$ and $q \geqslant \frac{12}{i_0 - 1} + 1$,

then the curve X has canonical ideal generated by quadratic polynomials.

Remark 3.3.1. Notice, that the missing cases in the above lemma which satisfy the KGB obstruction, are all either cyclic, D_3 or D_9, which are all known local Oort groups.

Proof. Using Petri's Theorem [67] it is enough to prove that the curve X has genus $g \geqslant 6$ provided that p or h is big enough. We will also prove that the curve X is not hyperelliptic nor trigonal.

Remark 3.3.2. Let us first recall that a cyclic group of order $q = p^h$ for $h \geqslant 2$ can not act on the rational curve, see [79, thm 1]. Also let us recall that a cyclic group of order p can act on a rational curve and in this case the first and only break in the ramification filtration is $i_0 = 1$. This latter case is excluded.

Consider first the case $p^h = p$ and $i_0 \neq 1$. In this case we compute the genus g of the HKG-curve X using Riemann-Hurwitz formula:

$$2g = 2 - 2mq + q(m-1) + qm - 1 + i_0(q-1),$$

where the contribution $q(m-1)$ is from the q-points above the unique tame ramified point, while $qm - 1 + i_0(q-1)$ is the contribution of the wild ramified point. This implies that,

$$2g = (i_0 - 1)(q-1),$$

therefore if $i_0 \geqslant 2$, it suffices to have $q = p^h \geqslant 13$ and more generally it is enough to have $q \geqslant \frac{12}{i_0-1} + 1$ in order to ensure that $g \geqslant 6$.

For the case $h \geqslant 2$, we can write a stronger inequality based on Riemann-Hurwitz theorem as (recall that $i_0 \equiv i_1 \bmod p$ so $i_0 - i_1 \geqslant p$)

$$2g \geqslant (i_0-1)(p^h-1) + (i_0 - i_1)(p^{h-1} - 1) \geqslant p^h - p, \tag{3.9}$$

which implies that $g \geqslant 6$ for $p > 3$ or $h > 3$.

In order to prove that the curve is not hyperelliptic we observe that hyperelliptic curves have a normal subgroup generated by the hyperelliptic involution j, so that $X \to X/\langle j \rangle = \mathbb{P}^1$. It is known that the automorphism group of a hyperelliptic curve fits in the short exact sequence

$$1 \to \langle j \rangle \to \mathrm{Aut}(X) \to H \to 1, \tag{3.10}$$

where H is a subgroup of $\mathrm{PGL}(2,k)$, see [15]. If m is odd then the hyperelliptic involution is not an element in C_m. If m is even, let σ be a generator of the cyclic group of order m and τ a generator of the group C_q. The involution $\sigma^{m/2}$ again can't be the hyperelliptic involution. Indeed, the hyperelliptic involution is central, while the conjugation action of σ on τ is faithful that is $\sigma^{m/2}\tau\sigma^{-m/2} \neq \tau$. In this case $G = C_q \rtimes C_m$ is a subgroup of H which should act on the rational function field. By the classification of such groups in [79, Th. 1] this is not possible. Thus X can't be hyperelliptic.

We will prove now that the curve is not trigonal. Using Clifford's theorem we can show [4, B-3 p.137] that a non-hyperelliptic curve of genus $g \geqslant 5$ cannot have two distinct g^1_3. Notice that we have already required the stronger condition $g \geqslant 6$. So if there is a g^1_3, then this is unique. Moreover, the g^1_3 gives rise to a map $\pi : X \to \mathbb{P}^1$ and every automorphism of the curve X fixes this map. Therefore, we obtain a morphism $\phi : C_q \rtimes C_m \to \mathrm{PGL}_2(k)$ and we arrive at the short exact sequence

$$1 \to \ker\phi \to C_q \rtimes C_m \to H \to 1,$$

for some finite subgroup H of $\mathrm{PGL}(2,k)$. If $\ker\phi = \{1\}$, then we have the tower of curves $X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\pi'} \mathbb{P}^1$, where π' is a Galois cover with group $C_q \rtimes C_m$. This implies that X is a rational curve contradicting remark 3.3.2. If $\ker\phi$ is a cyclic group of order 3, then we have that $3 \mid m$ and the tower $X \xrightarrow{\pi} \mathbb{P}^1 \xrightarrow{\pi'} \mathbb{P}^1$, where π is a cyclic Galois cover of order 3 and π' is a Galois cover with group $C_q \rtimes C_{m/3}$. As before this contradicts remark 3.3.2 and is not possible. ∎

3.4 Invariant subspaces of vector spaces

The $g \times g$ symmetric matrices $A_1, \ldots, A_r$ defining the quadratic canonical ideal of the curve X, define a vector subspace of the vector space V of $g \times g$ symmetric matrices. By Oort conjecture, we know that there are symmetric matrices $\tilde{A}_1, \ldots, \tilde{A}_r$ with entries in a local principal ideal domain R, which reduce to the initial matrices $A_1, \ldots, A_r$. These matrices $\tilde{A}_1, \ldots, \tilde{A}_r$ correspond to the lifted relative curve $\tilde{X}$. Moreover, the submodule $\tilde{V} = \langle \tilde{A}_1, \ldots, \tilde{A}_r \rangle$ is left invariant under the action of a lifting $\tilde{\rho}$ of the representation $\rho : C_q \to \mathrm{GL}_g(k)$.

Proposition 3.4.0.1. Let $\tilde{g}$ be the genus of the quotient curve X/H for a subgroup H of the automorphism group of a curve X in characteristic zero. We have

$$\dim H^0(X, \Omega_X^{\otimes d})^H = \begin{cases} \tilde{g} & \text{if } d = 1 \\ (2d-1)(\tilde{g}-1) + \sum_{P \in X/G} \left\lfloor d\left(1 - \frac{1}{e(\tilde{P})}\right)\right\rfloor & \text{if } d > 1 \end{cases}$$

Proof. See [28, eq. 2.2.3,2.2.4 p. 254]. □

Therefore, a generator of C_q acting on $H^0(X, \Omega_X)$ has no identity eigenvalues and m should divide g. This means that we have to consider liftings of indecomposable summands of the C_q-module $H^0(X, \Omega_X)$, which satisfy condition 3.1.3.b. with $a = 0$. We now assume that condition 3.1.3.b. of theorem 3.1.3 can be fulfilled, so there is a lifting of the representation

$$\begin{tikzcd} & \mathrm{GL}_g(R) \arrow[d, "\mathrm{mod}\, m_R"] \\ C_q \rtimes C_m \arrow[ur, "\tilde{\rho}"] \arrow[r, "\rho"'] & \mathrm{GL}_g(k) \end{tikzcd}$$

satisfying condition, see also the discussion in the introduction after the statement of this theorem after eq. (3.2).

We have to show that we can modify the space $\tilde{V} \subset \mathrm{Sym}_g(R)$ to a space $\tilde{V}'$ with the same reduction V modulo m_R so that $\tilde{V}$ is $C_q \rtimes C_m$-invariant.

Consider the sum of the free modules

$$W = \tilde{V} + \tilde{\rho}(\sigma)\tilde{V} + \tilde{\rho}(\sigma^2)\tilde{V} + \cdots + \tilde{\rho}(\sigma^{m-1})\tilde{V} \subset R^N.$$

Observe that W is an $R[C_q \rtimes C_m]$-module and also it is a free submodule of R^N and by the theory of modules over local principal ideal domain there is a basis $E_1, \ldots, E_N$ of R^N such that

$$W = E_1 \oplus \cdots \oplus E_r \oplus \pi^{a_{r+1}} E_{r+1} \oplus \cdots \oplus \pi^{a_N} E_N,$$

where $E_1, \ldots, E_r$ form a basis of $\tilde{V}$, while $\pi^{a_{r+1}} E_{r+1}, \ldots, \pi^{a_N} E_N$ form a basis of the kernel W_1 of the reduction modulo m_R. Since the reduction is compatible with the actions of $\rho, \tilde{\rho}$ we have that W_1 is an $R[C_q \rtimes C_m]$-module, while $\tilde{V}$ is just a C_q-module.

Let π be the $R[C_q]$-equivariant projection map $W = \tilde{V} \oplus_{R[C_q]-\text{modules}} W_1 \to W_1$. Since m is an invertible element of R, we can employ the proof of Mascke's theorem in order to construct a module $\tilde{V}'$, which is $R[C_q \rtimes C_m]$ stable and reduces to V modulo m_R, see also [1, I.3 p.12]. Indeed, consider the endomorphism $\bar{\pi} : W \to W$ defined by

$$\bar{\pi} = \frac{1}{m} \sum_{i=0}^{m-1} \tilde{\rho}(\sigma^i)\pi\tilde{\rho}(\sigma^{-i}).$$

We see that $\bar{\pi}$ is the identity on W_1 since π is the identity on W_1. Moreover $\tilde{V}' := \ker\bar{\pi}$ is both C_q and C_m invariant and reduces to V modulo m_R.

3.5 Galois module structure of holomorphic differentials, special fibre

Consider the group $C_q \rtimes C_m$. Let τ be a generator of C_q and σ a generator of C_m. It is known that $\mathrm{Aut}(C_q) \cong \mathbb{F}_p^* \times Q$, for some abelian group Q. The representation $\psi : C_m \to \mathrm{Aut}(C_q)$ given by the action of C_m on C_q is known to factor through a character $\chi : C_m \to \mathbb{F}_p^*$. The order of χ divides $p-1$ and $\chi^{p-1} = \chi^{-(p-1)}$ is the trivial one dimensional character. In our setting, using the definition of G given in eq. (3.4) and lemma 3.1.4.1 we have that the character χ is defined by

$$\chi(\sigma) = \alpha = \zeta_m^{a_0} \in \mathbb{F}_p. \tag{3.11}$$

For all $i \in \mathbb{Z}$, χ^i defines a simple $k[C_m]$-module of k dimension one, which we will denote by S_{χ^i}. For $0 \leqslant \ell \leqslant m-1$ denote by S_ℓ the simple module on which σ acts as ζ_m^ℓ. Both S_{χ^i}, S_ℓ can be seen as $k[C_q \rtimes C_m]$-modules using inflation. Finally for $0 \leqslant \ell \leqslant m-1$ we define $\chi^i(\ell) \in \{0, 1, \ldots, m-1\}$ such that $S_{\chi^i(\ell)} \cong S_\ell \otimes_k S_{\chi^i}$. Using eq. (3.11) we arrive at

$$S_{\chi^i(\ell)} = S_{\ell + i a_0}. \tag{3.12}$$

There are $q \cdot m$ isomorphism classes of indecomposable $k[C_q \rtimes C_m]$-modules and are all uniserial, i.e. the set of submodules are totally ordered by inclusion. An indecomposable $k[C_q \rtimes C_m]$-module U is uniquely determined by its socle, which is the kernel of the action of $\tau - 1$ on U, and its k-dimension. For $0 \leqslant \ell \leqslant m-1$ and $1 \leqslant \mu \leqslant q$, let $U_{\ell,\mu}$ be the indecomposable $k[C_q \rtimes C_m]$ module with socle S_ℓ and k-dimension μ. Then $U_{\ell,\mu}$ is uniserial and its μ ascending composition factors are the first μ composition factors of the sequence

$$S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}, S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}$$

Lemma 3.5.0.1. There is the following relation between the two different notations for indecomposable modules:

$$V_\alpha(\lambda, \kappa) = U_{(\lambda + a_0(\kappa-1)) \bmod m, \kappa)}$$

In particular, for the case of dihedral groups D_q we have the relation

$$V_\alpha(\lambda, \kappa) = U_{\lambda+\kappa-1 \bmod 2, \kappa}.$$

Proof. Indeed, in the $V_\alpha(\lambda, \kappa)$ notation we describe the action of σ on the generator e, by assuming that $\sigma e = \zeta_m^\lambda e$. We can then describe the action on every basis element $e_i = (\tau-1)^{i-1} e$, using the group relations

$$\sigma e_i = \sigma(\tau-1)^{i-1} e = (\tau^\alpha - 1)^{i-1} \sigma e = \zeta_m^\lambda (\tau^\alpha - 1)^{i-1} e$$

This allows us to prove, see lemma 4.3.0.2 that

$$\sigma e_i = \alpha^{i-1} \zeta_m^\lambda + \sum_{\nu=i+1}^{\kappa} a_\nu e_\nu$$

for some elements $a_\nu \in k$ and in particular

$$\sigma e_\kappa = \alpha^{\kappa-1} \zeta_m^\lambda.$$

Recall that the number $\alpha = \zeta_m^{a_0}$ for some natural number a_0, $0 \leqslant a_0 < m-1$, see also lemma 4.2.0.1. In the $U_{\mu,\kappa}$ notation, μ is the action on the one-dimensional socle which is the τ-invariant element $e_\kappa = (\tau-1)^{\kappa-1} e$, i.e. $\sigma(e_\kappa) = \zeta_m^\mu$. Putting all this together we have

$$\mu = \lambda + (\kappa-1) a_0 \bmod m.$$

In the case of dihedral group D_q, $m = 2$ and $\alpha = -1^{a_0}$, i.e. $a_0 = 1$, we have $V_\alpha(\lambda, \kappa) = U_{\lambda+\kappa-1 \bmod 2, \kappa}$. □

Assume that $X \to \mathbb{P}^1$ is an HKG-cover with Galois group $C_q \rtimes C_m$. The subgroup I generated by the Sylow p-subgroups of the inertia groups of all closed points of X is equal to C_q.

Definition 3.5.0.1. In [11] for each $0 \leqslant j \leqslant q-1$ the divisor

$$D_j = \sum_{y \in \mathbb{P}^1} d_{y,j} y,$$

is defined, where the integers $d_{y,j}$ are given as follows. Let x be a point of X above y and consider the i-th ramification group $I_{x,i}$ at x. The order of the inertia group at x is assumed to be $p^{n(x)}$ and t $i(x) = h - n(x)$ is defined. In our work we will have HKG-covers, where $n(x) = h$, so $i(x) = 0$. We will use this in order to simplify the notation in what follows.

Let $b_0, b_1, \dots, b_{h-1}$ be the jumps in the numbering of the lower ramification filtration subgroups of I_x. We define

$$d_{y,j} = \left\lfloor \frac{1}{p^h} \sum_{l=1}^{h} p^{h-l}\big(p - 1 + (p - 1 - a_{l,t})b_{l-1}\big) \right\rfloor$$

for all $j \geqslant 0$ with p-adic expansion

$$j = a_{1,j} + a_{2,j}p + \cdots + a_{h,j}p^{h-1}$$

In particular $D_{q-1} = 0$. Observe that $d_{y,j} \neq 0$ only for wildly ramified branch points.

Remark 3.5.1. For a divisor D on a curve Y define $\Omega_Y(D) = \Omega_Y \otimes \mathscr{O}_Y(D)$. In particular for $Y = \mathbb{P}^1$, and for $D = D_j = d_{P_\infty,j}P_\infty$, where D_j is a divisor supported at the infinity point P_∞ we have

$$H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_j)) = \{f(x)dx : 0 \leqslant \deg f(x) \leqslant d_{P_\infty,j} - 2\}.$$

For the sake of simplicity, we will denote $d_{P_\infty,j}$ by d_j. The space $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_j))$ has a basis given by $B = \{dx, xdx, \dots, x^{d_j-2}dx\}$. Therefore, the number $n_{j,\ell}$ of simple modules appearing in the decomposition $\Omega_{\mathbb{P}^1}(D_j)$ isomorphic to S_ℓ for $0 \leqslant \ell < m$, is equal to the number of monomials x^ν with

$$\nu \equiv \ell - 1 \bmod m, 0 \leqslant \nu \leqslant d_j - 2.$$

If $d_j \leqslant 1$ then $B = \emptyset$ and $n_{j,\ell} = 0$ for all $0 \leqslant \ell < m$. If $d_j > 1$, then we know that in the $d_j - 1$ elements of the basis B, the first $m\left\lfloor \frac{d_j-1}{m} \right\rfloor$ elements contribute to every representative modulo m. Thus, we have at least $\left\lfloor \frac{d_j-1}{m} \right\rfloor$ elements in isomorphic to S_ℓ for every $0 \leqslant \ell < m$. We will now count the rest elements, of the form $\{x^\nu dx\}$, where

$$m\left\lfloor \frac{d_j - 1}{m} \right\rfloor \leqslant \nu \leqslant d_j - 2 \text{ and } \nu \equiv \overline{\ell - 1} \bmod m,$$

where $\overline{\ell - 1}$ is the unique integer in $\{0, 1, \dots, m-1\}$ equivalent to $\ell - 1$ modulo m. We observe that the number $y_j(\ell)$ of such elements ν is given by

$$y_j(\ell) = \begin{cases} 1 & \text{if } \overline{\ell - 1} \leqslant d_j - 2 - m\left\lfloor \frac{d_j-1}{m} \right\rfloor \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$n_{j,\ell} = \begin{cases} \left\lfloor \frac{d_j-1}{m} \right\rfloor + y_j(\ell) & \text{if } d_j \geqslant 2 \\ 0 & \text{if } d_j \leqslant 1 \end{cases}$$

For example if $d_j = 9$ and $m = 3$, then a basis for $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(9P_\infty))$ is given by $\{dx, xdx, x^2dx, \dots x^7dx\}$. This basis has 8 elements, and each triple $\{dx, xdx, x^2dx\}$, $\{x^3dx, x^4dx, x^5dx\}$ contributes one to each class S_0, S_1, S_2, while there are two remaining basis elements $\{x^6dx, x^7dx, \}$, which contribute one to S_1, S_2. Notice that $\left\lfloor \frac{8}{3} \right\rfloor = 2$ and $y(\ell) = 1$ for $\ell = 1, 2$.

