Index theorems via groupoids, KK theory and cyclic cohomology

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Abstract

In the 1960s, Atiyah and Singer gave a vast generalisation of a series of results connecting Topology with Analysis. The Atiyah-Singer theorem generalises the classical Gauss-Bonnet theorem, Chern-Weil theory and the Riemann-Roch theorem. It gives a formula for the calculation of the analytic index of an elliptic (pseudo)differential operator using characteristic classes.

This dissertation presents the recent proof of the Atiyah-Singer theorem using Lie groupoids and K-theory. This proof arises from the observation that the analytic index depends only from the class of the principal symbol in K-theory. Starting from this, Alain Connes used a deformation groupoid and its associated extension of C* algebras to describe the relation of the elliptic operator with its principal symbol. The connecting map in K-theory is the analytic index.

Claire Debord showed that the topological index can also be expressed in K-theory using deformation groupoids and the Thom isomorphism. The proof of the equality of the two indices is a Poincare duality type theorem, expressed through Kasparov's KK-theory.

In this framework, the calculation of the index can be possible by the pairing of K-theory with cyclic cohomology. Partial results in this direction have been given by Pflaum-Posthuma-Tang.

Introduction

The overall purpose of this dissertation is to serve as a gentle introduction to index theory from the viewpoint of Noncommutative Geometry.

We discuss a vast generalization of the classical Atiyah-Singer index theorem, for families of elliptic pseudodifferential operators along a dynamical system, allowing the presence of **singularities**. Instances of this type of (hard!) Analysis are elliptic problems on statified manifolds, on families of manifolds, along an almost regular foliation, along orbits of a Lie group action, along the symplectic foliation of a Poisson manifold, and many other singular situations.

Lie groupoids provide a unified framework for the treatment of all cases as such, in the sense that they **desingularize** them. On the other hand, the structure of a groupoid lies at the core of pseudodifferential calculus, in fact at the core of Fourier Analysis per se (just think of the convolution formula). Moreover, Connes' tangent groupoid plays a fundamental role in the formulation of the correct K-theoretic proof of the classical Atiyah-Singer index theorem. This formulation is "correct" in the sense that it can be generalised to all the above situations. Indeed, Claire Debord showed exactly this in her habilitation thesis, which we follow here.

Once we prove the Atiyah-Singer index theorem in this very general context, the issue of calculation arises naturally. To this end, we discuss partial results in this direction, by Pflaum, Posthuma and Tang, involving the pairing of K-theory with cyclic cohomology.

We should point out that an open problem is to generalise the Atiyah-Singer index theorem to arbitrary singular foliations. One difficulty with this is that the groupoids arising in this case are no longer smooth. Nevertheless, the left-hand side of the Atiyah-Singer formula (analytic index) has already been formulated. For the right-hand side though, one first needs to develop characteristic classes along a singular foliation. Until the submission of this dissertation, no classes as such have been developed.

Overview of classical index theory

The fundamental problem is the following: Consider an elliptic pseudodifferential operator D on a compact manifold M. It acts as a bounded operator between Sobolev spaces. A consequence of the existence of a parametrix and Atkinson's theorem is that its kernel and cokernel have finite dimension. We desire to calculate its Fredholm index, namely the integer number $Ind(D) = dim(\ker D) - dim(coker D)$. This number carries information about the dimension of the space of solutions of differential equations expressed by D and the associated constraints. Moreover, it is well known that the spectrum of certain elliptic (pseudo)differential operators on a manifold M contains topological information for M. For example, Weyl showed that the dimension and volume of a manifold are determined by the eigenvalues of the Laplacian. The Fredholm index is the key tool we use to calculate the spectrum. For our exposition, it is very important to note that Fredholm operators have the following fundamental properties:

1. The Fredholm index is homotopy invariant.
2. The Fredholm index is a morphism of abelian groups.
3. The Fredholm index is invariant by compact perturbations.

Moreover, Hörmander made the following observation:

4. The Fredholm index depends only on the principal symbol $\sigma(D) \in C_0(T^*M)$. In other words, the principal symbol determines the operator up to compact operators.