In particular if $m = 2$, then $n_{j,\ell} = 0$ if $d_j \leqslant 1$ and for $d_j \geqslant 2$ we have

$$n_{j,\ell} = \begin{cases} \frac{d_j-1}{2} & \text{if } d_j \equiv 1 \bmod 2 \\ \frac{d_j}{2} - 1 & \text{if } \ell = 0 \text{ and } d_j \equiv 0 \bmod 2 \\ \frac{d_j}{2} & \text{if } \ell = 1 \text{ and } d_j \equiv 0 \bmod 2 \end{cases} \tag{3.13}$$

Lemma 3.5.1.1. Assume that $d_{j-1} = d_j + 1$. Then if $d_j \geqslant 2$

$$n_{j-1,\ell} - n_{j,\ell} = \begin{cases} 1 & \text{if } d_{j-1} \equiv 1 \bmod 2 \text{ and } \ell = 0 \\ & \text{or } d_{j-1} \equiv 0 \bmod 2 \text{ and } \ell = 1 \\ 0 & \text{if } d_{j-1} \equiv 1 \bmod 2 \text{ and } \ell = 1 \\ & \text{or } d_{j-1} \equiv 0 \bmod 2 \text{ and } \ell = 0 \end{cases}$$

If $d_j \leqslant 1$, then

$$n_{j-1,\ell} - n_{j,\ell} = \begin{cases} 0 & \text{if } d_j = 0 \text{ or } (d_j = 1 \text{ and } \ell = 0) \\ 1 & \text{if } d_j = 1 \text{ and } \ell = 1 \end{cases}$$

Proof. Assume that $d_j \geqslant 2$. We distinguish the following two cases, and we will use eq. (3.13)

- d_{j-1} is odd and d_j is even. Then, if $\ell = 0$

$$n_{j-1,\ell} - n_{j,\ell} = \frac{d_{j-1} - 1}{2} - \frac{d_j}{2} + 1 = 1$$

  while $n_{j-1,\ell} - n_{j,\ell} = 0$ if $\ell = 1$.

- d_{j-1} is even and d_j is odd. Then, if $\ell = 0$

$$n_{j-1,\ell} - n_{j,\ell} = \frac{d_{j-1}}{2} - 1 - \frac{d_j - 1}{2} = 0,$$

  while $n_{j-1,\ell} - n_{j,\ell} = 1$ if $\ell = 0$.

If now $d_j = 0$ and $d_{j-1} = 1$, then $n_{j-1,\ell} - n_{j,\ell} = 0$. If $d_j = 1$ and $d_{j-1} = 2$ then $n_{j,\ell} = 0$ while $n_{j-1,\ell} = 0$ if $\ell = 0$ and $n_{j-1,\ell} = 1$ if $\ell = 1$.

□

Theorem 3.5.2. Let $M = H^0(X, \Omega_X)$, let τ be the generator of C_q, and for all $0 \leqslant j < q$ we define $M^{(j)}$ to be the kernel of the action of $k[C_q](\tau-1)^j$. For $0 \leqslant a \leqslant m-1$ and $1 \leqslant b \leqslant q = p^h$, let $n(a, b)$ be the number of indecomposable direct $k[C_q \rtimes C_m]$-module summands of M that are isomorphic to $U_{a,b}$. Let $n_1(a, b)$ be the number of indecomposable direct $k[C_m]$-summands of $M^{(b)}/M^{(b-1)}$ with socle $S_{\chi^{-(b-1)}(a)}$ and dimension 1. Let $n_2(a, b)$ be the number of indecomposable direct $k[C_m]$-module summands of $M^{(b+1)}/M^{(b)}$ with socle $S_{\chi^{-b}(a)}$, where we set $n_2(a, b) = 0$ if $b = q$.

$$n(a, b) = n_1(a, b) - n_2(a, b).$$

The numbers $n_1(a, b), n_2(a, b)$ can be computed using the isomorphism

$$M^{(j+1)}/M^{(j)} \cong S_{\chi^{-j}} \otimes_k H^0(Y, \Omega_Y(D_j)),$$

where $Y = X/C_q$ and D_j are the divisors on Y, given in definition 3.5.0.1.

Proof. This theorem is proved in [11], see remark 4.4. □

Corolarry 3.5.2.1. Set $d_j = \left\lfloor \frac{1}{p^h} \sum_{l=1}^{h} p^{h-l}(p - 1 + (p - 1 - a_{l,t})b_{l-1}) \right\rfloor$. The numbers $n(a, b), n_1(a, b)$ and $n_2(a, b)$ are given by

$$n(a, b) = n_1(a, b) - n_2(a, b) = n_{b-1,a} - n_{b,a}.$$

Proof. We will treat the $n_1(a, b)$ case and the $n_2(a, b)$ follows similarly. By the equivariant isomorphism for $M = H^0(X, \Omega_X)$ we have that

$$M^{(b)}/M^{(b-1)} \cong S_{\chi^{-(b-1)}} \otimes_k H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_b)).$$

The number of idecomposable $k[C_m]$-summands of $M^{(b)}/M^{(b-1)}$ isomorphic to $S_{\chi^{-(b-1)(a)}} = S_{a-(b-1)a_0}$ equals to the number of idecomposable $k[C_m]$-summands of $H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}(D_j))$ isomorphic to S_a which is computed in remark 3.5.1. □

In [62, Th. 1.1] A. Obus and R. Pries described the upper jumps in the ramification filtration of $C_{p^h} \rtimes C_m$-covers.

Theorem 3.5.3. Let $G = C_{p^h} \rtimes C_m$, where $p \nmid m$. Let $m' = |\mathrm{Cent}_G(\sigma)|/p^h$, where $\langle\tau\rangle = C_{p^h}$. A sequence $u_1 \leqslant \cdots u_n$ of rational numbers occurs as the set of positive breaks in the upper numbering of the ramification filtration of a G-Galois extension of $k((t))$ if and only if:

(i) $u_i \in \frac{1}{m}\mathbb{N}$ for $1 \leqslant i \leqslant h$

(ii) $\gcd(m, mu_1) = m'$

(iii) $p \nmid mu_1$ and for $1 < i \leqslant h$, either $u_i = pu_{i-1}$ or both $u_i > pu_{i-1}$ and $p \nmid mu_i$.

(iv) $mu_i \equiv mu_1 \bmod m$ for $1 \leqslant i \leqslant n$.

Notice that in our setting $\mathrm{Cent}_G(\tau) = \langle\tau\rangle$, therefore $m' = 1$. Also the set of upper jumps of C_{p^h} is given by $w_1 = mu_1, \ldots, w_h = mu_h, w_i \in \mathbb{N}$, see [62, lemma 3.5].

The theorem of Hasse-Arf [72, p. 77] applied for cyclic groups, implies that there are strictly positive integers $\iota_0, \iota_1, \ldots, \iota_{h-1}$ such that

$$b_s = \sum_{\nu=0}^{s-1} \iota_\nu p^\nu, \text{ for } 0 \leqslant s \leqslant h-1$$

Also, the upper jumps for the C_q extension are given by

$$w_0 = i_0 - 1, w_1 = i_0 + i_1 - 1, \ldots, w_h = i_0 + i_1 + \cdots + u_h - 1. \tag{3.14}$$

Assume that for all $0 < \nu \leqslant h-1$ we have $w_\nu = pw_{\nu-1}$. Equation (3.14) implies that

$$i_1 = (p-1)w_0, i_2 = (p-1)pw_0, i_3 = (p-1)p^2 w_0, \ldots, u_{h-1} = (p-1)p^{h-2}w_0.$$

Therefore,

$$\begin{aligned}
b_\ell + 1 &= \sum_{\nu=0}^{\ell} i_\nu p^\nu \\
&= 1 + w_0 + (p-1)w_0 \cdot p + (p-1)pw_0 \cdot p^2 \cdots + (p-1)p^{\ell-1}w_0 \cdot p^\ell \\
&= 1 + u_0 + p(p-1)u_0 \left(\sum_{\nu=0}^{\ell-1} p^{2\nu} \right) = 1 + w_0 + p(p-1)w_0 \frac{p^{2\ell}-1}{p^2-1} \\
&= 1 + w_0 + pw_0 \frac{p^{2\ell}-1}{p+1} = 1 + w_0 \frac{p^{2\ell+1}+1}{p+1}
\end{aligned}$$

where we have used that $w_0 = b_0 = i_0 - 1$.

3.5a Examples of local actions that don't lift

Consider the curve with lower jumps $1, 21, 521$ and higer jumps $1, 5, 25$, acted on by $C_{125} \rtimes C_4$. According to eq. (3.5), the only possible values for α are $1, 57, 68, 124$. The value $\alpha = 1$ gives rise to a cyclic group G, while the value $\alpha = 124$ has order 2 modulo 125. The values $57, 68$ have order 4 modulo 125. The cyclic group $\mathbb{F}_5^*$ is generated by the primitive root 2 of order 4. We have that $57 \equiv 2 \bmod 5$, while $68 \equiv 3 \equiv 2^3 \bmod 5$.

Using corollary 3.5.2.1 together with remark 3.5.1 we have that $H^0(X, \Omega_X)$ is decomposed into the following indecomposable modules, each one appearing with multiplicity one:

$$U_{0,5}, U_{3,11}, U_{2,17}, U_{1,23}, U_{0,29}, U_{3,35}, U_{2,41}, U_{1,47}, U_{0,53}, U_{3,59},$$
$$U_{2,65}, U_{1,71}, U_{0,77}, U_{3,83}, U_{2,89}, U_{1,95}, U_{0,101}, U_{3,107}, U_{2,113}, U_{1,119}$$

We have that $119 \equiv 3 \bmod 4$ so the module $U_{1,119}$ can not be lifted by itself. Also it can't be paired with $U_{0,5}$ since $119+5 \equiv 4 \neq 1 \bmod 4$. All other modules have dimension d such that $d+119 > 125$. Therefore, the representation of $H^0(G, \Omega_X)$ cannot be lifted. Notice that this example has non-vanishing KGB obstruction, so our criterion does not give something new here.

The case of dihedral groups, in which the KGB-obstruction is always vanishing, is more difficult to find an example that does not lift. We have the following example.

The HKG-curve with lower jumps $9, 9 \cdot 21 = 189, 9 \cdot 521 = 4689$ has genus 11656 and the following modules appear in its decomposition, each one appearing with multiplicity one:

$$\begin{aligned}
&U_{0,1}, U_{1,1}, U_{0,2}, U_{1,2}, U_{1,3}, U_{0,4}, U_{1,4}, U_{0,5}, U_{1,6}, U_{0,7}, U_{1,7}, U_{0,8}, U_{1,8}, U_{0,9},\\
&U_{1,9}, U_{0,11}, U_{1,11}, U_{0,12}, U_{1,12}, U_{0,13}, U_{1,13}, U_{0,14}, U_{1,15}, U_{0,16}, U_{0,17}, U_{1,17},\\
&U_{0,18}, U_{1,18}, U_{0,19}, U_{1,19}, U_{0,21}, U_{1,21}, U_{0,22}, U_{1,22}, U_{0,23}, U_{1,23}, U_{1,24}, U_{0,25},\\
&U_{1,26}, U_{0,27}, U_{1,27}, U_{0,28}, U_{1,28}, U_{0,29}, U_{1,29}, U_{0,31}, U_{1,31}, U_{0,32}, U_{1,32}, U_{0,33},\\
&U_{0,34}, U_{1,34}, U_{1,35}, U_{0,36}, U_{0,37}, U_{1,37}, U_{0,38}, U_{1,38}, U_{0,39}, U_{1,39}, U_{0,41}, U_{1,41},\\
&U_{0,42}, U_{1,42}, U_{0,43}, U_{1,43}, U_{1,44}, U_{0,45}, U_{0,46}, U_{1,46}, U_{1,47}, U_{0,48}, U_{1,48}, U_{0,49},\\
&U_{1,49}, U_{0,51}, U_{1,51}, U_{0,52}, U_{1,52}, U_{0,53}, U_{0,54}, U_{1,54}, U_{1,55}, U_{0,56}, U_{0,57}, U_{1,57},\\
&U_{0,58}, U_{1,58}, U_{0,59}, U_{1,59}, U_{0,61}, U_{1,61}, U_{0,62}, U_{1,62}, U_{0,63}, U_{1,63}, U_{1,64}, U_{0,65},\\
&U_{0,66}, U_{1,66}, U_{1,67}, U_{0,68}, U_{1,68}, U_{0,69}, U_{1,69}, U_{0,71}, U_{1,71}, U_{0,72}, U_{1,72}, U_{0,73},\\
&U_{1,73}, U_{0,74}, U_{1,75}, U_{0,76}, U_{0,77}, U_{1,77}, U_{0,78}, U_{1,78}, U_{0,79}, U_{1,79}, U_{0,81}, U_{1,81},\\
&U_{0,82}, U_{1,82}, U_{0,83}, U_{1,83}, U_{1,84}, U_{0,85}, U_{1,86}, U_{0,87}, U_{1,87}, U_{0,88}, U_{1,88}, U_{0,89},\\
&U_{1,89}, U_{0,91}, U_{1,91}, U_{0,92}, U_{1,92}, U_{0,93}, U_{1,93}, U_{0,94}, U_{1,95}, U_{0,96}, U_{1,96}, U_{0,97},\\
&U_{0,98}, U_{1,98}, U_{0,99}, U_{1,99}, U_{0,101}, U_{1,101}, U_{0,102}, U_{1,102}, U_{1,103}, U_{0,104}, U_{1,104},\\
&U_{0,105}, U_{1,106}, U_{0,107}, U_{1,107}, U_{0,108}, U_{1,108}, U_{0,109}, U_{1,109}, U_{0,111}, U_{1,111},\\
&U_{0,112}, U_{1,112}, U_{0,113}, U_{1,113}, U_{0,114}, U_{1,115}, U_{0,116}, U_{1,116}, U_{0,117}, U_{0,118},\\
&U_{1,118}, U_{0,119}, U_{1,119}, U_{0,121}, U_{1,121}, U_{0,122}, U_{1,122}, U_{0,123}, U_{1,123}, U_{1,124},
\end{aligned}$$

The above formulas were computed using Sage 9.8 [74]. In order to be completely sure that are correct we will compute the values we need by hand also. We have

$$\begin{aligned}
d_j &= \left\lfloor \frac{1}{125}\left(5^2\big(4+(4-a_1)9\big) + 5\big(4+(4-a_2)189\big) + \big(4+(4-a_3)4689\big)\right)\right\rfloor \\
&= \left\lfloor \frac{1}{125}\left(23560 - 225a_1 - 945a_2 - 4689a_3\right)\right\rfloor
\end{aligned}$$

j	p−adic	d_j	$n_{j,0}$	$n_{j,1}$	$n_{j-1,0} - n_{j,0}$	$n_{j-1,1} - n_{j,1}$
0	$0,0,0$	$\lfloor \frac{23560}{125} \rfloor = 188$	93	94	–	–
1	$1,0,0$	$\lfloor \frac{23335}{125} \rfloor = 186$	92	93	1	1
2	$1,0,0$	$\lfloor \frac{23110}{125} \rfloor = 184$	91	92	1	1
3	$1,0,0$	$\lfloor \frac{22885}{125} \rfloor = 183$	91	91	0	1
4	$1,0,0$	$\lfloor \frac{22660}{125} \rfloor = 181$	90	90	1	1
5	$0,1,0$	$\lfloor \frac{22615}{125} \rfloor = 180$	89	90	1	0
6	$1,1,0$	$\lfloor \frac{22390}{125} \rfloor = 179$	89	89	0	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
120	$0,4,4$	$\lfloor \frac{1024}{125} \rfloor = 8$	3	4		
121	$1,4,4$	$\lfloor \frac{799}{125} \rfloor = 6$	2	3	1	1
122	$2,4,4$	$\lfloor \frac{574}{125} \rfloor = 4$	1	2	1	1
123	$3,4,4$	$\lfloor \frac{349}{125} \rfloor = 2$	0	1	1	1
124	$4,4,4$	$\lfloor \frac{124}{125} \rfloor = 0$	0	0	0	1

Notice that $U_{1,123}, U_{0,123}$ can be paired with $U_{1,0}, U_{1,1}$, and then for $U_{0,121}$, $U_{1,121}$ there is only one $U_{1,3}$ to be paired with. The lift is not possible.

3.5b Examples of actions that lift

Our aim now is to prove the following

Proposition 3.5.3.1. Assume that the first lower jump equals $b_0 = w_0 = 1$ and each other lower jump is given by
$$b_\ell = \frac{p^{2\ell+1}+1}{p+1}. \tag{3.15}$$
Then, the local action of the dihedral group D_{p^h} lifts.

Remark 3.5.4. Notice that in this case if $d_{j-1} > d_j$ then $d_{j-1} = d_j + 1$.

Remark 3.5.5. This set of upper jumps was constructed by assuming that $w_0 = 1$ and $w_\nu = pw_{\nu-1}$ for all $0 < w_\nu \leqslant h-1$. Hence the above proposition is a special case of [61, cor. 1.20], for $m = 2$.

Definition 3.5.5.1. For an integer j with p-adic expansion $j = a_1 + a_2 p + \cdots + a_h p^{h-1}$ we define
$$B(j) = \sum_{\ell=1}^{h} a_\ell b_{\ell-1} p^{h-\ell}.$$

Lemma 3.5.5.1. Write
$$\begin{aligned} j-1 &= (p-1) + (p-1)p + \cdots + (p-1)p^{s-2} + a_s p^{s-1} + \cdots \\ j &= (a_s+1)p^{s-1} + \cdots \end{aligned}$$
where $1 \leqslant s \leqslant h$ is the smallest integer such that the corresponding coefficient a_s in the p-adic expansion of $j-1$ satisfies $0 \leqslant a_s < p-1$. Then
$$B(j) - B(j-1) = p^{h-s}. \tag{3.16}$$

Proof. By definition of the function $B(j)$ and using the values of b_ℓ from eq (3.15), we have
$$\begin{aligned} B(j) - B(j-1) &= b_{s-1}p^{h-s} - (p-1)(b_0 p^{h-1} + \cdots + b_{s-2}p^{h-s+1}) \\ &= \frac{p^{2s-1}+1}{p+1}p^{h-s} - (p-1)\sum_{\nu=1}^{s-1} p^{h-\nu}\frac{p^{2\nu-1}+1}{p+1} \\ &= p^{h-s}. \end{aligned}$$
□

Definition 3.5.5.2. We will call the element j of type s if all p-adic coefficients $a = \nu$ in the p-adic expansion of j for $1 \leqslant \nu \leqslant s-1$ are $p-1$, while a_s is not $p-1$. For example $j-1$ in lemma 3.5.5.1 is of type s, while j is of type 1.

Proposition 3.5.5.1. Write $\pi_j = \left\lfloor \frac{B(j)}{p^h} \right\rfloor$. Then,
$$\pi_j = \begin{cases} \pi_{j-1}+1 & \text{if } j = k(p+1) \\ \pi_{j-1} & \text{otherwise} \end{cases}$$
Also $p^h \nmid B(j)$ for all $1 \leqslant j \leqslant p^h - 1$.

Proof. Equation (3.16) implies that $B(j) > B(j-1)$ hence $\pi_j \geqslant \pi_{j-1}$. Write $B(j) = \pi_j p^h + \nu_j$, $0 \leqslant \nu_j < p^h$ for each $0 \leqslant j \leqslant p^{h-1}$. We observe first that
$$B(j) - B(j-1) = (\pi_j - \pi_{j-1})p^h + \nu_j - \nu_{j-1}$$

therefore

$$\pi_j - \pi_{j-1} = \frac{1}{p^s} - \frac{\nu_{j-1} - \nu_j}{p^h}$$

Notice that $|\nu_j - \nu_{j-1}| < p^h$, thus $|\pi_j - \pi_{j-1}| < 2$. Since $\pi_j \geqslant \pi_{j-1}$ we have either $\pi_j = \pi_{j-1}$ or $\pi_j = \pi_{j-1} + 1$.