The implication of these properties is profound: The first property suggests that Fredholm index has topological nature. Hence it should be possible to realise it as an element of an appropriate cohomology theory. As it happens, K-theory is the correct cohomology theory which accomodates both the principal symbol and the Fredholm index. To see this, first note that, an apropriate change of the norm of the Sobolev spaces makes every positive order pseudodifferential operator a zero order operator. Now Hörmander's observation can be phrased as an exact sequence of C^*-algebras defined by the principal symbol map principal symbol map:

$$0 \to \mathcal{K}(L^2(M)) \to \Psi(M) \xrightarrow{\sigma} C_0(T^*M) \to 0$$

where $\Psi(M)$ is the closure of zero-order pseudodifferential operators in the multiplier algebra of the compact operators $\mathcal{K}(L^2(M))$. In K-theory, exact sequences as such, amount to decompositions of a topological space which are necessary in ordinary cohomology theories for the purpose of applying the Meyer-Vietoris sequence. Thanks to Bott periodicity in K-theory the long exact sequence associated with any exact sequence of C^*-algebras has only 6 terms. The boundary map associated with the above exact sequence gives rise to the "analytic index map"

$$\mathrm{ind}_{an} : K_0(T^*M) \to K_0(\mathcal{K}(L^2(M)))$$

Noting that $K_0(\mathcal{K}(L^2(M))) = \mathbb{Z}$, we find that the analytic index map ind_{an} sends the class $[\sigma(D)]$ of the principal symbol to the Fredholm index of D.

The celebrated theorem of Atiyah and Singer calculates the analytic index using topological invariants of the underlying manifold. In particular, Atiyah and Singer gave the following formula:

$$\text{ind}_{an}(D) = \int_{T^*M} ch(\sigma(D))Td(M)$$

This formula connects Analysis with Topology and is the culmination of the theory of characteristic classes and Chern-Weil theory. It generalises simultaneously the classical Gauss-Bonnet theorem as well as the Hirzenbruch-Riemann-Roch theorem. It has numerous implications in various fields of mathematics, such as representation theory, number theory, spectral theory, etc.

Remarkably, the proof of the Atiyah-Singer theorem is given entirely in the realm of K-theory. The ingredients of the "topological index" appearing in its right-hand side arise from:

- The Chern character map $ch : K(M) \to H^{ev}(M, \mathbb{Q})$
- The Todd class, which arises as the failure for a diagram involving the Chern character to commute.

As for the equality of the analytic and the topological index, it is proven by means of Kasparov's KK-theory, in particular the associated Kasparov product. Alternatively, the equality can be seen as the "wrong-way functoriality" property, which is the appropriate version of Poincare duality in K-theory.

An alternative viewpoint of the Atiyah-Singer formula is by the pairing of K-theory with (periodic) cyclic cohomology, by means of a trace map. We will use both the "K-theoretic" and the "cyclic" viewpoints here.

The use of Lie groupoids in classical index theory

Lie groupoids are geometric realizations of internal and external symmetries. Their simplest form, Lie groups, appear naturally in Fourier Analysis - this is evident already in the convolution formula. Let us see how they arise in pseudodifferential calculus and index theory.

In fact, the simplest Lie groupoid has already been used in the above context. This is just the pair groupoid $M \times M$ (over M). Namely, vieweing a pair $(y, x) \in M \times M$ as an arrow with source x and range y, we have an obvious multiplication $(z, y)(y, x) = (z, x)$, inversion $(y, x)^{-1} = (x, y)$ and unit inclusion given by the diagonal map $M \to M \times M$. This structure is fundamental, as the following observations show:

1. Pseudodifferential operators are completely determined by their Schwarz kernels. Such a kernel is a distribution on $M \times M$ which is smooth away from the diagonal.
2. The groupoid multiplication and inversion appears in the Schwarz kernel formulas for the product and adjoint of pseudodifferential operators respectively.
3. Likewise for the associated asymptotic expansions.

Motivated by the stationary phase method (see Guillemin and Sternberg), Alain Connes realised that the analytic index is strongly related with a deformation. Namely, if we consider the family of pseudodifferential operators $\{tD\}_{t\in\mathbb{R}}$ on $M \times \mathbb{R}$, then the limit as $t \to 0$ turns out to be the principal symbol $\sigma(D)$.

The correct formulation of this observation is in terms of a deformation groupoid, known as "Connes' tangent groupoid". This is the groupoid which deforms the pair groupoid $M \times M$ to its infinitesinal counterpart, the tangent bundle TM, viewed as a Lie groupoid (over M), with addition along the fibers. Specifically, using a tubular neighbourhood map, we put a topology on the space

$$\mathbb{T}M = (TM \times \{0\}) \cup (M \times M \times \mathbb{R}^*)$$

which makes it a Lie groupoid over $M \times \mathbb{R}$.

Now it is easy to extract the analytic index map from this groupoid. Restricting to $[0, 1]$, and bearing in mind that the C^*-algebra of the pair groupoid is the compact operators $\mathcal{K}(L^2(M))$, we obtain the short exact sequence

$$0 \to \mathcal{K}(L^2(M)) \otimes C_0((0, 1]) \to C^*(\mathbb{T}M) \xrightarrow{ev_0} C_0(T^*M) \to 0$$

The algebra $C_0((0, 1])$ is contractible (and nuclear), whence passing to K-theory, the map ev_0 is invertible. Composing with evaluation at $t = 1$ we obtain a map

$$[ev_1] \circ [ev_0]^{-1} : K(T^*(M)) \to K(\mathcal{K}(L^2)(M))$$

It turns out that this is exactly the analytic index map ind_{an}. Even more remarkably, the topological index map can also be formulated in terms of deformation groupoids, similar to Connes' tangent groupoid. This was clarified by C. Debord in her habilitation thesis.

Overview of this dissertation

As we mentioned in the beginning, our goal is to display how the above results can be generalised to dynamical systems, possibly with singularities. It turns out that Lie groupoids provide the correct framework in which this can be achieved.

That is because Lie groupoids *desingularize* dynamical systems as such. Here are a few instances of this:

- Let $\pi : N \to M$ be a submersion and assume that we are interested in the involutive distribution $\ker d\pi$ of N. That is to say, we want to study families of pseudodifferential operators along the fibers of π. The fibered product $N \times_\pi N$ is a Lie subgroupoid of the pair groupoid $N \times N$. Applying the target map to the source-fiber at $n \in N$ we recover the fiber $\pi^{-1}(\pi(n))$.
- For an example presenting singularities, consider an arbitrary action of a Lie group G on a manifold M. We are interested in the dynamical system formulated by the infinitesimal generators of the action, which are vector fields tangent to the orbits of this action. The cartesian product $G \times M$ carries the structure of a Lie groupoid over M, with source the projection map and target the action map. Given any $x \in M$, the source-fiber $s^{-1}(x)$ is diffeomorphic to the Lie group G, whence it has constant dimension. By applying the tartget map, we obtain the orbit at x.

Whence, the term "desingularization" means that, although the manifold M may be partitioned to submanifolds with varying dimension, the Lie groupoid G allows us to replace them with the source-fibers, which have constant dimension. This way, every time we are interested in an index problem along the given dynamical system on M, we can lift it to an index problem along the source-fibers of G, which is much better behaved dynamical system.

We will show that pseudodifferential calculus can be defined on a Lie groupoid $\mathcal{G}$. To this end, the associated Lie theory is crucial: Every Lie groupoid $\mathcal{G}$ has a Lie algebroid $A\mathcal{G}$. A pseudodifferential operator on $\mathcal{G}$ should be thought of as a family of pseudodifferential operators along the source-fibers. Its symbol is a function on $A^*\mathcal{G}$. Moreover, all the apparatus we discussed above can be generalised to groupoids:

1. A C^*-algebra can be attached to $\mathcal{G}$. Its elements are families of pseudodifferential operators along the source-fibers, with negative order. In other words, we have an exact sequence
$$0 \to C^*(\mathcal{G}) \to \Psi(\mathcal{G}) \xrightarrow{\sigma} C_0(A^*\mathcal{G}) \to 0$$
2. A deformation groupoid
$$\mathbb{T}\mathcal{G} = (A\mathcal{G} \times \{0\}) \cup (\mathcal{G} \times \mathbb{R}^*)$$
3. The analytic index map
$$[ev_1] \circ [ev_0]^{-1} : K(A^*(\mathcal{G})) \to K(C^*(\mathcal{G}))$$
4. Certain deformation groupoids, which make sense of the topological index map.

With these ingredients in hand, as well as the notion of Morita equivalence, we will discuss how Claire Debord was able to generalise the proof of the Atiyah-Singer index theorem in the context of Lie groupoids. There we will perform several explicit KK theory calculations.