In the following table we present the change on $B(j)$ after increasing $j-1$ to j, where $j-1$ has type s, using lemma 3.5.5.1.

j	$B(j)$	$\left\lfloor \frac{B(j)}{p^h} \right\rfloor$
0	0	0
1	p^{h-1}	0
$a_1 = 2, \dots, p-1$	$a_1 p^{h-1}$	0
p	$(p-1)p^{h-1} + p^{h-2}$	0
$\boxed{p+1}$	$p^h + p^{h-2}$	$\boxed{1}$
$p+2$	$p^h + p^{h-2} + p^{h-1}$	1
$p + a_1, a_1 = 3, \dots, p-1$	$p^h + p^{h-2} + (a_1 - 1)p^{h-1}$	1
$2p$	$p^h + 2p^{h-2} + (p-2)p^{h-1}$	1
$2p+1$	$p^h + 2p^{h-2} + (p-1)p^{h-1}$	1
$\boxed{2p+2}$	$2p^h + 2p^{h-2}$	$\boxed{2}$
$2p+3$	$2p^h + 2p^{h-2} + p^{h-1}$	2
$2p + a_1$	$2p^h + 2p^{h-2} + (a_1 - 2)p^{h-1}$	2
$3p$	$2p^h + 3p^{h-2} + (p-3)p^{h-1}$	2
$\dots$	$\dots$	$\dots$
$(p-1)p$	$(p-2)p^h + (p-1)p^{h-2} + p^{h-1}$	$p-2$
$\dots$	$\dots$	$\dots$
$(p-1) + (p-1)p$	$(p-1)p^h + (p-1)p^{h-2}$	$p-1$
p^2	$(p-1)p^h + (p-1)p^{h-2} + p^{h-3}$	$p-1$
$\dots$	$\dots$	$\dots$
$(p-1) + p^2$	$(p-1)p^h + (p-1)p^{h-2} + p^{h-3} + (p-1)p^{h-1}$	$p-1$
$\boxed{p + p^2}$	$p^{h+1} + p^{h-3}$	$\boxed{p}$
$1 + p + p^2$	$p^{h+1} + p^{h-1} + p^{h-3}$	p

Indeed, if the type of $j-1$ is $s = 1$ then $B(j) = B(j-1) + p^{h-1}$, therefore $\pi_j = \pi_{j-1}$. It is clear from the above table that $\pi_j = \pi_{j-1} + 1$ at $j = kp + k$, for $1 \leqslant k \leqslant p$. These integers are put in a box in the table above.

We will prove the result in full generality by induction. Observe that if $j-1$ is of type s, and $\pi_j = \pi_{j-1} + 1$, then $B(j) = B(j-1) + p^{h-s}$ and moreover

$$\begin{aligned} B(j-1) &= (p-1)p^{h-1} + (p-1)p^{h-2} + \cdots + (p-1)p^{h-s} + \pi_{j-1}p^h + u \\ B(j) &= p^h + \pi_{j-1}p^h + u \end{aligned}$$

for some

$$u = u_j - (p-1)p^{h-1} + (p-1)p^{h-2} + \cdots + (p-1)p^{h-s} = \sum_{\nu=0}^{h-s-1} \gamma_\nu p^\nu,$$

for some integers $0 \leqslant \gamma_\nu < p$, $0 \leqslant \nu \leqslant h-s-1$. Set $T = \pi_{j-1}p^h + u$. Assume by induction that this jump occurs at $j = k(p+1)$. We will prove that the next jump will occur at $j = k(p+1) + (p+1) = (k+1)(p+1)$. Indeed, j has the zero p-adic coefficient a_0 equal to 0, so it is of type 1 and we have

$$\begin{aligned} B(j+1) &= B(j) + p^{h-1} + T \\ B(j+2) &= B(j) + 2p^{h-1} + T \\ &\dots \\ B(j + (p-1)) &= B(j) + (p-1)p^{h-1} + T \qquad \longleftarrow \text{ type } 2 \\ B(j+p) &= B(j) + (p-1)p^{h-1} + p^{h-2} + T \\ B(j+p+1) &= B(j) + p^h + T + p^{h-2}. \end{aligned} \tag{3.17}$$

Therefore, $\pi_j = \pi_{j+1} = \cdots = \pi_{j+p} < \pi_{j+(p+1)} = \pi_j + 1$, i.e. the desired result.

In order to prove that $p^h \nmid B(j)$ we observe first that all values of $B(j)$ given in the table are not divisible by p^h. The result can be proved by induction. Indeed, we can assume that $B(j)$ is not divisible by p^h and then we add p^{h-1}. Therefore all values in equation (3.17) when divided by p^h have non-zero residue either $\nu p^{h-1}+u$ for $\nu = 1, \dots, (p-1)$ or $p^{h-2}+u$. □

Theorem 3.5.6. Assume that $w_0 = 1$, and the jumps of the C_q action are as in proposition 3.5.3.1. Then each direct summand $U(\epsilon, j)$ of $H^0(X, \Omega_X)$ has a compatible pair according to criterion 3.1.4, which is given by

$$U(\epsilon', p^h - 1 - j) \text{ if } h \text{ is odd}$$
$$U(\epsilon', p^h - p - j) \text{ if } h \text{ is even}$$

Proof. For every $1 \leqslant j \leqslant p^h - 1$, set $\tilde{j} = p^h - 1 - j$. For every $1 \leqslant j \leqslant p^h - 1$ write $B(j) = \pi_j p^h + \nu_j$, $0 \leqslant \nu_j < p^h$. Recall that

$$d_j = \left\lfloor \frac{p^h - 1 + B(p^h-1) - B(j)}{p^h} \right\rfloor = \left\lfloor \frac{p^h - 1 + B(\tilde{j})}{p^h} \right\rfloor = 1 + \pi_{\tilde{j}} + \left\lfloor \frac{-1+\nu_j}{p^h} \right\rfloor .$$

Since $\nu_j \neq 0$, we have that $\left\lfloor \frac{-1+\nu_j}{p^h} \right\rfloor = 0$. Therefore, $d_{j-1} > d_j$ if and only if $\pi_{\tilde{j}+1} > \pi_{\tilde{j}}$ that is

$$\tilde{j} + 1 = k(p+1) \Rightarrow \tilde{j} = k(p+1) - 1. \tag{3.18}$$

Observe now that if $d_{j-1} = d_j + 1$, that is $\tilde{j} = k(p+1) - 1$, then

$$j = p^h - 1 - \tilde{j} = p^h - k(p+1). \tag{3.19}$$

- If h is odd, then by the right hand side of eq. (3.18) we have

$$\tilde{j} = p^h - (1 + p^h) + k(p+1) = p^h - k'(p+1)$$

  for some integer $k' = \frac{p^h+1}{p+1} - k$, since in this case $p+1 \mid p^h + 1$. This proves that $d_{\tilde{j}-1} = d_{\tilde{j}} + 1$, using proposition 3.5.5.1, since both $j, \tilde{j}$ are of the same form. Using $\tilde{\tilde{j}} = j$ we can assume that $j < \tilde{j}$. Then $d_j - d_{\tilde{j}}$ is the number of jumps between $d_j, d_{\tilde{j}}$, that is the number of elements $x = p^h - l_x(p+1) \in \mathbb{N}$ of the form

$$j = p^h - k(p+1) < p^h - l_x(p+1) \leqslant p^h - k'(p+1)$$

  that is $k' \leqslant l_x < k$. This number equals $k - k' = 2k - \frac{p^h+1}{p+1}$, which is odd since $\frac{p^h+1}{p+1} = \sum_{\nu=0}^{h-1}(-p)^\nu$ is odd.

- If h is even, then we set $j' = p^h - p - j$ and using eq. (3.19) we have

$$j' = p^h - p - j = p^h - (p + p^h) + k(p+1) = p^h - k'(p+1)$$

  for some integer $k' = \frac{p^h+p}{p+1} - k$, since in this case $p+1 \mid p^h + p$. As in the h odd case, this proves that $d_{\tilde{j}-1} = d_{\tilde{j}} + 1$, using proposition 3.5.5.1, since both j, j' are of the same form. Again since $j'' = j$ we can assume that $j < j'$. As in the odd h case, the difference $d_j - d_{j'}$ is the number of jumps between $d_j, d_{j'}$, which equals to $2k - \frac{p^h+p}{p+1}$ which is odd since $\frac{p^h+p}{p+1} = p\frac{p^{h-1}+1}{p+1}$ is odd.

Observe that we have proved in both cases that d_j is odd if and only if $d_{\tilde{j}}$ (resp. d_j') is even. The change of ϵ to ϵ' follows by lemma 3.5.1.1, which implies that if we have the indecomposable summand $U(\epsilon, d_j)$, where $\epsilon \in \{0,1\}$, then we also have $U(\epsilon', d_{\tilde{j}})$ (resp. $U(\epsilon', d_{j'})$) with $\epsilon' \in \{0,1\} - \{\epsilon\}$ and $d_j + d_{\tilde{j}} \leqslant q^h$ (resp. $d_j + d_{j'} \leqslant q^h$), that is criterion 3.1.4 is satisfied. □

Part II

Representations of metacyclic group

Chapter 4

The lifting of representations of a metacyclic group

4.1 Introduction

The lifting problem for a representation

$$\rho : G \to \mathrm{GL}_n(k),$$

where k is a field of characteristic $p > 0$, is about finding a local ring R of characteristic 0, with maximal ideal $\mathfrak{m}_R$ such that $R/\mathfrak{m}_R = k$, so that the following diagram is commutative:

$$\begin{tikzcd} & \mathrm{GL}_n(R) \arrow[d] \\ G \arrow[ur] \arrow[r] & \mathrm{GL}_n(k) \end{tikzcd}$$

Equivalently one asks if there is a free R-module V, which is also an $R[G]$-module such that $V \otimes_R R/\mathfrak{m}_R$ is the $k[G]$-module corresponding to our initial representation. We know that projective $k[G]$-modules lift in characteristic zero, [71, chap. 15], but for a general $k[G]$-module such a lifting is not always possible, for example, see proposition 2.2.2.1. In the second part we study the lifting problem for the group $G = C_q \rtimes C_m$, where C_q is a cyclic group of order p^h and C_m is a cyclic group of order m, $(p, m) = 1$ and give necessary and sufficient condition in order to lift. We assume that the local ring R contains the q-roots of unity and k is algebraically closed, and we might need to consider a ramified extension of R, in order to ensure that certain q-roots of unit are distant in the $\mathfrak{m}_R$-topology, see remark 4.5.6. An example of such a ring R is the ring of Witt vectors $W(k)[\zeta_q]$ with the q-roots of unity adjoined to it.

We notice that a decomposable $R[G]$-module V gives rise to a decomposable R-module modulo $\mathfrak{m}_R$ and also an indecomposable $R[G]$-module can break in the reduction modulo $\mathfrak{m}_R$ into a direct sum of indecomposable $k[G]$-summands. We also give a classification of $k[C_q \rtimes C_m]$-modules in terms of Jordan decomposition and give the relation with the more usual uniserial description in terms of their socle [1].

Our interest to this problem comes from the problem of lifting local actions. The local lifting problem considers the following question: Does there exist an extension $\Lambda/W(k)$, and a representation

$$\tilde{\rho} : G \hookrightarrow \mathrm{Aut}(\Lambda[[T]]),$$

such that if t is the reduction of T, then the action of G on $\Lambda[[T]]$ reduces to the action of G on $k[[t]]$?

If the answer to the above question is positive, then we say that the G-action lifts to characteristic zero. A group G for which every local G-action on $k[[t]]$ lifts to characteristic zero is called a local Oort group for k. Notice that cyclic groups are always local Oort groups. This result was known as the "Oort conjecture", which was recently proved by F. Pop [66] using the work of A. Obus and S. Wewers [63].

There are a lot of obstructions that prevent a local action to lift in characteristic zero. Probably the most important of these obstructions in the KGB-obstruction [20]. It is believed that this is the only obstruction for the local lifting problem, see [59], [61]. In theorem 3 we gave a criterion for the local lifting, which involves the lifting of a linear representation of the same group. The case $G = C_q \rtimes C_m$ and especially the case of dihedral groups $D_q = C_q \rtimes C_2$, is a problem of current interest in the theory of local liftings, see [61], [23], [80]. For more details on the local lifting problem we refer to [19], [20], [21], [59].

Keep also in mind that the $C_q \rtimes C_m$ groups were important to the study of group actions in holomorphic differentials of curves defined over fields of positive characteristic p, where the group involved has cyclic p-Sylow subgroup, see [11].

Let us now describe the method of proof. For understanding the splitting of indecomposable $R[G]$-modules modulo $\mathfrak{m}_R$, we develop a version of Jordan normal form in lemma 4.5.0.3 for endomorphisms $T : V \to V$ of order p^h, where V is a free module of rank d. We give a way to select this basis, by selecting an initial suitable element $E \in V$, see lemma 4.5.0.2. The normal form (as given in eq. (4.9)) of the element T of order q, determines the decomposition of the reduction. We show that for every indecomposable summand V_i of V, we can select E as an eigenvalue of the generator σ of C_m and then by forcing the relation $\Gamma T = T^\alpha \Gamma$ to hold, we see how the action of σ can be extended recursively to an action of σ on V_i, this is done in lemma 4.5.4.2. Proving that this gives indeed a well defined action is a technical computation and is done in lemmata 4.5.4.3, 4.5.4.4, 4.5.4.5, 4.5.5.1, 4.5.5.2.

The important thing here is that the definition of the action of σ on E is the "initial condition" of a dynamical system that determines the action of C_m on the indecomposable summand V_i. The $R[C_q \rtimes C_m]$ indecomposable module V_i can break into a direct sum $V_\alpha(\epsilon_\nu, \kappa_\nu)$-modules $1 \leqslant \nu \leqslant s$ (for a precise definition of them see definition 4.3.0.1, notice that κ_i denotes the dimension). The action of σ on each $V_\alpha(\epsilon_\nu, \kappa_\nu)$ can be uniquely determined by the action of σ on an initial basis element as shown in section 4.3, again by a "dynamical system" approach, where we need s initial conditions, one for each $V_\alpha(\epsilon_\nu, \kappa_\nu)$. The lifting condition essentially means that the indecomposable summands $V_\alpha(\epsilon, \kappa)$ of the special fibre, should be able to be rearranged in a suitable way, so that they can be obtained as reductions of indecomposable $R[C_q \rtimes C_m]$-modules. The precise expression of our lifting criterion is given in the following proposition:

Proposition 4.1.0.1. Consider a $k[G]$-module M which is decomposed as a direct sum

$$M = V_\alpha(\epsilon_1, \kappa_1) \oplus \cdots \oplus V_\alpha(\epsilon_s, \kappa_s).$$

The module lifts to an $R[G]$-module if and only if the set $\{1, \ldots, s\}$ can be written as a disjoint union of sets I_ν, $1 \leqslant \nu \leqslant t$ so that

a. $\sum_{\mu \in I_\nu} \kappa_\mu \leqslant q$, for all $1 \leqslant \nu \leqslant t$.

b. $\sum_{\mu \in I_\nu} \kappa_\mu \equiv a \bmod m$ for all $1 \leqslant \nu \leqslant t$, where $a \in \{0, 1\}$.

c. For each ν, $1 \leqslant \nu \leqslant t$ there is an enumeration $\sigma : \{1, \ldots, \# I_\nu\} \to I_\nu \subset \{1, .., s\}$, such that

$$\epsilon_{\sigma(2)} = \epsilon_{\sigma(1)} \alpha^{\kappa_{\sigma(1)}}, \epsilon_{\sigma(3)} = \epsilon_{\sigma(3)} \alpha^{\kappa_{\sigma(3)}}, \ldots, \epsilon_{\sigma(s)} = \epsilon_{\sigma(s-1)} \alpha^{\kappa_{\sigma(s-1)}}$$

In the above proposition, each set I_ν corresponds to a collection of modules $V_\alpha(\epsilon_\mu, \kappa_\mu)$, $\mu \in I_\nu$ which come as the reduction of an indecomposable $R[C_q \rtimes C_m]$-module V_ν of V.

4.2 Notation

Let τ be a generator of the cyclic group C_q and σ be a generator of the cyclic group C_m. The group G is given in terms of generators and relations as follows:

$$G = \langle \sigma, \tau | \tau^q = 1, \sigma^m = 1, \sigma\tau\sigma^{-1} = \tau^\alpha \text{ for some } \alpha \in \mathbb{N}, 1 \leqslant \alpha \leqslant p^h - 1, (\alpha, p) = 1 \rangle.$$

The integer α satisfies the following congruence:

$$\alpha^m \equiv 1 \bmod q \tag{4.1}$$

as one sees by computing $\tau = \sigma^m \tau \sigma^{-m} = \tau^{\alpha^m}$. Also the integer α can be seen as an element in the finite field $\mathbb{F}_p$, and it is a $(p-1)$-th root of unity, not necessarily primitive. In particular the following holds:

Lemma 4.2.0.1. Let $\zeta_m \in k$ be a fixed primitive m-th root of unity. There is a natural number a_0, $0 \leqslant a_0 < m-1$ such that $\alpha = \zeta_m^{a_0}$.

Proof. The integer α if we see it as an element in k is an element in the finite field $\mathbb{F}_p \subset k$, therefore $\alpha^{p-1} = 1$ as an element in $\mathbb{F}_p$. Let $\mathrm{ord}_p(\alpha)$ be the order of α in $\mathbb{F}_p^*$. By eq. (4.1) we have that $\mathrm{ord}_p(\alpha) \mid p-1$ and $\mathrm{ord}_p(\alpha) \mid m$, that is $\mathrm{ord}_p(\alpha) \mid (p-1, m)$.

The primitive m-th root of unity ζ_m generates a finite field $\mathbb{F}_p(\zeta_m) = \mathbb{F}_{p^\nu}$ for some integer ν, which has cyclic multiplicative group $\mathbb{F}_{p^\nu} \backslash \{0\}$ containing both the cyclic groups $\langle \zeta_m \rangle$ and $\langle \alpha \rangle$. Since for every divisor δ of the order of a cyclic group C there is a unique subgroup $C' < C$ of order δ we have that $\alpha \in \langle \zeta_m \rangle$, and the result follows. □

Definition 4.2.0.1. For each $p^i \mid q$ we define $\mathrm{ord}_{p^i} \alpha$ to be the smallest natural number o such that $\alpha^o \equiv 1 \bmod p^i$.