Last, notice that our index map takes values in the K-theory of the groupoid C^*-algebra. This K-theory may be much harder to compute than the K-theory of compact operators. So, in this case, we really need to discuss pairing with cyclic cohomology. The difficulty with this is that there is no trace map for groupoid C^*-algebras. Instead of this, in the end we will discuss some partial results by Pflaum, Posthuma and Tang. The second part of this thesis can be summarized as follows ,it is possible to calculate the index of operators using traces.Here we introduce the purely algebraic theory of cyclic homology and cohomology which serves as generalized traces. Indeed cyclic cocycles are the higher dimensional generalization of traces. Then the chern charachter is introduced and we show that the index pairing between K homology and K theory can be calculated by passing to cyclic homology and cohomology and taking their natural pairing.A proof for the index theorem for $\mathbb{R}^n$ using deformation of pseudodifferential operators but cyclic cocycles instead is presented.
Lastly we present connes moscovici localized index theorem .The naturall setting for the ideas here is cyclic cohomology and groupoids. To generalize the localized index for groupoids one needs a trace ,towards research we give the construction of this trace using a form of transversal density and prove it is actually a trace.
The selection of topics will give the reader the basic ideas needed for future developments.Here we won't be discussing the topic of deformation quantization (or quantization) in general and algebraic index theorems also we won't be discussing equivariant KK theory and connections to the novikov conjecture because the inclusion of these topics would make the present thesis more extensive than intended. As stated earlier the purpose here is to present the flow of ideas and so I will not go into unnecessary detail.

A large part of this thesis is textbook material conveniently gathered in one place.For example I dedicate a section to Brown Douglas Fillmore theory which is a motivation for K homology and KK theory stemming from operator theory. A few things that the reader is not going to find in other places are the following , I give a complete and detailed proof of the equality of the analytic and fredholm indices which ,I make a few remarks and fill in a few details in the proof of the atiyah singer index theorem presented here. , In the end I give a proof that the trace for groupoids from [38] is indeed a trace.

Contents

Chapter 1

1.1 Pseudodifferential operators

Pseudodifferential operators are a generalization of differential operators. They can be defined on smooth manifolds , so a pseudodifferential operator is an operator.

$$P : C^\infty(M) \to C^\infty(M)$$

More generally they act on smooth sections of vector bundles and we also consider their action on sobolev spaces. Pseudodifferential operators have a distributional schwarz kernel that is smooth outside the diagonal in other words they are pseudolocal and this is one of the properties we can define them on manifolds in the first place.(we only care what the operator does in a neighborhood of a point). Their most important feature is the principal symbol which is a function on the cotangent bundle the principal symbol has an order and it roughly tells how much they reduce the differentiability of a function.

The composition and adjoints of pseudodifferential operator are still pseudodifferential and the corresponding principal symbols are given by compostion and adjoint respectively. Therefore pseudodifferential operators are roughly a quantization of the cotangent bundle and they form a filtered $*$ algebra (under the order m of the symbol-operator). This algebra is denoted by $\Psi^\infty(M) = \bigcup \Psi^m(M)$.
An important property of pseudodifferential operators is ellipticity ,when the principal symbol in which case it is shown that they have a parametrix ,a two sided inverse modulo compact operators , and many interesting properties follow.Notably elliptic operators have finite dimensional kernel and cokernel and therefore a well defined index.

1.1.1 Pseudodifferential operators on $\mathbb{R}^n$

A pseudodifferential operator on $\mathbb{R}^n$ is given via the fourier transform.

$$Pf(x) = \int p(x,\xi)e^{i\langle x,\xi\rangle}\hat{f}(\xi)d\xi = \int p(x,\xi)e^{i\langle x-y,\xi\rangle}f(y)dyd\xi$$

$p(x,\xi)$ is by definition the symbol of the operator(The measure we use throughout this section is normalized by $(2\pi)^{n/2}$). Denote by S^m the class of symbols $a(x,\xi) \in C^\infty(R^n \times R^n)$

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|}$$

and by $S^{-\infty}$ the intersection of all S^m i.e. rapidly decaying symbols.
The kernel of a pseudodifferntial operator is the distribution on $\mathbb{R}^n \times \mathbb{R}^n$ given by

$$K(x,y) = \int p(x,\xi)e^{i\langle x-y,\xi\rangle}d\xi$$

For (x,y) outside the diagonal this is an oscillatory integral that gives a smooth function so a pseudodifferential operator is only interesting on the diagonal and uninteresting as a propagator.For example think about differential operators.
We will also denote the pseudodifferential operator associated to a symbol p as Op_p

1.1.2 Asymptotic expansion

A symbol has an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ if $a_j \in S^{m_j}$ if $m_j \to -\infty$ and $a - \sum_{j=0}^k a_j \in S^{m_{k+1}}$. Converely any such sequence of symbols is the asymptotic expansion to a symbol in S^{m_0} which is unique modulo $S^{-\infty}$ This can be achieved if we define $a(x,\xi) = \sum(1-\chi(\epsilon_j\xi))a_j(x,\xi)$ for a cutoff function and sufficiently fast

decaying ϵ_j . If the a_j in the expansion are homogeneous of degree m_j :$a_j(x,t\xi) = t^{m_j} a_j(x,\xi)$ for $|t|,|\xi| \geq 1$ then we refer to a polyhomogeneous or classical symbol . If a pseudodifferential operator has a symbol in S^m then we refer to it as an order m pseudodifferential operator. The principal symbol of an order m pseudodifferential operator (with symbol $p(x,\xi)$) is refered to the class of p in S^m/S^{m-1}.This is refered to a symbol map.
We mention a simple device that will be of use to us later for recovering the principal symbol of an operator .

$$\sigma_P(x,\xi) = e^{-i\langle x,\xi\rangle} P(x,D)(e^{i\langle y,\xi\rangle})$$

represents the principal symbol.(the integrals involved are defined as oscillatory)

1.1.3 Composition ,adjoint

composition

The composition of pseudodifferential operators P_1, P_2 with symbols $a_1 \in S^{m_1}, a_2 \in S^{m_2}$ is a pseudodifferential operator with a symbol $a = a_1 \diamond a_2$ in $S^{m_1+m_2}$ and has asymptotic expansion :

$$a(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha a_1(x,\xi) \partial_x^\alpha a_2(x,\xi)$$

It is explicitly given by:

$$a(x,\xi) = \iint e^{-i\langle x-y,\xi-\eta\rangle} a_1(x,\eta) a_2(y,\xi) dy d\eta$$

where the expression is understood as an oscillatory integral.This formula can be shown to have the asymptotic expansion above by standard methods (stationary phase,taylor expansion around (x,ξ))
This formula is suggested by the following calculations ,(assume that the symbols a_1, a_2 are compactly supported with respect to x.This is the case we are usuallly going to use.) Note that

$$\mathrm{Op}_{a_2}(u)^\wedge(\eta) = \int \widehat{a_2}(\eta-\xi,\xi)\widehat{u}(\xi) d\lambda(\xi)$$

(we mean fourier transform with respect to x ,$\widehat{a_2}$ is rapidly decaying with respect to $\eta - \xi$.).Therefore:

$$\begin{aligned}
\mathrm{Op}_{a_1} \circ \mathrm{Op}_{a_2}(u)(x) &= \int a_1(x,\eta) e^{i\langle x,\eta\rangle} \mathrm{Op}_{a_2}(u)^\wedge(\eta) d\eta \\
&= \int a_1(x,\eta) e^{i\langle x,\eta\rangle} \left(\int \widehat{a_2}(\eta-\xi,\xi)\widehat{u}(\xi) d\lambda(\xi) \right) d\eta \\
&= \int \left(\int a_1(x,\eta)\widehat{a_2}(\eta-\xi,\xi) e^{i\langle x,\eta-\xi\rangle} d\eta \right) e^{i\langle x,\xi\rangle} \widehat{u}(\xi) d\xi \\
&= \int \left(\iint e^{-i\langle x-y,\xi-\eta\rangle} a_1(x,\eta) a_2(y,\xi) dy d\eta \right) e^{i\langle x,\xi\rangle} \widehat{u}(\xi) d\xi
\end{aligned}$$

Notably the principal symbol of the commutator $[a(x,D), b(x,D)]$ is given by $a(x,\xi), b(x,\xi)$ (the poisson bracket for the cotangent bundle of R^n).

adjoint

The adjoint operator (with respect to the usual sesquilinear inner product) of an operator with symbol in S^m is also a pseudodifferential operator with symbol in S^m with asymptotic expansion:

$$a^\dagger(x,\xi) \sim \sum \frac{1}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha \bar{a}(x,\xi)$$

and is explicitly given by

$$a^\dagger(x,\xi) = \iint e^{-i\langle x-y,\xi-\eta\rangle} \bar{a}(y,\eta) dy d\eta$$