It is clear that for $\nu \in \mathbb{N}$

$$\alpha^\nu \equiv 1 \bmod p^i \Rightarrow \alpha^\nu \equiv 1 \bmod p^j \text{ for all } j \leqslant i.$$

Therefore

$$\mathrm{ord}_{p^j} \alpha \mid \mathrm{ord}_{p^i} \alpha \text{ for } j \leqslant i.$$

On the other hand $\alpha \in \mathbb{N}$ and $\alpha^{p-1} \equiv 1 \bmod p$ so $\mathrm{ord}_p \alpha \mid p-1$. Also since $\sigma^t \tau \sigma^{-t} = \tau^{\alpha^t}$ we have that $\alpha^m \equiv 1 \bmod p^h$, therefore $\mathrm{ord}_p \alpha \mid \mathrm{ord}_{p^i} \alpha \mid \mathrm{ord}_{p^h} \alpha \mid m$, for $1 \leqslant i \leqslant h$.

Lemma 4.2.0.2. The center $\mathrm{Cent}_G(\tau) = \langle \tau, \sigma^{\mathrm{ord}_{p^h} \alpha} \rangle$. Moreover

$$\frac{|\mathrm{Cent}_G(\tau)|}{p^h} = \frac{m}{\mathrm{ord}_{p^h}(\alpha)} =: m'$$

Proof. The result follows by observing $(\tau^\nu \sigma^t)\tau(\tau^\nu \sigma^t)^{-1} = \tau^{\alpha^t}$, for all $1 \leqslant \nu \leqslant q$, $1 \leqslant t \leqslant m$. □

Remark 4.2.1. If $\mathrm{ord}_p \alpha = m$ then $\mathrm{ord}_{p^i} \alpha = m$ for all $1 \leqslant i \leqslant h$.

Lemma 4.2.1.1. If the group $G = C_q \rtimes C_m$ is a subgroup of $\mathrm{Aut}(k[[t]])$, then all orders $\mathrm{ord}_{p^i} \alpha = m/m'$, for all $1 \leqslant i \leqslant h$.

Proof. We will use the notation of the book of J.P.Serre on local fields [72]. By [62, Th.1.1b] we have that the first gap i_0 in the lower ramification filtration of the cyclic group C_q satisfies $(m, i_0) = m'$.

The ramification relation [72, prop. 9 p. 69]

$$\alpha \theta_{i_0}(\tau) = \theta_{i_0}(\tau^\alpha) = \theta_{i_0}(\sigma\tau\sigma^{-1}) = \theta_0(\sigma)^{i_0}\theta_{i_0}(\tau),$$

implies that $\theta_0(\sigma)^{i_0} = \alpha \in \mathbb{N}$. From $(m, i_0) = m'$ and the fact that $\mathrm{ord}\theta_0(\sigma) = m$ we obtain

$$\frac{m}{m'} = \mathrm{ord}\theta_0(\sigma)^{i_0} = \mathrm{ord}_p(\alpha).$$

Thus

$$\frac{m}{m'} = \mathrm{ord}_p \alpha | \mathrm{ord}_{p^i} \alpha | \mathrm{ord}_{p^h} \alpha = \frac{m}{m'}.$$

Hence all orders $\mathrm{ord}_{p^i} \alpha = m/m'$. □

Remark 4.2.2. If the KGB-obstruction vanishes and $\alpha \neq 1$, then by [59][prop. 5.9] $i_0 \equiv -1 \bmod m$ and $\mathrm{ord}_{p^i} \alpha = m$ for all $1 \leqslant i \leqslant h$.

4.3 Indecomposable $C_q \rtimes C_m$odules, modular representation theory

In this section we will describe the indecomposable $C_q \rtimes C_m$-modules. We will give two methods in studying them. The first one is needed since it is in accordance to the method we will give in order to describe indecomposable $R[C_q \rtimes C_m]$-modules. The second one, using the structure of the socle, is the standard method of describing $k[C_q \rtimes C_m]$-modules in modular representation theory.

4.3a Linear algebra method.

The indecomposable modules of the C_q are determined by the Jordan normal forms of the generator τ of the cyclic group C_q. So for each $1 \leqslant \kappa \leqslant p^h$ there is exactly one C_q indecomposable module denoted by J_κ. Therefore we have the following decomposition of an indecomposable $C_q \rtimes C_m$-module M considered as a C_q-module.

$$M = J_{\kappa_1} \oplus \cdots \oplus J_{\kappa_r}. \tag{4.2}$$

Lemma 4.3.0.1. In the indecomposable module J_κ for every element E such that

$$(\tau - \mathrm{Id}_{\kappa_i})^{\kappa_i - 1} E \neq 0$$

the elements $B = \{E, (\tau - \mathrm{Id}_\kappa)E, \ldots, (\tau - \mathrm{Id}_\kappa)^{\kappa-1}E\}$ form a basis of J_κ such that the matrix of τ with respect to this basis is given by

$$\tau = \mathrm{Id}_\kappa + \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & 1 & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}. \tag{4.3}$$

Proof. Since the set B has k-elements it is enough to prove that it consists of linear independent elements. Indeed, consider a linear relation

$$\lambda_0 E + \lambda_1(\tau - \mathrm{Id}_\kappa)E + \cdots + \lambda_{\kappa-1}(\tau - \mathrm{Id}_\kappa)^{\kappa-1}E = 0.$$

By applying $(\tau - \mathrm{Id}_\kappa)^{\kappa-1}$ we obtain $\lambda_0(\tau - \mathrm{Id}_\kappa)^{\kappa-1} = 0$, which gives us $\lambda_0 = 0$. We then apply $(\tau - \mathrm{Id}_\kappa)^{\kappa-2}$ to the linear relation and by the same argument we obtain $\lambda_1 = 0$ and we continue this way proving that $\lambda_0 = \cdots = \lambda_{\kappa-1} = 0$. The matrix form of τ in this basis is immediate. □

We will now prove that σ acts on each J_κ of eq. (4.2) proving that $r = 1$. Since the field k is algebraically closed and $(m, p) = 1$ we know that there is a basis of M consisting of eigenvectors of σ. There is an eigenvector E of σ, which is not in the kernel of $(\tau - \mathrm{Id}_\kappa)^{\kappa_1 - 1}$. Then the elements of the set $B = \{E, (\tau - \mathrm{Id}_\kappa)E, \ldots, (\tau - \mathrm{Id}_\kappa)^{\kappa_1-1}E\}$ are linearly independent and form a direct C_q summand of M isomorphic to J_{κ_1}.

We will now show that this module is an $k[C_q \rtimes C_m]$-module. For this, we have to show that the generator σ of C_m acts on the basis B. Observe that for every $0 \leqslant i \leqslant \kappa_1 - 1 < p^h$

$$\sigma(\tau - 1)^{i-1} = (\tau^\alpha - 1)^{i-1}\sigma.$$

Set $e = E_1$ and $\kappa = \kappa_1$. This means that the action of σ on e determines the action of σ on all other basis elements $e_\nu := (\tau - 1)^{\nu-1}e$, $1 \leqslant \nu \leqslant \kappa_1$.

Let us compute:

$$\sigma e_{i+1} = \sigma(\tau - 1)^i e = (\tau^\alpha - 1)^i \zeta_m^\lambda e$$

On the basis $\{e_1, \ldots, e_{\kappa_1}\}$ the matrix τ is given by eq. (4.3) hence using the binomial formula we compute

$$\tau^{\alpha} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ \binom{\alpha}{1} & 1 & \ddots & & & \vdots \\ \binom{\alpha}{2} & \binom{\alpha}{1} & \ddots & \ddots & & \vdots \\ \binom{\alpha}{3} & \binom{\alpha}{2} & \ddots & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \binom{\alpha}{1} & 1 & 0 \\ \binom{\alpha}{k} & \binom{\alpha}{k-1} & \cdots & \binom{\alpha}{2} & \binom{\alpha}{1} & 1 \end{pmatrix}. \tag{4.4}$$

Thus $\tau^{\alpha} - 1$ is a nilpotent matrix $A = (a_{ij})$ of the form:

$$a_{ij} = \begin{cases} \binom{\alpha}{\mu} & \text{if } j = i - \mu \text{ for some } \mu, 1 \leqslant \mu \leqslant \kappa \\ 0 & \text{if } j \geqslant i \end{cases}$$

The ℓ-th power $A^{\ell} = (a_{ij}^{(\ell)})$ of A is then computed by (keep in mind that $a_{ij} = 0$ for $i \leqslant j$)

$$a_{ij}^{(\ell)} = \sum_{i < \nu_1 < \cdots < \nu_{\ell-1} < j} a_{i,\nu_1} a_{\nu_1,\nu_2} a_{\nu_2,\nu_3} \cdots a_{\nu_{\ell-1},j}$$

This means that $i - j > \ell$ in order to have $a_{ij} \neq 0$. Moreover for $i = j + \ell$ (which is the the first non zero diagonal below the main diagonal) we have

$$a_{i,i+\ell} = a_{i,i+1} a_{i+1,i+2} \cdots a_{i+\ell-1,i+\ell} = \binom{\alpha}{1}^{\ell} = \alpha^{\ell}.$$

Therefore, the matrix of A^{ℓ} is of the following form:

$$\begin{array}{c} \overbrace{\hphantom{0 \quad \cdots \quad \cdots \quad 0}}^{k-\ell} \quad \overbrace{\hphantom{0 \quad \cdots \quad 0}}^{\ell} \\ \begin{pmatrix} 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ \alpha^{\ell} & \ddots & & 0 & \vdots & & \vdots \\ * & \alpha^{\ell} & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ * & \cdots & * & \alpha^{\ell} & 0 & \cdots & 0 \end{pmatrix} \end{array} \tag{4.5}$$

Definition 4.3.0.1. We will denote by $V_{\alpha}(\lambda, \kappa)$ the indecomposable κ-dimensional G-module given by the basis elements $\{(\tau - 1)^{\nu} e, \nu = 0, \ldots, \kappa - 1\}$, where $\sigma e = \zeta_m^{\lambda} e$.

This definition is close to the notation used in [42].

Lemma 4.3.0.2. The action of σ on the basis element e_i of $V_{\alpha}(\lambda, \kappa)$ is given by:

$$\sigma e_i = \alpha^{i-1} \zeta_m^{\lambda} e_i + \sum_{\nu=i+1}^{\kappa} a_{\nu} e_{\nu}, \tag{4.6}$$

for some coefficients $a_i \in k$. In particular the matrix of σ with respect to the basis $e_1, \ldots, e_{\kappa}$ is lower triangular.

Proof. Recall that $e_i = (\tau - 1)^{i-1} e_1$. Therefore

$$\sigma e_i = \sigma(\tau - 1)^{i-1} e_1 = (\tau^{\alpha} - 1)^{i-1} \sigma e_1 = \zeta_m^{\lambda} (\tau^{\alpha} - 1)^{i-1} e_1.$$

The result follows by eq. (4.5) $\square$

We have constructed a set of indecomposable modules $V_\alpha(\lambda,\kappa)$. Apparently $V_\alpha(\lambda,\kappa)$ can not be isomorphic to $V_\alpha(\lambda',\kappa')$ if $\kappa\neq\kappa'$, since they have different dimensions.

Assume now that $\kappa=\kappa'$. Can the modules $V_\alpha(\lambda,\kappa)$ and $V_\alpha(\lambda',\kappa)$ be isomorphic for $\lambda\neq\lambda'$?

The eigenvalues of the prime to p generator σ on $V_\alpha(\lambda,\kappa)$are

$$\zeta_m^\lambda, \alpha\zeta_m^\lambda, \ldots, \alpha^{\kappa-1}\zeta_m^\lambda.$$

Similarly the eigenvalues for σ when acting on $V_\alpha(\lambda',\kappa)$ are

$$\zeta_m^{\lambda'}, \alpha\zeta_m^{\lambda'}, \ldots, \alpha^{\kappa-1}\zeta_m^{\lambda'}.$$

If the two sets of eigenvalues are different then the modules can not be isomorphic. But even if $\lambda\neq\lambda' \bmod n$ the two sets of eigenvalues can still be equal. Even in this case the modules can not be isomorphic.

Lemma 4.3.0.3. The modules $V_\alpha(\lambda_1,\kappa)$ and $V_\alpha(\lambda_2,\kappa)$ are isomorphic if and only if $\lambda_1\equiv\lambda_2 \bmod m$.

Proof. Indeed, the module $V_\alpha(\lambda_1,\kappa)$ has an eigenvector for the action of σ which generates the $V_\alpha(\lambda_1,\kappa)$ by powers of $(\tau-1)$, i.e. the vectors

$$e, (\tau-1)e, (\tau-1)^2e, \ldots, (\tau-1)^{\kappa-1}e \tag{4.7}$$

form a basis of $V_\alpha(\lambda_1,\kappa)$.

The elements E which can generate $V_\alpha(\lambda_1,\kappa)$ by powers of $(\tau-1)$ are linear combinations

$$E=\sum_{\nu=0}^{\kappa-1}\lambda_i(\tau-1)^\nu e,$$

for $\lambda_i\in k$ and $\lambda_0\neq 0$.

On the other hand using eq. (4.6) we see that σ with respect to the basis given in eq. (4.7) admits the matrix form:

$$\begin{pmatrix} \zeta_m^\lambda & 0 & \cdots & \cdots & 0 \\ 0 & \alpha\zeta_m^\lambda & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \alpha^{\kappa-1}\zeta_m^\lambda \end{pmatrix}.$$

It is now easy to see from the above matrix that every eigenvector of the eigenvalue $\alpha^\nu\lambda_1$, $\nu>1$ is expressed as a linear combination of the basis given in eq. (4.7), where the coefficient of e is zero.

Therefore, the eigenvector of the eigenvalue $\alpha^\nu\zeta_m$ can not generate the module $V_\alpha(\lambda,\kappa)$ by powers of $(\sigma-1)^\nu$. □

4.3b The uniserial description

We will now give an alternative description of the indecomposable $C_q\rtimes C_m$-modules, which is used in [11].

It is known that $\mathrm{Aut}(C_q)\cong\mathbb{F}_p^*\times Q$, for some abelian p-group Q. The representation $\psi: C_m\to\mathrm{Aut}(C_q)$ given by the action of C_m on C_q is known to factor through a character $\chi: C_m\to\mathbb{F}_p^*$. The order of χ divides $p-1$ and $\chi^{p-1}=\chi^{-(p-1)}$ is the trivial one dimensional character.

For all $i\in\mathbb{Z}$, χ^i defines a simple $k[C_m]$-module of k dimension one, which we will denote by S_{χ^i}. For $0\leqslant\ell\leqslant m-1$ denote by S_ℓ the simple module where on which σ acts as ζ_m^ℓ. Both S_{χ^i}, S_ℓ can be seen as $k[C_q\rtimes C_m]$-modules using inflation. Finally for $0\leqslant\ell\leqslant m-1$ we define $\chi^i(\ell)\in\{0,1,\ldots,m-1\}$ such that $S_{\chi^i(\ell)}\cong S_\ell\otimes_k S_{\chi^i}$.

There are $q\cdot m$ isomorphism classes of indecomposable $k[C_q\rtimes C_m]$-modules and are all uniserial. An indecomposable $k[C_q\rtimes C_m]$-module U is unique determined by its socle, which is the kernel of the action of $\tau-1$ on U, and its k-dimension. For $0\leqslant\ell\leqslant m-1$ and $1\leqslant\mu\leqslant q$, let $U_{\ell,\mu}$ be the indecomposable $k[C_q\rtimes C_m]$ module with socle S_a and k-dimension μ. Then $U_{\ell,\mu}$ is uniserial and its μ ascending composition factors are the first μ composition factors of the sequence

$$S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}, S_\ell, S_{\chi^{-1}(\ell)}, S_{\chi^{-2}(\ell)}, \ldots, S_{\chi^{-(p-2)}(\ell)}.$$

Notice that in our notation $V_\alpha(\lambda,\kappa)=U_{\lambda+\kappa,\kappa}$.

Remark 4.3.1. The condition $\mathrm{ord}_{p^i} = m$ for all $1 \leqslant i \leqslant h$, is equivalent to requiring that $\psi_i : C_m \to \mathrm{Aut}(C_{p^i})$ is faithful for all i.

4.4 Lifting of representations

Proposition 4.4.0.1. Let $G = C_q \rtimes C_m$. Assume that for all $1 \leqslant i \leqslant h$, $\mathrm{ord}_{p^i} a = m$. If the G-module V lifts to an $R[G]$-module $\tilde{V}$, where $K = \mathrm{Quot}(R)$ is a field of characterstic zero, then

$$m \mid \left(\dim(\tilde{V} \otimes_R K) - \dim(\tilde{V} \otimes_R K)^{C_q}\right).$$

Moreover, if $\tilde{V}(\zeta_q^{\alpha^i \kappa})$ is the eigenspace of the eigenvalue $\zeta_q^{\alpha^i \kappa}$ of T acting on $\tilde{V}$, then

$$\dim \tilde{V}(\zeta_q^{\kappa}) = \dim \tilde{V}(\zeta_q^{\alpha\kappa}) = \dim \tilde{V}(\zeta_q^{\alpha^2\kappa}) = \cdots = \dim \tilde{V}(\zeta_q^{\alpha^{m-1}\kappa}).$$

Proof. Consider a lifting $\tilde{V}$ of V. The generator τ of the cyclic part C_q has eigenvalues $\lambda_1, \ldots, \lambda_s$ which are p^n-roots of unity. Let ζ_q be a primitive q-root of unity. Consider any eigenvalue $\lambda \neq 1$. It is of the form $\lambda = \zeta_q^\kappa$ for some $\kappa \in \mathbb{N}, q \nmid \kappa$. If E is an eigenvector of T corresponding to λ, that is $\tau E = \zeta_q^\kappa E$ then

$$\tau\sigma^{-1}E = \sigma^{-1}\tau^{\alpha}E = \zeta_q^{\kappa\alpha^{m-1}}\sigma^{-1}E$$

and we have a series of eigenvectors $E, \sigma^{-1}E, \sigma^{-2}E, \cdots$ with corresponding eigenvalues $\zeta_q^\kappa, \zeta_q^{\kappa\alpha}, \zeta_q^{\kappa a^2} \cdots, \zeta_q^{\kappa\alpha^o}$, where $o = \mathrm{ord}_{q/(q,k)}$. Indeed, the integer o satisfies the

$$\kappa\alpha^o \equiv \kappa \bmod q \Rightarrow \alpha^m \equiv 1 \bmod \frac{q}{(q,k)}.$$

Therefore the eigenvalues $\lambda \neq 1$ form orbits of size m, while the eigenspace of the eigenvalue 1 is just the invariant space V^G and the result follows. □

4.5 Indecomposable $C_q \rtimes C_m$odules, integral representation theory

From now on V be a free R-module, where R is an integral local principal ideal domain with maximal ideal m_R, R has characteristic zero and that R contains all q-th roots of unity and has characteristic zero. Let $K = \mathrm{Quot}(R)$.

The indecomposable modules for a cyclic group both in the ordinary and in the modular case are described by writing down the Jordan normal form of a generator of the cyclic group. Since in integral representation theory there are infinitely many non-isomorphic indecomposable C_q-modules for $q = p^h$, $h \geqslant 3$, one is not expecting to have a theory of Jordan normal forms even if one works over complete local principal ideal domains [38], [39].

Lemma 4.5.0.1. Let T be an element of order $q = p^h$ in $\mathrm{End}(V)$, then the minimal polynomial of T has simple eigenvalues and T is diagonalizable when seen as an element in $\mathrm{End}(V \otimes K)$.

Proof. Since $T^q = \mathrm{Id}_V$, the minimal polynomial of T divides $x^q - 1$, which has simple roots over a field of characteristic zero. This ensures that $T \in \mathrm{End}(V \otimes K)$ is diagonalizable. □

Lemma 4.5.0.2. Let $f(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_d)$ be the minimal polynomial of T on V. There is an element $E \in V$, such that

$$E, (T - \lambda_1 \mathrm{Id}_V)E, (T - \lambda_2 \mathrm{Id}_V)(T - \lambda_1 \mathrm{Id}_V)E, \ldots, (T - \lambda_{d-1}\mathrm{Id}_V) \cdots (T - \lambda_1 \mathrm{Id}_V)E$$

are linear independent elements in $V \otimes K$.

Proof. Consider the endomorphisms for $i = 1, \dots, d$

$$\Pi_i = \prod_{\substack{\nu=1 \\ \nu \neq i}}^{d} (T - \lambda_\nu \mathrm{Id}_V).$$

In the above product notice that $T - \lambda_i \mathrm{Id}_V$, $T - \lambda_j \mathrm{Id}_V$ are commuting endomorphisms. Since the minimal polynomial of T has degree d all R-modules $\mathrm{Ker}\Pi_i$ are strictly less than V. Moreover there is an element E such that $E \notin \mathrm{Ker}(\Pi_i)$ for all $1 \leqslant i \leqslant d$. Consider a relation

$$\sum_{\mu=0}^{d} \gamma_\mu \prod_{\nu=0}^{\mu} (T - \lambda_\mu \mathrm{Id}_V) E, \tag{4.8}$$

where $\prod_{\nu=0}^{0} (T - \lambda_\nu \mathrm{Id}_V) E = E$. We fist apply the operator $\prod_{\nu=2}^{d} (T - \lambda_\nu \mathrm{Id}_V)$ to eq. (4.8) and we obtain

$$0 = \gamma_0 \Pi_1 E,$$

and by the selection of E we have that $a_0 = 0$. We now apply $\prod_{\nu=3}^{d} (T - \lambda_\nu \mathrm{Id}_V)$ to eq. (4.8). We obtain that

$$0 = \gamma_1 \prod_{\nu=3}^{d} (T - \lambda_\nu \mathrm{Id}_V)(T - \lambda_1 \mathrm{Id}_V) = \gamma_1 \Pi_2 E,$$

and by the selection of E we have that $\gamma_1 = 0$. We now apply $\prod_{\nu=4}^{d} (T - \lambda_\nu \mathrm{Id}_V)$ to eq. (4.8) and we obtain

$$0 = \gamma_2 \prod_{\nu=4}^{d} (T - \lambda_\nu \mathrm{Id}_V)(T - \lambda_2 \mathrm{Id}_V)(T - \lambda_1 \mathrm{Id}_V) E = \gamma_2 \Pi_3 E$$

and by the selection of E we obtain $\gamma_3 = 0$. Continuing this way we finally arrive at $\gamma_0 = \gamma_1 = \cdots = \gamma_{d-1} = 0$. □

Lemma 4.5.0.3. Let V be a free R-module of rank R acted on by an automorphism $T : V \to V$ of order p^h. Assume that the minimal polynomial of T is of degree d and has roots $\lambda_1, \dots, \lambda_d$. Then T can be written as a matrix with respect to the basis as follows:

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ a_1 & \lambda_2 & \ddots & & \vdots \\ 0 & a_2 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{d-1} & \lambda_d \end{pmatrix} \tag{4.9}$$

Proof. By lemma 4.5.0.2 the elements

$$E, (T - \lambda_1 \mathrm{Id}_V)E, (T - \lambda_2 \mathrm{Id}_V)(T - \lambda_1 \mathrm{Id}_V)E, \dots, (T - \lambda_{d-1} \mathrm{Id}_V) \cdots (T - \lambda_1 \mathrm{Id}_V)E$$

form a free submodule of V of rank d. The theory of submodules of principal ideal domains, there is a basis $E_1, E_2, \dots, E_d$ of the free module V such that

$$\begin{aligned} E_1 &= E, \\ a_1 E_2 &= (T - \lambda_1 \mathrm{Id}_V) E_1, \\ a_2 E_3 &= (T - \lambda_2 \mathrm{Id}_V) E_2, \\ &\cdots \\ a_{s-1} E_d &= (T - \lambda_{d-1} \mathrm{Id}_V) E_{d-1}. \end{aligned} \tag{4.10}$$

Let us consider the module $V_1 = \langle E_1, \dots, E_d \rangle \subset V$. By construction, the map T restricts to an automorphism $V_1 \to V_1$ with respect to the basis $E_1, \dots, E_d$ has the desired form. We then consider the free module V/V_1 and we repeat the procedure for the minimal polynomial of T, which again acts on V/V_1. The desired result follows. □

Remark 4.5.1. The element T as defined in eq. (4.9) has order equal to the higher order of the eigenvalues $\lambda_1, \ldots, \lambda_d$ involved. Indeed, since we have assumed that the eigenvalues are different the matrix is diagonalizable in $\mathrm{Quot}(R)$ and has order equal to the maximal order of the eigenvalues involved. In particular it has order q if there is at least one λ_i that is a primitive q-root of unity. The statement about the order of T is not necessarily true if some of the eigenvalues are the same. For instance the matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ has infinite order over a field of characteristic zero.

Remark 4.5.2. The number of indecomposable $R[T]$-summands of V is given by $\#\{i : a_i = 0\} + 1$.

A lift of a sum of indecomposable kC_q-modules $J_{\kappa_1} \oplus \cdots \oplus J_{\kappa_n}$ can form an indecomposable RC_q-module. For example the indecomposable module where the generator T of C_q has the form

$$T = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ a_1 & \lambda_2 & \ddots & & \vdots \\ 0 & a_2 & \lambda_3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{s-1} & \lambda_d \end{pmatrix}$$

where $a_1 = \cdots = a_{\kappa_1 - 1} = 1$, $a_{\kappa_1} \in m_R$, $a_{\kappa_1+1}, \ldots, a_{\kappa_2+\kappa_1-1} = 1$, $a_{\kappa_2+\kappa_1} \in m_R$, etc reduces to a decomposable direct sum of Jordan normal forms of sizes $J_{\kappa_1}, J_{\kappa_2 - \kappa_1}, \cdots$.

Remark 4.5.3. It is an interesting question to classify these matrices up to conjugation with a matrix in $\mathrm{GL}_d(R)$. It seems that the valuation of elements a_i should also play a role.

Definition 4.5.3.1. Let $h_i(x_1, \ldots, x_j)$ be the complete symmetric polynomial of degree i in the variables $x_1, \ldots, x_j$. For instance

$$h_3(x_1, x_2, x_3) = x_1^3 + x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^3 + x_2^2 x_3 + x_2 x_3^2 + x_3^3.$$

Set

$$L(\kappa, j, \nu) = h_\kappa(\lambda_j, \lambda_{j+1}, \ldots, \lambda_{j+\nu})$$
$$A(i, j) = \begin{cases} a_i a_{i+1} \cdots a_{i+j} & \text{if } j \geqslant 0 \\ 0 & \text{if } j < 0 \end{cases}$$

Lemma 4.5.3.1. The matrix $T^\alpha = (t_{ij}^{(\alpha)})$ is given by the following formula:

$$t_{ij}^{(\alpha)} = \begin{cases} \lambda_i^\alpha & \text{if } i = j \\ A(j, i-j-1) \cdot L(\alpha - (i-j), j, i-j) & \text{if } j < i \\ 0 & \text{if } j > i \end{cases}$$

Proof. For $j \geqslant i$ the proof is trivial. When $j < i$ and $\alpha = 1$ it is immediate, since $L(x, \cdot, \cdot) \equiv 0$, for every $x \leqslant 0$. Assume this holds for $\alpha = n$. If $\alpha = n + 1$,

$$\begin{aligned} t_{ij}^{(n+1)} &= t_{ij}^{(n)} t_{ij} = \sum_{k=1}^{r} t_{ik}^{(\alpha)} t_{kj} = \lambda_j t_{ij}^{(\alpha)} + a_j t_{ij+1}^{(\alpha)} = \lambda_j A(j, i-j-1) L(\alpha - (i-j), j, i-j) + \\ &+ a_j A(j+1, i-j-2) L(\alpha - (i-j-1), j+1, i-j-1) = \\ &= A(j, i-j-1) \left(\lambda_j h_{\alpha-(i-j)}(\lambda_j, \ldots, \lambda_j) + h_{\alpha-(i-j)+1}(\lambda_{j+1}, \ldots, \lambda_i) \right) = \\ &= A(j, i-j-1) h_{\alpha-(i-j)+1}(\lambda_j, \ldots, \lambda_i) = \\ &= A(j, i-j-1) L(\alpha - (i-j) + 1, i, i-j). \end{aligned}$$

$\square$

Remark 4.5.4. The space of homogeneous polynomials of degree k in n-variables has dimension $\binom{n-1+c}{n-1}$. Since all q-roots of unity are reduced to 1 modulo $\mathfrak{m}_R$ the quantity $L(\alpha-(i-j),j,i-j)$ is reduced to $n=(i-j)+1$, $c=\alpha-(i-j)$

$$\binom{n-1+c}{n-1}=\binom{\alpha}{i-j}.$$

This equation is compatible with the computation of τ^α given in eq. (4.4).

Lemma 4.5.4.1. There is an eigenvector E of the generator σ of the cyclic group C_m which is not an element in $\bigcup_{i=1}^{s}\mathrm{Ker}(\Pi_i\otimes K)$.

Proof. The eigenvectors $E_1,\dots,E_d$ of σ form a basis of the space $V\otimes K$. By multiplying by certain elements in R, if necessary, we can assume that all E_i are in V and their reductions $E_i\otimes R/\mathfrak{m}_R$, $1\leqslant i\leqslant d$ give rise to a basis of eigenvectors of a generator of the cyclic group C_m acting on $V\otimes R/\mathfrak{m}_R$. If every eigenvector E_i is an element of some $\mathrm{Ker}(\Pi_\nu)$ for $1\leqslant i\leqslant d$, then their reductions will be elements in $\mathrm{Ker}(T-1)^{d-1}$, a contradiction since the later kernel has dimension $<d$. □

Lemma 4.5.4.2. Let V be a free $C_q\rtimes C_m$-module, which is indecomposable as a C_q-module. Consider the basis given in lemma 4.5.0.3. Then the value of $\sigma(E_1)$ determines $\sigma(E_i)$ for $2\leqslant i\leqslant d$.

Proof. Let σ be a generator of the cyclic group C_m. We will use the notation of lemma 4.5.0.2. We use lemma 4.5.4.1 in order to select a suitable eigenvector of E_1 of σ and then form the basis $E_1,E_2,\dots,E_d$ as given in eq. (4.10). We can compute the action of σ on all basis elements E_i by

$$\sigma(a_{i-1}E_i)=\sigma(T-\lambda_{i-1}\mathrm{Id}_V)E_{i-1}=(T^a-\lambda_{i-1}\mathrm{Id}_V)\sigma(E_{i-1}). \tag{4.11}$$

This means that one can define recursively the action of σ on all elements E_i. Indeed, assume that

$$\sigma(E_{i-1})=\sum_{\nu=1}^{d}\gamma_{\nu,i-1}E_\nu.$$

We now have

$$\begin{aligned}(T^a-\lambda_{i-1}\mathrm{Id}_V)E_\nu&=\sum_{\mu=1}^{d}t^{(\alpha)}_{\mu,\nu}E_\mu-\lambda_{i-1}E_\nu\\&=(\lambda_\nu^\alpha-\lambda_{i-1})E_\nu+\sum_{\mu=\nu+1}^{d}t^{(\alpha)}_{\mu,\nu}E_\mu\end{aligned}$$

We combine all the above to

$$\begin{aligned}a_{i-1}\sigma(E_i)&=\sum_{\nu=1}^{d}\gamma_{\nu,i-1}(\lambda_\nu^\alpha-\lambda_{i-1})E_\nu+\sum_{\nu=1}^{d}\gamma_{\nu,i-1}\sum_{\mu=\nu+1}^{d}t^{(\alpha)}_{\mu,\nu}E_\mu\\&=\sum_{\nu=1}^{d}\tilde{\gamma}_{\nu,i}E_\nu,\end{aligned} \tag{4.12}$$

for a selection of elements $\gamma_{\nu,i}\in R$, which can be explicitly computed by collecting the coefficients of the basis elements $E_1,\dots,E_d$.

Observe that the quantity on the right hand side of eq. (4.12) must be divisible by a_{i-1}. Indeed, let v be the valuation of the local principal ideal domain R. Set

$$e_0=\min_{1\leqslant\nu\leqslant d}\{v(\tilde{\gamma}_{\nu,i})\}.$$

If $e_0 < v(a_{i-1})$ then we divide eq. (4.12) by π^{e_0} where π is the local uniformizer of R, that is $\mathfrak{m}_R = \pi R$. We then consider the divided equation modulo $\mathfrak{m}_R$ to obtain a linear dependence relation among the elements $E_i \otimes k$, which is a contradiction. Therefore $e_0 \geqslant v(a_{i-1})$ and we obtain an equation

$$\sigma(E_i) = \sum_{\nu=1}^{d} \frac{\tilde{\gamma}_{\nu,i}}{a_{i-1}} E_\nu = \sum_{\nu=1}^{d} \gamma_{\nu,i} E_\nu.$$

□

For example $\sigma(E_1) = \zeta_m^\epsilon E_1$. We compute that

$$a_1 \sigma(E_2) = (T^\alpha - \lambda_1 \mathrm{Id})\sigma(E_1)$$

and

$$\begin{aligned}
\sigma(E_2) &= \frac{(\lambda_1^\alpha - \lambda_1)}{a_1} \zeta_\mu^\epsilon E_1 + \zeta_m^\epsilon \sum_{\mu=2}^{d} \frac{t_{\mu,1}^{(\alpha)}}{a_1} E_\mu \\
&= \frac{(\lambda_1^\alpha - \lambda_1)}{a_1} \zeta_\mu^\epsilon E_1 + \zeta_m^\epsilon \sum_{\mu=2}^{d} \frac{A(1,\mu-2)L(\alpha-(\mu-1),1,\mu-1)}{a_1} E_\mu \\
&= \frac{(\lambda_1^\alpha - \lambda_1)}{a_1} \zeta_\mu^\epsilon E_1 + \zeta_m^\epsilon \sum_{\mu=2}^{d} \frac{a_1 a_2 \cdots a_{\mu-1} h_{\alpha-(\mu-1)}(\lambda_1,\lambda_2,\ldots,\lambda_\mu)}{a_1} E_\mu.
\end{aligned}$$

Proposition 4.5.4.1. Assume that no element $a_1, \ldots, a_{d-1}$ given in eq. (4.9) is zero. Given $\alpha \in \mathbb{N}, \alpha \geqslant 1$ and an element E_1, which is not an element in $\bigcup_{i=1}^{d} \mathrm{Ker}(\Pi_i \otimes K)$, if there is a matrix $\Gamma = (\gamma_{ij})$, such that $\Gamma T \Gamma^{-1} = T^\alpha$ and $\Gamma E_1 = \zeta_m^\epsilon E_1$, then this matrix Γ is unique.

Proof. We will use the idea leading to equation (4.11) replacing σ with Γ. We will compute recursively and uniquely the entries $\gamma_{\mu,i}$, arriving at the explicit formula of eq. (4.18).

Observe that trivially $\gamma_{\nu,1} = 0$ for all $\nu < 1$ since we only allow $1 \leqslant \nu \leqslant d$. We compute

$$\begin{aligned}
\tilde{\gamma}_{\mu,i} &= \gamma_{\mu,i-1}(\lambda_\mu^\alpha - \lambda_{i-1}) + \sum_{\nu=1}^{\mu-1} \gamma_{\nu,i-1} t_{\mu,\nu}^{(\alpha)} \\
&= \gamma_{\mu,i-1}(\lambda_\mu^\alpha - \lambda_{i-1}) + \sum_{\nu=1}^{\mu-1} \gamma_{\nu,i-1} A(\nu,\mu-\nu-1) L\big(\alpha-(\mu-\nu),\nu,\mu-\nu\big) \\
&= \gamma_{\mu,i-1}(\lambda_\mu^\alpha - \lambda_{i-1}) + \sum_{\nu=1}^{\mu-1} \gamma_{\nu,i-1} a_\nu a_{\nu+1} \cdots a_{\mu-1} h_{\alpha-\mu+\nu}(\lambda_\nu,\lambda_{\nu+1},\ldots,\lambda_\mu)
\end{aligned} \tag{4.13}$$

Define

$$[\lambda_m^\alpha - \lambda_x]_i^j = \prod_{x=i}^{j} (\lambda_\mu^\alpha - \lambda_x)$$

$$[a]_i^j = \prod_{x=i}^{j} a_x$$

for $i \leqslant j$. If $i > j$ then both of the above quantities are defined to be equal to 1.