Again one obtains the asymptotic expansion from the above oscillatory integral by standard methods.
The above formula is justified by the following:
We should have that

$$\langle Op_a(u), v\rangle = \langle u, Op_{a^\dagger}(v)\rangle = \overline{\langle Op_{a^\dagger}(v), u\rangle}$$

But

$$\langle Op_a(u), v\rangle = \int \mathrm{Op}_a(u)(x)\overline{v(x)}dx = \int\int a(x,\xi)e^{i\langle x,\xi\rangle}\widehat{u}(\xi)\overline{v(x)}dxd\xi$$

Fourier transform with respect to x gives that:

$$\langle Op_a(u), v\rangle = \int \widehat{a}(\eta-\xi,\xi)\widehat{u}(\xi)\overline{\widehat{v}(\eta)}d\xi d\eta$$

Similarly we get (with ξ,η interchanged) that:

$$\langle Op_{a^\dagger}(v), u\rangle = \int \widehat{a}^\dagger(\xi-\eta,\eta)\overline{\widehat{u}(\xi)}\widehat{v}(\eta)d\xi d\eta$$

It follows that

$$\widehat{a}^\dagger(\xi-\eta,\eta) = \overline{\widehat{a}}(\eta-\xi,\xi) = \widehat{\overline{a}}(\xi-\eta,\xi)$$

After a substitution :$(\eta,\xi)\mapsto(\xi,\xi+\eta)$ we get:

$$\widehat{a}^\dagger(\eta,\xi) = \widehat{\overline{a}}(\eta,\xi+\eta)$$

From which we obtain:

$$a^\dagger(x,\xi) = \int e^{i\langle x,\eta\rangle}\widehat{a}^\dagger(\eta,\xi)d\eta = \int e^{i\langle x,\eta\rangle}\widehat{\overline{a}}(\eta,\xi+\eta)d\eta = \iint e^{-i\langle x-y,\xi-\eta\rangle}\overline{a}(y,\eta)dyd\eta$$

We usually consider pseudodifferential operators acting between hermitian vector bundles ,In that case in the adjoint formula replace $\overline{a}$ by the adjoint operator a^* . The composition formula gives another way to prove that the kernel is smooth outside the diagonal. for $\phi,\psi \in C_c^\infty(M)$ with $\operatorname{supp}\phi\cap\operatorname{supp}\psi = 0$ $\phi P\psi$ has zero asymptotic expansion.

1.1.4 Action on functions

We mention the following facts about the action of pseudodifferential operators on functions spaces.(The following applies to manifolds) The proofs can be found in any book about pseudodifferential operators.A pseudodifferential operator P in S^m.

- maps schwartz functions to schwarz functions.
- extends to a bounded linear operator between sobolev spaces $P: H^s \to H^{s-m}$ (this is what we meant by reducing differentiability)

We also get that a pseudodifferential operator with a symbol in $S^{-\infty}$ always outputs smooth functions.This is an obvious consequence of the sobolev lemma:
(Sobolev lemma):The intersection of all sobolev spaces H^s consists the smooth functions:

$$\bigcap H^s = C^\infty$$

We shall refer to such operators as regularising operators denoted $\mathcal{R}$.Regularizing operators are also compact (when viewed as maps between any two sobolev spaces)

1.1.5 Wavefront set

Being pseudolocal a pseudodifferential operator preserves the singularities of a distribution.
On R^n The singular support of a distribution u (singsupp(u))is the complement of the set of points close to which u is a C^∞ function. The wavefront set $WF(u)$ also indicates the directions at which a distribution is singular. The precise definition is: for $u\in\mathcal{D}'(\mathbb{R}^n)$.

$$WF(u) = \{(x,\xi)|\xi\in\Sigma_x(u)$$

Where $\Sigma_x(u)$ is the set of singular directions at x ,the complement of the set of directions ξ for which there exists a bump function $\phi(x)=1$ such that $\widehat{\phi u}(\xi') < c_N(1+|\xi'|)^{-N}$ for ξ' in a cone around ξ. It is easy to see that $WF(fu)\subset WF(u)$ for any smooth f.
singsupp(u) is easily seen to be the set of x for which Σ_x is nonemtpy (at these points is easy to prove by a compactness argument that there is ϕ such that ϕu has rapidly decaying fourier transform ,thus is smooth). Using this and the fact that the kernel of a pseudodifferential operator has singular support on the diagonal, we immediately get that singsupp(Pu) $\subset$ singsupp(u) (a more precise result is the propagation of singularities theorem). We are going to generalize these notions on manifolds in the next section ,the symbol is going to be a function on the cotangent bundle and the wavefront set a subset thereof,