Observe that for $\mu = 1$ eq. (4.13) becomes

$$\gamma_{1,i} = \frac{1}{a_{i-1}} \gamma_{1,i-1}(\lambda_1^\alpha - \lambda_{i-1}) \tag{4.14}$$

and we arrive at (assuming that $\Gamma(E_1) = \zeta_m^\epsilon E_1$)

$$\gamma_{1,i} = \frac{\zeta_m^\epsilon}{a_1 a_2 \cdots a_{i-1}} \prod_{x=1}^{i-1} (\lambda_1^\alpha - \lambda_x) = \frac{\zeta_m^\epsilon}{a_1 a_2 \cdots a_{i-1}} [\lambda_1^\alpha - \lambda_x]_1^{i-1}. \tag{4.15}$$

For $\mu \geqslant 2$ we have $\gamma_{\mu,1} = 0$, since by assumption $TE_1 = \zeta_m^{\epsilon} E_1$. Therefore eq. (4.13) gives us

$$\begin{aligned}
\gamma_{\mu,i} &= \sum_{\kappa_1=0}^{i-2} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-\kappa_1}^{i-1}}{[a]_{i-1-\kappa_1}^{i-1}} \sum_{\mu_2=1}^{\mu-1} \gamma_{\mu_2,i-1-\kappa_1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \\
&= \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \sum_{\kappa_1=0}^{i-2} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-\kappa_1}^{i-1}}{[a]_{i-1-\kappa_1}^{i-1}} \gamma_{\mu_2,i-1-\kappa_1}.
\end{aligned} \tag{4.16}$$

We will now prove eq. (4.16) by induction on i. For $i = 2, \mu \geqslant 2$ we have

$$\begin{aligned}
\gamma_{\mu,2} &= \frac{1}{a_1}\gamma_{\mu,1}(\lambda_\mu^\alpha - \lambda_1) + \frac{1}{a_1}\sum_{\mu_2=1}^{\mu-1} \gamma_{\mu_2,1}[a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \\
&= \frac{1}{a_1}[a]_1^{\mu-1} h_{\alpha-\mu+1}(\lambda_1,\dots,\lambda_\mu)\gamma_{1,1}.
\end{aligned}$$

Assume now that eq. (4.16) holds for computing $\gamma_{\mu,i-1}$. We will treat the $\gamma_{\mu,i}$ case. We have

$$\begin{aligned}
\gamma_{\mu,i} &= \frac{(\lambda_\mu^\alpha - \lambda_{i-1})}{a_{i-1}}\gamma_{\mu,i-1} + \frac{1}{a_{i-1}} \sum_{\mu_2=1}^{\mu-1} \gamma_{\mu_2,i-1}[a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \\
&= \frac{(\lambda_\mu^\alpha - \lambda_{i-1})}{a_{i-1}} \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \sum_{\kappa_1=0}^{i-3} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-1-\kappa_1}^{i-2}}{[a]_{i-2-\kappa_1}^{i-2}} \gamma_{\mu_2,i-2-\kappa_1} \\
&+ \frac{1}{a_{i-1}} \sum_{\mu_2=1}^{\mu-1} \gamma_{\mu_2,i-1}[a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \\
&= \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \sum_{\kappa_1=0}^{i-3} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-1-\kappa_1}^{i-1}}{[a]_{i-2-\kappa_1}^{i-1}} \gamma_{\mu_2,i-2-\kappa_1} \\
&+ \frac{1}{a_{i-1}} \sum_{\mu_2=1}^{\mu-1} \gamma_{\mu_2,i-1}[a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \\
&= \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \sum_{\kappa_1=1}^{i-2} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-\kappa_1}^{i-1}}{[a]_{i-1-\kappa_1}^{i-1}} \gamma_{\mu_2,i-1-\kappa_1} \\
&+ \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \frac{1}{a_{i-1}} \gamma_{\mu_2,i-1} \\
&= \sum_{\mu_2=1}^{\mu-1} [a]_{\mu_2}^{\mu-1} h_{\alpha-\mu+\mu_2}(\lambda_{\mu_2},\dots,\lambda_\mu) \sum_{\kappa_1=0}^{i-2} \frac{[\lambda_\mu^\alpha - \lambda_x]_{i-\kappa_1}^{i-1}}{[a]_{i-1-\kappa_1}^{i-1}} \gamma_{\mu_2,i-1-\kappa_1}
\end{aligned}$$

and equation (4.16) is now proved.

We proceed recursively applying eq. (4.16) to each of the summands $\gamma_{\mu_2,i-1-\kappa_1}$ if $\mu_2 > 1$ and $i-1-\kappa_1 > 1$. If $\mu_2 = 1$, then $\gamma_{\mu_2,i-1-\kappa_1}$ is computed by eq. (4.14) and if $\mu_2 > 1$ and $i-1-\kappa_1 \leqslant 1$ then $\gamma_{\mu_2,i-1-\kappa_1} = 0$. We can classify all iterations needed by the set Σ_μ of sequences $(\mu_s, \mu_{s-1}, \dots, \mu_3, \mu_2)$ such that

$$1 = \mu_s < \mu_{s-1} < \cdots < \mu_3 < \mu_2 < \mu = \mu_1. \tag{4.17}$$

For example for $\mu = 5$ the set of such sequences is given by

$$\Sigma_\mu = \{(1), (1,2), (1,3), (1,2,3), (1,4), (1,2,4), (1,3,4), (1,2,3,4)\}$$

corresponding to the tree of iterations given in figure 4.1. The length of the sequence $(\mu_s, \mu_{s-1}, \dots, \mu_2)$ is given in eq. (4.17) is $s-1$. In each iteration the i changes to $i-1-k$ thus we have the following sequence of indices

$$i_1 = i \to i_2 = i-1-\kappa_1 \to i_3 = i-2-(\kappa_1+\kappa_2) \to \cdots \to i_s = i-(s-1)-(\kappa_1+\cdots+\kappa_{s-1})$$

For the sequence $i_1, i_2, \dots,$ we might have $i_t = 1$ for $t < s-1$. But in this case, we will arrive at the

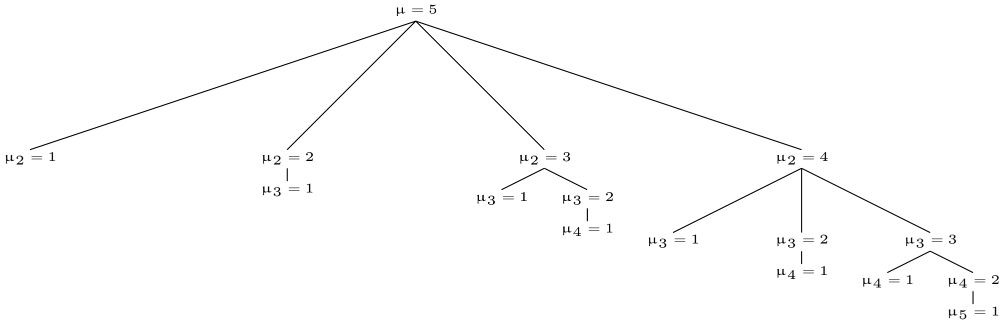


Figure 4.1: Iteration tree for $\mu = 5$

element $\gamma_{\mu_{t+1},i_t} = \gamma_{\mu_t,1} = 0$ since $\mu_t > 1$. This means that we will have to consider only selections $\kappa_1, \dots, \kappa_{s-1}$ such that $i_{s-1} \geqslant 1$. Therefore we arrive at the following expression for $\mu \geqslant 2$

$$\begin{aligned}
\gamma_{\mu,i} = & \sum_{(\mu_s,\dots,\mu_2)\in\Sigma_\mu} [a]^{\mu-1}_{\mu_2}[a]^{\mu_2-1}_{\mu_3}\cdots[a]^{\mu_{s-1}-1}_{\mu_s} \prod_{\nu=2}^{s} h_{\alpha-\mu_{\nu-1}+\mu_\nu}(\lambda_{\mu_\nu},\dots,\lambda_{\mu_{\nu-1}}) \\
& \cdot \sum_{i=i_1>i_2>\cdots>i_s\geqslant 1} \prod_{\nu=1}^{s-1} \frac{[\lambda^\alpha_{\mu_\nu}-\lambda_x]^{i_\nu-1}_{i_{\nu+1}+1}}{[a]^{i_\nu-1}_{i_{\nu+1}}} \cdot \gamma_{1,i_s}. \\
= & \sum_{(\mu_s,\dots,\mu_2)\in\Sigma_\mu} \prod_{\nu=2}^{s} h_{\alpha-\mu_{\nu-1}+\mu_\nu}(\lambda_{\mu_\nu},\dots,\lambda_{\mu_{\nu-1}}) \\
& \cdot \sum_{i=i_1>i_2>\cdots>i_s\geqslant 1} \frac{[a]^{\mu-1}_1}{[a]^{i-1}_{i_s}} \prod_{\nu=1}^{s-1}[\lambda^\alpha_{\mu_\nu}-\lambda_x]^{i_\nu-1}_{i_{\nu+1}+1} \frac{\zeta^\epsilon_m[\lambda^\alpha_1-\lambda_x]^{i_s-1}_1}{[a]^{i_s-1}_1} \\
= & \sum_{(\mu_s,\dots,\mu_2)\in\Sigma_\mu} \prod_{\nu=2}^{s} h_{\alpha-\mu_{\nu-1}+\mu_\nu}(\lambda_{\mu_\nu},\dots,\lambda_{\mu_{\nu-1}}) \frac{[a]^{\mu-1}_1}{[a]^{i-1}_1}\zeta^\epsilon_m \sum_{i=i_1>i_2>\cdots>i_s\geqslant 1} \prod_{\nu=1}^{s}[\lambda^\alpha_{\mu_\nu}-\lambda_x]^{i_\nu-1}_{i_{\nu+1}+1}
\end{aligned} \tag{4.18}$$

where $i_{s+1}+1 = 1$ that is $i_{s+1} = 0$. □

We will now prove that the matrix Γ of lemma 4.5.4.1 exists by cheking that $\Gamma T = T^\alpha\Gamma$. Set $(a_{\mu,i}) = \Gamma T$, $(b_{\mu,i}) = T^\alpha\Gamma$. For $i < d$ we have

$$\begin{aligned}
a_{\mu,i} &= \sum_{\nu=1}^{d} \gamma_{\mu,\nu}t_{\nu,i} = \gamma_{\mu,i}t_{ii} + \gamma_{\mu,i+1}t_{i+1,i} \\
&= \gamma_{\mu,i}\lambda_i + \gamma_{\mu,i}(\lambda^\alpha_\mu - \lambda_i) + \sum_{\nu=1}^{\mu-1}\gamma_{\nu,i}t^{(\alpha)}_{\mu,\nu} \\
&= \gamma_{\mu,i}\lambda^\alpha_\mu + \sum_{\nu=1}^{\mu-1}\gamma_{\nu,i}t^{(\alpha)}_{\mu,\nu} = \sum_{\nu=1}^{\mu} t^{(\alpha)}_{\mu,\nu}\gamma_{\nu,i} = b_{\mu,i}.
\end{aligned}$$

For $i = d$ we have:

$$a_{\mu,d} = \sum_{\nu=1}^{d}\gamma_{\mu,\nu}t_{\nu,d} = \gamma_{\mu,d}t_{d,d} = \gamma_{\mu,d}\lambda_d$$

while

$$b_{\mu,d} = \sum_{\nu=1}^{d} t^{(\alpha)}_{\mu,\nu}\gamma_{\nu,d} = \sum_{\nu=1}^{\mu-1} t^{(\alpha)}_{\mu,\nu}\gamma_{\nu,d} + \lambda^\alpha_\mu\gamma_{\mu,d}$$

This gives us the relation

$$(\lambda_d - \lambda^a_\mu)\gamma_{\mu,d} = \sum_{\nu=1}^{\mu-1} t^{(\alpha)}_{\mu,\nu}\gamma_{\nu,d} \tag{4.19}$$

For $\mu = 1$ using eq. (4.15) we have

$$\gamma_{1,d}\lambda_d = \gamma_{1,d}\lambda_1^{\alpha} \Rightarrow [\lambda_1^{\alpha} - \lambda_x]_1^d = 0.$$

This relation is satisfied if λ_1^{α} is one of $\{\lambda_1, \dots, \lambda_d\}$. Without loss of generality we assume that

$$\lambda_i^{(a)} = \begin{cases} \lambda_{i+1} & \text{if } m \nmid i \\ \lambda_{i-m+1} & \text{if } m \mid i \end{cases} \tag{4.20}$$

We have the following conditions:

$$\begin{array}{ll}
\mu = 2 & (\lambda_d - \lambda_2^{\alpha})\gamma_{2,d} = t_{2,1}^{(\alpha)}\gamma_{1,d} \\
\mu = 3 & (\lambda_d - \lambda_3^{\alpha})\gamma_{3,d} = t_{3,1}^{(\alpha)}\gamma_{1,d} + t_{3,2}^{(\alpha)}\gamma_{2,d} \\
\mu = 4 & (\lambda_d - \lambda_4^{\alpha})\gamma_{4,d} = t_{4,1}^{(\alpha)}\gamma_{1,d} + t_{4,2}^{(\alpha)}\gamma_{2,d} + t_{4,3}^{(\alpha)}\gamma_{3,d} \\
\vdots & \vdots \\
\mu = d-1 & (\lambda_d - \lambda_{d-1}^{\alpha})\gamma_{d-1,d} = t_{d-1,1}^{(\alpha)}\gamma_{1,d} + t_{d-1,2}^{(\alpha)}\gamma_{2,d} + \cdots + t_{d-1,d-2}^{(\alpha)}\gamma_{d-1,d}
\end{array}$$

All these equations are true provided that $\gamma_{1,d}, \dots, \gamma_{d-2,d} = 0$. Finally, for $\mu = d$, we have

$$(\lambda_d - \lambda_d^{\alpha})\gamma_{d,d} = \sum_{\nu=1}^{d-1} t_{d,\nu}^{(\alpha)}\gamma_{\nu,d} \tag{4.21}$$

which is true provided that $(\lambda_d - \lambda_d^{\alpha})\gamma_{d,d} = t_{d,d-1}^{(a)}\gamma_{d-1,d}$.

Lemma 4.5.4.3. For $n \geqslant 2$ the vertical sum S_n of the products of every line of the following array

$$\begin{array}{c|cccccc}
y & & & & & & \\
\hline
1 & 1 & (x_1 - x_2) & (x_1 - x_3) & \cdots & \cdots & (x_1 - x_n) \\
2 & (z - x_1) & 1 & (x_1 - x_3) & \cdots & \cdots & (x_1 - x_n) \\
3 & (z - x_1) & (z - x_2) & 1 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & & \vdots \\
\vdots & \vdots & & & \ddots & & \vdots \\
n-1 & (z - x_1) & (z - x_2) & \cdots & (z - x_{n-2}) & 1 & (x_1 - x_n) \\
n & (z - x_1) & (z - x_2) & \cdots & (z - x_{n-2}) & (z - x_{n-1}) & 1
\end{array}$$

is given by

$$S_n = \sum_{y=1}^{n} \prod_{\nu=y+1}^{n} (x_1 - x_\nu) \prod_{\mu=1}^{y-1} (z - x_\mu) = (z - x_2) \cdots (z - x_n).$$

In particular when $z = x_n$ the sum is zero.

Proof. We will prove the lemma by induction. For $n = 2$ we have $S_2 = (x_1 - x_2) + (z - x_1) = z - x_2$. Assume that the equality holds for n. The sum S_{n+1} corresponds to the array:

$$\begin{array}{c|cccccc}
y & & & & & & \\
\hline
1 & 1 & (x_1 - x_2) & (x_1 - x_3) & \cdots & (x_1 - x_n) & (x_1 - x_{n+1}) \\
2 & (z - x_1) & 1 & (x_1 - x_3) & \cdots & (x_1 - x_n) & (x_1 - x_{n+1}) \\
3 & (z - x_1) & (z - x_2) & 1 & \ddots & \vdots & \vdots \\
\vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\
n-1 & (z - x_1) & \cdots & (z - x_{n-2}) & 1 & (x_1 - x_n) & (x_1 - x_{n+1}) \\
n & (z - x_1) & (z - x_2) & \cdots & (z - x_{n-1}) & 1 & (x_1 - x_{n+1}) \\
n+1 & (z - x_1) & (z - x_2) & \cdots & (z - x_{n-1}) & (z - x_n) & 1
\end{array}$$

We have by definition $S_{n+1} = S_n(x_1 - x_{n+1}) + (z - x_1)(z - x_2)\cdots(z - x_n)$, which by induction gives

$$\begin{aligned} S_{n+1} &= (z - x_2)\cdots(z - x_n)(x_1 - x_{n+1}) + (z - x_1)(z - x_2)\cdots(z - x_n) \\ &= (z - x_2)\cdots(z - x_n)(x_1 - x_{n+1} + z - x_1) \end{aligned}$$

and gives the desired result. □

Lemma 4.5.4.4. Consider $A < l < L < B$. The quantity

$$\sum_{l \leqslant y \leqslant L} [\lambda_a - \lambda_x]_A^{y-1} \cdot [\lambda_b - \lambda_x]_{y+1}^B$$

equals to

$$[\lambda_a - \lambda_x]_A^{l-1} \cdot [\lambda_b - \lambda_x]_{L+1}^B \cdot \frac{[\lambda_a - \lambda_x]_l^L - [\lambda_b - \lambda_x]_l^L}{(\lambda_a - \lambda_b)}$$

Proof. We write

$$\begin{aligned} &\sum_{l \leqslant y \leqslant L} [\lambda_a - \lambda_x]_A^{y-1} \cdot [\lambda_b - \lambda_x]_{y+1}^B \\ &= [\lambda_a - \lambda_x]_A^{l-1} \cdot [\lambda_b - \lambda_x]_{L+1}^B \cdot \sum_{l \leqslant y \leqslant L} [\lambda_a - \lambda_x]_l^{y-1} \cdot [\lambda_b - \lambda_x]_{y+1}^L \end{aligned}$$

The last sum can be read as the vertical sum S of the products of every line in the following array:

$$\begin{array}{c|cccccc}
y & & & & & & \\
\hline
l & 1 & (\lambda_b - \lambda_{l+1}) & (\lambda_b - \lambda_{l+2}) & \cdots & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
l+1 & (\lambda_a - \lambda_l) & 1 & (\lambda_b - \lambda_{l+2}) & \cdots & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
l+2 & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & 1 & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
L-2 & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & 1 & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
L-1 & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & (\lambda_a - \lambda_{L-2}) & 1 & (\lambda_b - \lambda_L) \\
L & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & (\lambda_a - \lambda_{L-2}) & (\lambda_a - \lambda_{L-1}) & 1
\end{array}$$

If $l = b$, then lemma 4.5.4.3 implies that $S = [\lambda_a - \lambda_x]_{b+1}^L$. Furthermore, if $L = a$ then $S = 0$.

The quantity S cannot be directly computed using lemma 4.5.4.3, if $l \neq b$. We proceed by forming the array:

$$\begin{array}{c|ccc|cccccc}
y & & & & & & & & & \\
\hline
b & 1 & (\lambda_b - \lambda_{b+1}) & \cdots & (\lambda_b - \lambda_l) & \cdots & \cdots & \cdots & \cdots & (\lambda_b - \lambda_L) \\
\vdots & & & & \vdots & & & & & \vdots \\
l-1 & (\lambda_a - \lambda_b) & \cdots & 1 & (\lambda_b - \lambda_l) & \cdots & \cdots & \cdots & \cdots & (\lambda_b - \lambda_L) \\
\hline
l & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & 1 & (\lambda_b - \lambda_{l+1}) & (\lambda_b - \lambda_{l+2}) & \cdots & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
l+1 & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & (\lambda_a - \lambda_l) & 1 & (\lambda_b - \lambda_{l+2}) & \cdots & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
l+2 & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & 1 & & \vdots & \vdots \\
\vdots & & & & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
L-2 & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & 1 & (\lambda_b - \lambda_{L-1}) & (\lambda_b - \lambda_L) \\
L-1 & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & (\lambda_a - \lambda_{L-2}) & 1 & (\lambda_b - \lambda_L) \\
L & (\lambda_a - \lambda_b) & \cdots & (\lambda_a - \lambda_{l-1}) & (\lambda_a - \lambda_l) & (\lambda_a - \lambda_{l+1}) & \cdots & (\lambda_a - \lambda_{L-2}) & (\lambda_a - \lambda_{L-1}) & 1
\end{array}$$

The value of this array is computed using lemma 4.5.4.3 to be equal to $[\lambda_a - \lambda_x]_{b+1}^L$. We observe that the sum of the products of the top left array can be computed using lemma 4.5.4.3, while the sum of the products of the lower right array is S.

$$[\lambda_a - \lambda_x]_b^{l-1} \cdot S + [\lambda_a - \lambda_x]_{b+1}^{l-1} \cdot [\lambda_b - \lambda_x]_l^L = [\lambda_a - \lambda_x]_{b+1}^L$$

we arrive at

$$[\lambda_a - \lambda_x]_b^{l-1} S = [\lambda_a - \lambda_x]_{b+1}^{l-1} \left([\lambda_a - \lambda_x]_l^L - [\lambda_b - \lambda_x]_l^L\right)$$

or equivalently

$$(\lambda_a - \lambda_b) \cdot S = [\lambda_a - \lambda_x]_l^L - [\lambda_b - \lambda_x]_l^L$$

□

Lemma 4.5.4.5. For all $1 \leqslant \mu \leqslant d-2$ we have $\gamma_{\mu,d} = 0$.

Proof. Let $\mu_1 = \mu > \mu_2 > \cdots > \mu_s = 1 \in \Sigma_\mu$ be a selection of iterations and $d = i_1 > i_2 > \cdots\cdots i_s \geqslant 1 > i_{s+1} = 0$ be the sequence of i's. Using eq. (4.20) we see that the quantity $[\lambda^\alpha_{\mu_\nu} - \lambda_x]^{i_\nu - 1}_{i_{\nu+1}+1} \neq 0$ if and only if one of the following two inequalities hold:

$$\text{either} \qquad i_{\nu+1} > \mu_\nu - mf(\mu_\nu) \tag{4.22}$$

$$\text{or} \qquad i_\nu < \mu_\nu + 2 - mf(\mu_\nu), \tag{4.23}$$

where

$$f(x) = \begin{cases} 1 & \text{if } m \mid x \\ 0 & \text{if } m \nmid x \end{cases}$$

We will denote the above two inequalities by $(4.22)_\nu$,$(4.23)_\nu$ when applied for the integer ν. Assume, that for all $1 \leqslant \nu \leqslant s$ one of the two inequalities $(4.22)_\nu$,$(4.23)_\nu$ hold, that is $[\lambda^\alpha_{\mu_\nu} - \lambda_x]^{i_\nu - 1}_{i_{\nu+1}+1} \neq 0$. Inequality $(4.22)_s$ can not hold for $\nu = s$ since it gives us $0 = i_{s+1} > 1 = \mu_s$, we have $m \nmid 1 = \mu_s$.

We will keep the sequence $\bar{\mu} : \mu_1 > \mu_2 > \cdots > \mu_s$ fixed and we will sum over all possible selections of sequences of $i_1 > \cdots i_s > i_{s+1} = 0$, that is we will show that the sum

$$\Gamma_{\bar{\mu},i} := \sum_{i = i_1 > i_2 > \cdots > i_s \geqslant 1} \prod_{\nu=1}^{s} [\lambda^\alpha_{\mu_\nu} - \lambda_x]^{i_\nu - 1}_{i_{\nu+1}+1} \tag{4.24}$$

is zero, which will show that $\gamma_{\mu,d} = 0$ using eq. (4.18).

Observe now that if $(4.23)_\nu$ holds and $m \nmid \nu, \nu-1$, then $(4.23)_{\nu-1}$ also holds. Indeed the combination of $(4.23)_\nu$ and $(4.22)_{\nu-1}$ gives the impossible inequality

$$\mu_\nu + 2 \overset{(4.23)_\nu}{>} i_\nu \overset{(4.22)_{\nu-1}}{>} \mu_{\nu-1}.$$

Assume now that $m \mid \nu$ and $(4.23)_\nu$ holds, then $(4.23)_{\nu-1}$ also holds. Indeed the combination of $(4.23)_\nu$ and $(4.22)_{\nu-1}$ gives us

$$\mu_\nu + 2 - m \overset{(4.23)_\nu}{>} i_\nu \overset{(4.22)_{\nu-1}}{>} \mu_{\nu-1} - mf(\mu_{\nu-1}).$$

If $m \nmid \mu_{\nu-1}$, then the above inequality is impossible since it implies that

$$\mu_\nu + 2 - m > \mu_{\nu-1} > \mu_\nu.$$

If $m \mid \mu_{\nu-1}$, then the inequality is also impossible since it implies that $\mu_\nu + 2 > \mu_{\nu-1}$ so if we write $\mu_{\nu-1} = k'm$ and $\mu_\nu = km$, $k, k' \in \mathbb{N}$, $k' > k$, we arrive at $2 > (k'-k)m \geqslant m$. This proves the following

Lemma 4.5.4.6. The inequality $(4.22)_{\nu-1}$ might be correct only in cases where $m \mid \mu_{\nu-1}$, $m \nmid \mu_\nu$.

Assume that for all ν inequality (4.23) holds. Then for $\nu = 1$ it gives us (recall that $\mu \leqslant d-2$)

$$\mu + 2 \leqslant d = i_1 < \mu_1 + 2 - mf(\mu_1) = \mu + 2 - mf(\mu), \tag{4.25}$$

which is impossible. Therefore either there are ν such that none of the two inequalities $(4.22)_\nu$, $(4.23)_\nu$ hold (in this case the contribution to the sum is zero) or there are cases where (4.22) holds.

The sumands appearing in eq. (4.24) can be zero, for example the sequence $\mu_1 = m > \mu_2 = 1$ with $i_2 = 2 < i_1 = d$, $s = 2$ give the contribution

$$[\lambda^\alpha_{\mu_2} - \lambda_x]^{i_2-1}_1 [\lambda^\alpha_{\mu_1} - \lambda_x]^{d-1}_{i_2} = [\lambda^\alpha_1 - \lambda_x]^1_1 [\lambda^\alpha_m - \lambda_x]^{d-1}_{i_2+1} = (\lambda_2 - \lambda_1)[\lambda_1 - \lambda_x]^{d-1}_3$$

while for $i_2 = 1 < i_1 = d$ it gives the contribution

$$[\lambda^\alpha_{\mu_2} - \lambda_x]^{i_2-1}_1 [\lambda^\alpha_{\mu_1} - \lambda_x]^{d-1}_{i_2+1} = [\lambda^\alpha_1 - \lambda_x]^0_1 [\lambda^\alpha_m - \lambda_x]^{d-1}_2 = [\lambda_1 - \lambda_x]^{d-1}_2$$

It is clear that these non-zero contributions cancel out when added.

Lemma 4.5.4.7. Assume that $m \mid \mu_{\nu_0-1}$ and $m \nmid \mu_{\nu_0}$, where $(4.23)_{\nu_0}$ and $(4.22)_{\nu_0-1}$ hold. Then, we can eliminate μ_{ν_0-1} and i_{ν_0} from both selections of the sequence of μ's and i's, i.e. we can form the sequence of length $s-1$

$$\bar{\mu}_{s-1} = \mu_s < \bar{\mu}_{s-2} = \mu_{s-1} < \cdots < \bar{\mu}_{\nu_0-1} = \mu_{\nu_0} < \bar{\mu}_{\nu_0-2} = \mu_{\nu_0-2} < \cdots < \bar{\mu}_1 = \mu_1.$$

and the corresponding sequence of equal length

$$\bar{i}_{s-1} = i_s < \bar{i}_{s-2} = i_{s-1} < \cdots < \bar{i}_{\nu_0-1} = i_{\nu_0-1} < \bar{i}_{\nu_0} = i_{\nu_0+1} < \cdots < \bar{i}_1 = i_1 = d,$$

so that

$$\Gamma_{\bar{\mu},i} = \sum_{i_1 > \cdots > i_s} \prod_{\nu=1}^{s} [\lambda^{\alpha}_{\mu_\nu} - \lambda_x]^{i_\nu - 1}_{i_{\nu+1}+1} = (\star) \sum_{\bar{i}_1 > \cdots > \bar{i}_{s-1}} \prod_{\substack{\nu=1 \\ \nu \neq \nu_0 - 1}}^{s} [\lambda^{\alpha}_{\mu_\nu} - \lambda_x]^{i_\nu - 1}_{i_{\nu+1}+1},$$

where $(\star)$ is a non zero element.

Proof. (of lemma 4.5.4.7) We are in the case $m \mid \mu_{\nu_0-1}$ and $m \nmid \mu_{\nu_0}$, where $(4.23)_{\nu_0}$ and $(4.22)_{\nu_0-1}$ hold,

$$\mu_{\nu_0-1} - m \overset{(4.22)_{\nu_0-1}}{<} i_{\nu_0} \overset{(4.23)_{\nu_0}}{<} \mu_{\nu_0} + 2, \tag{4.26}$$

or equivalently

$$\mu_0 := \mu_{\nu_0-1} - m + 1 \leqslant i_{\nu_0} \leqslant \mu_{\nu_0} + 1$$

For i_{ν_0+1} the inequality $(4.22)_{\nu_0}$ $i_{\nu_0+1} > \mu_{\nu_0} - mf(\mu_{\nu_0})$ can not hold, since it implies

$$i_{\nu_0+1} < i_{\nu_0} \overset{(4.23)_{\nu_0}}{<} \mu_{\nu_0} + 2 < i_{\nu_0+1} + 2.$$

Observe that also

$$i_{\nu_0+1} + 1 \leqslant i_{\nu_0} \leqslant i_{\nu_0-1} - 1.$$

Set $l = \max\{\mu_0, i_{\nu_0+1}+1\}$ and $L = \min\{\mu_{\nu_0}+1, i_{\nu_0-1}-1\}$. Then $y = i_{\nu_0}$ satisfies

$$l \leqslant y \leqslant L.$$

By lemma 4.5.4.4 the quantity

$$\sum_{l \leqslant y \leqslant L} [\lambda_{\mu_{\nu_0}+1} - \lambda_x]^{y-1}_{i_{\nu_0+1}+1} \cdot [\lambda_{\mu_0} - \lambda_x]^{i_{\nu_0-1}-1}_{y+1}$$

equals to

$$[\lambda_{\mu_{\nu_0}+1} - \lambda_x]^{l-1}_{i_{\nu_0+1}+1} \cdot [\lambda_{\mu_0} - \lambda_x]^{i_{\nu_0-1}-1}_{L+1} \cdot \frac{[\lambda_{\mu_{\nu_0}+1} - \lambda_x]^L_l - [\lambda_{\mu_0} - \lambda_x]^L_l}{(\lambda_{\mu_{\nu_0}+1} - \lambda_{\mu_0})}$$

$$\frac{[\lambda_{\mu_{\nu_0}+1} - \lambda_x]^L_{i_{\nu_0+1}+1} \cdot [\lambda_{\mu_0} - \lambda_x]^{i_{\nu_0-1}-1}_{L+1} - [\lambda_{\mu_{\nu_0}+1} - \lambda_x]^{l-1}_{i_{\nu_0+1}+1} \cdot [\lambda_{\mu_0} - \lambda_x]^{i_{\nu_0-1}-1}_{l}}{(\lambda_{\mu_{\nu_0}+1} - \lambda_{\mu_0})} \tag{4.27}$$

Case A1 $l = \mu_0 \geqslant i_{\nu_0+1} + 1$. Then $[\lambda_{\mu_0} - \lambda_x]^L_l = 0$.
Case A2 $l = i_{\nu_0+1} + 1 > \mu_0$. We set $z := i_{\nu_0+1}$, which is bounded by eq. $(4.23)_{\nu_0+1}$ that is

$$\mu_0 \overset{\text{Case A2}}{\leqslant} z \overset{(4.23)_{\nu_0+1}}{\leqslant} \mu_{\nu_0+1} + 1.$$

Notice that in this case $m \nmid \mu_{\nu_0+1}$. Indeed, we have assumed that inequality $(4.23)_{\nu_0+1}$ holds wich gives us

$$\mu_{\nu_0-1} - m = \mu_0 - 1 \overset{\text{(Case A2)}}{<} i_{\nu_0+1} \overset{(4.23)_{\nu_0+1}}{<} \mu_{\nu_0+1} + 2 - m,$$

which implies that $\mu_{\nu_0-1} < \mu_{\nu_0+1} + 2$, a contradiction. Thus for $l = z+1$ we compute

$$\sum_{\mu_0 \leqslant z \leqslant \mu_{\nu_0+1}+1} [\lambda^{\alpha}_{\mu_{\nu_0+1}} - \lambda_x]^{i_{\nu_0+1}-1}_{i_{\nu_0+2}+1} \cdot [\lambda_{\mu_0} - \lambda_x]^L_l =$$

$$= \sum_{\mu_0 \leqslant z \leqslant \mu_{\nu_0+1}+1} [\lambda_{\mu_{\nu_0+1}+1} - \lambda_x]_{i_{\nu_0+2}+1}^{z-1} \cdot [\lambda_{\mu_0} - \lambda_x]_{z+1}^{L} =$$

$$= (\star) \cdot \frac{[\lambda_{\mu_{\nu_0+1}+1} - \lambda_x]_{\mu_0}^{\mu_{\nu_0+1}+1} - [\lambda_{\mu_0} - \lambda_x]_{\mu_0}^{\mu_{\nu_0+1}+1}}{\lambda_{\mu_{\nu_0+1}+1} - \lambda_{\mu_0+1}} = 0.$$

Case B1 $L = \mu_{\nu_0} + 1 \leqslant i_{\nu_0-1} - 1$. In this case $[\lambda_{\mu_{\nu_0}+1} - \lambda_x]_l^L = 0$.
Case B2 $L = i_{\nu_0-1} - 1 < \mu_{\nu_0} + 1$. In this case eq. (4.27) is reduced to

$$\frac{[\lambda_{\mu_{\nu_0}+1} - \lambda_x]_{i_{\nu_0+1}+1}^{i_{\nu_0-1}-1}}{(\lambda_{\mu_{\nu_0}+1} - \lambda_{\mu_0})}$$

This means that we have erased the μ_{ν_0-1} from the product and we have

$$\sum_{i_1>\cdots>i_s} \prod_{\nu=1}^{s} [\lambda_{\mu_\nu}^{\alpha} - \lambda_x]_{i_{\nu+1}+1}^{i_\nu-1} = (\star) \sum_{i_1>\cdots>i_s} \prod_{\substack{\nu=1 \\ \nu\neq\nu_0-1}}^{s} [\lambda_{\mu_\nu}^{\alpha} - \lambda_x]_{i_{\nu+1}+1}^{i_\nu-1},$$

where $(\star)$ is a non zero element. This procedure gives us that the original quantity

$$[\lambda_{\mu_{\nu_0}}^{\alpha} - \lambda_x]_{i_{\nu_0+1}+1}^{i_{\nu_0}-1} \cdot [\lambda_{\mu_{\nu_0-1}}^{\alpha} - \lambda_x]_{i_{\nu_0}+1}^{i_{\nu_0-1}-1}$$

after summing over i_{ν_0} becomes the quantity

$$[\lambda_{\mu_{\nu_0}}^{\alpha} - \lambda_x]_{i_{\nu_0+1}+1}^{i_{\nu_0-1}-1} = [\lambda_{\bar{\mu}_{\nu_0-1}}^{\alpha} - \lambda_x]_{\bar{i}_{\nu_0}+1}^{\bar{i}_{\nu_0-1}-1},$$

that is we have eliminated the μ_{ν_0-1} and i_{ν_0} from both selections of the sequence of μ's and i's, i.e. we have the sequence of length $s-1$

$$\bar{\mu}_{s-1} = \mu_s < \bar{\mu}_{s-2} = \mu_{s-1} < \cdots < \bar{\mu}_{\nu_0-1} = \mu_{\nu_0} < \bar{\mu}_{\nu_0-2} = \mu_{\nu_0-2} < \cdots < \bar{\mu}_1 = \mu_1.$$

and the corresponding sequence of equal length

$$\bar{i}_{s-1} = i_s < \bar{i}_{s-2} = i_{s-1} < \cdots < \bar{i}_{\nu_0-1} = i_{\nu_0-1} < \bar{i}_{\nu_0} = i_{\nu_0+1} < \cdots < \bar{i}_1 = i_1 = d.$$

□

Remark 4.5.5. One should be careful here since $\bar{i}_{\nu_0-1} = i_{\nu_0-1} > i_{\nu_0} > \bar{i}_{\nu_0} = i_{\nu_0+1}$, so $\bar{i}_{\nu_0-1} > \bar{i}_{\nu_0} + 1$. This means that the new sequence of $\bar{i}_{s-1} > \cdots > \bar{i}_1$ satisfies a stronger inequality in the ν_0 position, unless $\nu_0 - 1 = d$ in the computation of $\gamma_{d,d}$.

Consider the set $s, s-1, \ldots, \nu_0$ such that $m \nmid \mu_\nu$ for $s \geqslant \nu \geqslant \nu_0$ and assume that $m \mid \mu_{\nu_0-1}$ and $(4.23)_{\nu_0}$ and $(4.22)_{\nu_0-1}$ hold. We apply lemma 4.5.4.7 and we obtain a new sequence of μ's with μ_{ν_0-1} removed, provided that $\nu_0 - 1 > 1$. We continue this way and in the sequence of μ's we eliminate all possible inequalities like (4.26) obtaining a series of μ which involves only inequalities of type (4.23). But this is not possible if $\mu \leqslant d-2$, according to equation (4.25). This proves that all $\gamma_{\mu,d} = 0$ for $1 \leqslant \mu \leqslant d-2$, this completes the proof of lemma 4.5.4.5. □

Lemma 4.5.5.1. If $\mu_2 \neq d-1$, then the contribution of the corresponding summand $\Gamma_{\bar{\mu},i}$ to $\gamma_{d,d}$ is zero.

Proof. We are in the case $\mu = d = i$. We begin the procedure of eliminating all sequences of inequalities of the form $(23)_{\nu_0}, (22)_{\nu_0-1}$, where $m \mid \nu_0 - 1$, $m \nmid \nu_0$, using lemma 4.5.4.7. For $\nu = 1$ inequality $(4.23)_1$ can not hold since it implies the impossible inequality $d = i_1 < d + 2 - m$. Therefore, $(4.22)_1$ holds, that is $i_2 > d - m$. On the other hand we can assume that $(4.23)_2$ holds by the elimination process, so we have

$$d - m \overset{(4.22)_1}{<} i_2 \overset{(4.23)_2}{<} \mu_2 + 2.$$

Following the analysis of the proof of lemma 4.5.4.5 we see that the contribution to $\gamma_{d,d}$ is non zero if case B2 holds, that is ($\nu_0 = 2$ in this case) $d - 1 = i_{\nu_0-1} - 1 < \mu_2 + 1$, obtaining that $\mu_2 = d - 1$. □

Lemma 4.5.5.2. Equation (4.21) holds, that is

$$(\lambda_d - \lambda_d^\alpha)\gamma_{d,d} = \sum_{\nu=1}^{d-1} t_{d,\nu}^{(\alpha)}\gamma_{\nu,d} = t_{d,d-1}^{(\alpha)}\gamma_{d-1,d}.$$

Proof. We will use the procedure of the proof of lemma 4.5.4.7. We recall that for each fixed sequence of $\mu_s > \cdots > \mu_1$ we summed over all possible sequences $i_1 > \cdots > i_{s+1} = 0$. In the final step the inequality (4.26) appears, for $\nu_0 = 2$, and $\mu_{\nu_0} = \mu_2 = d-1$ and $\nu_0 - 1 = 1$ and $\mu_{\nu_0-1} = \mu = d$, that is:

$$0 = \mu_{\nu_0-1} - m \overset{(4.22)_2}{<} i_{\nu_0} \overset{(4.23)_1}{<} \mu_{\nu_0} + 2 = d+1.$$

As in the proof of lemma 4.5.4.7 we sum over $y = i_\nu$ and the result is either zero in case B1 or in the B2 case, where $\mu_{\nu_0} = \mu_2 = d-1$ and $\mu_0 = \mu_{\nu_0-1} - m + 1 = d - m + 1$, the contribution is computed to be equal to

$$\frac{[\lambda^\alpha_{\mu_{\nu_0}+1} - \lambda_x]_{i_{\nu_0+1}+1}^{i_{\nu_0-1}-1}}{(\lambda_{\mu_{\nu_0}+1} - \lambda_{m_0})} = \frac{[\lambda_d^\alpha - \lambda_x]_{i_3+1}^{d-1}}{\lambda_d - \lambda_d^\alpha}.$$

The last $\mu_{\nu_0-1} = \mu_1 = d$ is eliminated in the above expression. This means that for a fixed sequence $\mu_1 > \ldots > \mu_s$ the contribution of the inner sum in eq. (4.24) is given by

$$\frac{1}{\lambda_d - \lambda_d^\alpha} \cdot \sum_{d-1=i_2>i_3>\cdots>i_s\geqslant 1} \prod_{\nu=2}^{s} [\lambda^\alpha_{\mu_\nu} - \lambda_x]_{i_{\nu+1}+1}^{i_\nu - 1}.$$

Observe that $\mu_1 = d$ does not appear in this expression and this expression corresponds to the sequence $\bar{\mu}_1 = \mu_2 = d-1 > \bar{\mu}_2 = \mu_3 > \cdots > \bar{\mu}_{s-1} = \bar{\mu}_s = 1$. Notice, also that the problem described in remark 4.5.5 does not appear here, sence we erased i_1 which is not between some i's but the first one. Therefore, we can relate it to a similar expression that contributes to $\gamma_{d-1,d}$. Conversely every contribution of $\gamma_{d-1,d}$ gives rise to a contribution in $\gamma_{d,d}$, by multiplying by $\lambda_d - \lambda_d^\alpha$. The desired result follows by the expression of $\gamma_{\mu,d}$ given in eq. (4.18). □

We have shown so far how to construct matrices Γ, T so that

$$T^q = 1, \Gamma T \Gamma^{-1} = T^\alpha. \tag{4.28}$$

We will now prove that Γ has order m. By equation (4.28) Γ^k should satisfy equation

$$\Gamma^k T \Gamma^{-k} = T^{\alpha^k}.$$

Using proposition 4.5.4.1 asserting the uniqueness of such Γ^k with α replaced by α^k we have that the matrix multiplication of the entries of Γ giving rise to $(\gamma_{\mu,i}^{(k)}) = \Gamma^k$ coincide to the values by the the recursive method of proposition (4.28) applied for $\Gamma' = \Gamma^k$, $\alpha' = \alpha^k$ and $\Gamma' E_1 = \zeta_m^{\epsilon k} E_1$. In particular for $k = m$, we have $\alpha^m \equiv 1 \bmod p^\nu$ for all $1 \leqslant \nu \leqslant h$, that is the matrix Γ^k should be recursively constructed using proposition (4.28) for the relation $\Gamma^m T \Gamma^m = T$, $\Gamma^m E_1 = E_1$, leading to the conclusion $\Gamma^m = \mathrm{Id}$. Notice that the first eigenvalue of Γ is a primitive root of unity, therefore Γ has order exactly m.

By lemma 4.3.0.2 the action of σ in the special fibre is given by a lower triangular matrix. Therefore, we must have

$$\gamma_{\nu,i} \in \mathfrak{m}_r \text{ for } \nu < i. \tag{4.29}$$

Proposition 4.5.5.1. If

$$v(\lambda_i - \lambda_j) > v(a_\nu) \text{ for all } 1 \leqslant i,j \leqslant d \text{ and } 1 \leqslant \nu \leqslant d-1 \tag{4.30}$$

then the matrix $(\gamma_{\mu,i})$ has entries in the ring R and is lower triangular modulo $\mathfrak{m}_R$.

Proof. Assume that the condition of eq. (4.30) holds. In equation (4.18) we compute the fraction

$$\frac{[a]_1^{\mu-1}}{[a]_1^{i-1}} = \begin{cases} \frac{1}{[a]_\mu^{i-1}} & \text{if } i > \mu \\ 1 & \text{if } i = \mu \\ [a]_i^{\mu-1} & \text{if } i < \mu \end{cases} \tag{4.31}$$

The number of $(\lambda_\mu^\alpha - \lambda_x)$ factors in the numerator is equal to (recall that $i_{s+1} = 0$)

$$\sum_{\nu=1}^{s} (i_\nu - 1 - i_{\nu+1} - 1 + 1) = i - s,$$

and $i > \mu \geqslant s$, so $i - s > 0$. Therefore, for the upper part of the matrix $i > \mu$ we have $i - s$ factors of the form $(\lambda_i^\alpha - \lambda_j)$ in the numerator and $i - \mu$ factors a_x in the denominator. Their difference is equal to $(i - s) - (i - \mu) = \mu - s \geqslant 0$. By assumption the matrix reduces to an upper triangular matrix modulo $\mathfrak{m}_R$. □

Remark 4.5.6. The condition given in equation (4.30) can be satisfied in the following way: It is clear that $\lambda_i - \lambda_j \in \mathfrak{m}_R$. Even in the case $v_{\mathfrak{m}_R}(\lambda_i - \lambda_j) = 1$ we can consider a ramified extension R' of the ring R with ramification index e, in order to make the valuation $v_{\mathfrak{m}_{R'}}(\lambda_i - \lambda_j) = e$ and then there is space to select $v_{\mathfrak{m}_{R'}}(a_i) < v_{\mathfrak{m}_{R'}}(\lambda_i - \lambda_j)$.

Proposition 4.5.6.1. We have that

$$\gamma_{i,i} \equiv \zeta_m^{\epsilon} \alpha^{i-1} \bmod \mathfrak{m}_R \tag{4.32}$$

Let $A = \{a_1, \ldots, a_{d-1}\} \in R$ be the set of elements below the diagonal in eq. (4.9). If $a_i \in \mathfrak{m}_R$, then

$$\gamma_{\mu,i} \in \mathfrak{m}_R \text{ for } \mu \neq i,$$

that is E_i is an eigenvector for the reduced action of Γ modulo $\mathfrak{m}_R$. If $a_{\kappa_1}, \ldots, a_{\kappa_r}$ the elements of the set A which are in $\mathfrak{m}_R$, then the reduced matrix of Γ has the form:

$$\begin{pmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma_r \end{pmatrix}$$

where $\Gamma_1, \Gamma_2, \ldots, \Gamma_{r+1}$ for $1 \leqslant \nu \leqslant r+1$ are $(\kappa_\nu - \kappa_{\nu-1}) \times (\kappa_\nu - \kappa_{\nu-1})$ lower triangular matrices (we set $\kappa_0 = 0, \kappa_{r+1} = d$).

Proof. Consider the matrix Γ:

$$\begin{pmatrix} \boxed{\begin{matrix} \gamma_{11} & & \\ \vdots & \ddots & \\ \gamma_{\kappa_1,1} & \cdots & \gamma_{\kappa_1,\kappa_1} \end{matrix}} & & & \mathbf{0} \\ \boxed{\begin{matrix} \gamma_{\mu,i} \\ 1 \leqslant i \leqslant \kappa_1 < m \leqslant d \end{matrix}} & \boxed{\begin{matrix} \gamma_{\kappa_1+1,\kappa_1+1} & & \\ \vdots & \ddots & \\ \gamma_{\kappa_2,\kappa_1+1} & \cdots & \gamma_{\kappa_2,\kappa_2} \end{matrix}} & & \\ & \boxed{\begin{matrix} \gamma_{\mu,i} \\ \kappa_1 < i \leqslant \kappa_2 < \mu \leqslant d \end{matrix}} & \ddots & \\ & & \cdots & \boxed{\begin{matrix} \gamma_{\kappa_r+1,\kappa_r+1} & & \\ \vdots & \ddots & \\ \gamma_{d,\kappa_r+1} & \cdots & \gamma_{d,d} \end{matrix}} \end{pmatrix}$$

We have that $\mu = i$ and the only element in Σ_μ which does not have any factor of the form $(\lambda_y^\alpha - \lambda_x)$ is the sequence

$$1 = \mu_s = \mu_{s-1} - 1 < \mu_{s-1} < \cdots < \mu_2 = \mu_1 - 1 < \mu_1 = \mu$$

For this sequence eq. (4.18) becomes

$$\gamma_{i,i} = \prod_{\nu=2}^{s} h_{\alpha-1}(\lambda_{\mu_\nu}, \lambda_{\mu_{\nu-1}})\zeta_m^{\epsilon} \bmod \mathfrak{m}_R,$$

which gives the desired result since $h_{\alpha-1}(\lambda_{\mu_\nu}, \lambda_{\mu_{\nu-1}}) \equiv \binom{\alpha}{1} = \alpha \bmod \mathfrak{m}_R$.

For proving that all entries $\gamma_{\mu,i} \in \mathfrak{m}_R$ for $\kappa_\nu < i \leqslant \kappa_{\nu+1} < \mu \leqslant d$, that is for all entries bellow the central blocks, we observe that from equation (4.18) combined with eq. (4.31) that $\gamma_{\mu,i}$ is divisible by $[a]_i^{\mu-1} = a_i a_{i+1} \cdots a_{\kappa_\nu+1} \cdots a_{\mu-1} \in \mathfrak{m}_R$.

□

Recall that by lemma 4.2.0.1 there is an $1 \leqslant a_0 \leqslant m$ such that $\alpha = \zeta_m^{a_0}$.

Proposition 4.5.6.2. The indecomposable module V modulo $\mathfrak{m}_R$ breaks into a direct sum of $r+1$ indecomposable $k[C_q \rtimes C_m]$ modules V_ν, $1 \leqslant \nu \leqslant r+1$. Each V_ν is isomorphic to $V_\alpha(\epsilon + a_0\kappa_{\nu-1}, \kappa_\nu - \kappa_{\nu-1})$.

Proof. By eq. (4.32) the first eigenvalue of the reduced matrix block Γ_ν is

$$\zeta_m^{\epsilon}\alpha^{\kappa_{\nu-1}} = \zeta_m^{\epsilon+(\kappa_{\nu-1})a_0}.$$

Since that first eigenvalue together with the size of the block determine the last eigenvalue, that is the action of C_m on the socle the reduced block is uniquely determined up to isomorphism. □

This way we arrive at a new obstruction. Assume that the indecomposable representation given by the matrix T as in lemma 4.5.0.3 reduces modulo $\mathfrak{m}_R$ to a sum of Jordan blocks. Then the σ action on the leading elements of each Jordan block in the special fibre should be described by the corresponding action of σ on the leading eigenvector E of V. The corresponding actions on the special fibre should be compatible.

This observation is formally given in proposition 4.1.0.1, which we now prove: Each set I_ν, $1 \leqslant \nu \leqslant t$ corresponds to an indecomposable $R[G]$-module, which decomposes to the indecomposables $V_\alpha(\epsilon_\mu, \kappa_\mu)$, $\nu \in I_\nu$ of the special fiber. Indecomposable summands have different roots of unity in R, therefore $\sum_{\mu \in I_\nu} k_\nu \leqslant q$, this is condition (4.1.0.1.a.). The second condition (4.1.0.1.b.) comes from proposition 4.4.0.1. If 1 is one of the possible eigenvalues of the lift T, then $\sum_{\mu\in I_\nu} \kappa_\mu \equiv 1 \bmod m$. If all eigenvalues of the lift T are different than one, then $\sum_{\mu \in I_\nu} \kappa_\mu \equiv 0 \bmod m$. If $\# I_\nu = q$, then there is one zero eigenvalue and the sum equals $1 \bmod m$.

It is clear by eq. (4.32) that condition (4.1.0.1.c.) is a necessary condition. On the other hand if (4.1.0.1.c.) is satisfied we can write (after a permutation if necessary) the set $\{1, \ldots, S\}$, $S = \sum_{\nu=1}^{t} \# I_\nu$ as

$$J_1 = \{1, 2, \ldots, \kappa_1^{(1)}, \kappa_1^{(1)} + 1, \ldots, \kappa_1^{(1)} + \kappa_2^{(1)}, \ldots, \sum_{j=1}^{r_1} \kappa_j^{(1)} = b_1\}, I_1 = \{\kappa_1^{(1)}, \ldots, \kappa_{r_1}^{(1)}\}$$

$$J_2 = \{b_1 + 1, b_1 + 2, \ldots, b_2 = b_1 + \sum_{j=1}^{r_2} \kappa_j^{(2)}\}, I_2 = \{\kappa_1^{(2)}, \ldots, \kappa_{r_2}^{(2)}\}$$

......

$$J_s = \{b_{s-1} + 1, b_{t-1} + 2, \ldots, b_t = S\}, I_s = \{\kappa_1^{(s)}, \ldots, \kappa_{r_s}^{(s)}\}$$

The matrix given in eq. (4.9), where

$$a_i = \begin{cases} 0 & \text{if } i \in \{b_1, \ldots, b_{s-1}\} \\ \pi & \text{if } i \in \{\kappa_1^{(\nu)}, \kappa_1^{(\nu)} + \kappa_2^{(\nu)}, \kappa_1^{(\nu)} + \kappa_2^{(\nu)} + \kappa_3^{(\nu)}, \ldots, \kappa_1^{(\nu)} + \kappa_2^{(\nu)} + \cdots + \kappa_{r_\nu - 1}^{(\nu)}\} \\ 1 & \text{otherwise} \end{cases}$$

lifts the τ generator, and by (4.12) there is a well defined extended action of the σ as well.
Example: Consider the group $q = 5^2, m = 4, \alpha = 7$,

$$G = C_{5^2} \rtimes C_4 = \langle \sigma, \tau | \sigma^4 = \tau^{25} = 1, \sigma\tau\sigma^{-1} = \tau^7 \rangle.$$

Observe that $\operatorname{ord}_5 7 = \operatorname{ord}_{5^2} 7 = 4$.

- The module $V(\epsilon, 25)$ is projective and is known to lift in characteristic zero. This fits well with proposition 4.1.0.1, since $4 \mid 25 - 1 = 4 \cdot 6$.

- The modules $V(\epsilon, \kappa)$ do not lift in characteristic zero if $4 \nmid \kappa$ or $4 \nmid \kappa - 1$. Therefore only $V(\epsilon, 1)$, $V(\epsilon, 4)$, $V(\epsilon, 5)$, $V(\epsilon, 8)$, $V(\epsilon, 9)$, $V(\epsilon, 12)$, $V(\epsilon, 13)$, $V(\epsilon, 16)$, $V(\epsilon, 17)$, $V(\epsilon, 20)$, $V(\epsilon, 21)$, $V(\epsilon, 24)$, $V(\epsilon, 25)$ lift.

- The module $V(1, 2) \oplus V(3, 2)$ lift to characteristic zero, where the matrix of T with respect to a basis E_1, E_2, E_3, E_4 is given by
$$T = \begin{pmatrix} \zeta_q & 0 & 0 & 0 \\ 1 & \zeta_q^2 & 0 & 0 \\ 0 & \pi & \zeta_q^3 & 0 \\ 0 & 0 & 1 & \zeta_q^4 \end{pmatrix}$$
and $\sigma(E_1) = \zeta_q E_1$.

- The module $V(1, 2) \oplus V(1, 2)$ does not lift in characteristic zero. There is no way to permute the direct summands so that the eigenvalues of σ are given by $\zeta_m^\epsilon, \alpha\zeta_m^\epsilon, \alpha^2\zeta_m^\epsilon, \alpha^3\zeta_m^\epsilon$. Notice that $\alpha = 2 = \zeta_m$.

- The module $V(\epsilon_1, 21) \oplus V(2^{21} \cdot \epsilon_1, 23)$ does not lift in characteristic zero. The sum $21 + 24$ is divisible by 4, $\epsilon_2 = 2^{21}\epsilon_1$ is compatible, but $21 + 23 = 44 > 25$ so the representation of T in the supposed indecomposable module formed by their sum can not have different eigenvalues which should be 25-th roots of unity.

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