1.1.6 Definition on manifolds

Since pseudodifferential operators are truly local objects they can be defined on manifolds,we give the details.First we verify coordinate invariance. If $k; X_1 \to X_2$ is a diffeomorphism between bounded domains in euclidean space and $a(x,D) : C^\infty(X_1) \to C^\infty(X_1)$ is a pseudodifferential operator then the pushforward operator $k_*a(x,D) : C^\infty(X_2) \to C^\infty(X_2)$ given by

$$k_*a(-,D)g(x) = a(-,D)g \circ k(k^{-1}(x))$$

$$\begin{array}{ccc} C^\infty(X_1) & \xrightarrow{a(x,D)} & C^\infty(X_1) \\ {\scriptstyle k^*}\uparrow & & \downarrow{\scriptstyle (k^{-1})^*} \\ C^\infty(X_2) & \xrightarrow{k_*a(x,D)} & C^\infty(X_2) \end{array}$$

is also a pseudodifferential operator in S^m. Assuming that it indeed is a pseudodifferential operator we recover the symbol using the device mention before which suggests that the symbol k_*a should be given by.

$$k_*a(x,\xi) = e^{-i\langle x,\xi\rangle}[a(-,D)e^{i\langle k(x),\xi\rangle}]_{k^{-1}(x)}$$

We are going to elaborate on this ,the pushforward symbol given above indeed gives the pushforward operator:(assume that $a(-,D)$ is actually given by $\phi_1 a(-,D)\phi_2$ where we mean multiplication by ϕ compactly supported functions in X_1 which are 1 around a point of interest ,this is okay because we only need to see local behaviour and this is the sort of operators we need consider with when we define pseudodifferential operators on manifolds.
The pushforward operator acting on a schwarz function $g \in C^\infty(X_2)$ gives:

$$\phi_1(k^{-1}(x)) \int a(k^{-1}(x),\xi)e^{i\langle k^{-1}(x),\xi\rangle}\widehat{\phi_2 g \circ k}(\xi)d\xi$$

Whereas the pseudodifferential operator associated to the pushforward symbol is given by:

$$\int e^{i\langle x,\xi\rangle}k_*a(x,\xi)\hat{g}(\xi)d\xi =$$

$$\int e^{i\langle x,\xi\rangle}\left(e^{-i\langle x,\xi\rangle}\phi_1(k^{-1}(x))\int a(k^{-1}(x),\eta)e^{i\langle k^{-1}(x),\eta\rangle}\left(\int e^{-i\langle z,\eta\rangle}\phi_2(z)e^{i\langle k(z),\xi\rangle}dz\right)d\eta\right)\hat{g}(\xi)d\xi =$$

$$\phi_1(k^{-1}(x))\int a(k^{-1}(x),\eta)e^{i\langle k^{-1}(x),\eta\rangle}\left(\int e^{-i\langle z,\eta\rangle}\left(\phi_2(z)\int e^{i\langle k(z),\xi\rangle}\hat{g}(\xi)d\xi\right)dz\right)d\eta =$$

$$\phi_1(k^{-1}(x))\int a(k^{-1}(x),\eta)e^{i\langle k^{-1}(x),\eta\rangle}\left(\int e^{-i\langle z,\eta\rangle}\left(\phi_2(z)g(k(z))\right)dz\right)d\eta$$

This is exactly the same as before. It remains to prove that the pushforward symbol belongs to S^m ,one way to prove this is is found in hormander using the stationary phase method.It is equivalent to prove that $k_*a(k(x),\xi)$ belongs to S^m Note that

$$k_*a(k(x),\xi) = \phi_1(x)e^{-i\langle k(x),\xi\rangle}\int a(x,\eta)e^{i\langle x,\eta\rangle}\left(\int e^{-i\langle z,\eta\rangle}\phi_2(z)e^{i\langle k(z),\xi\rangle}dz\right)d\eta =$$

Inspect the term $\Phi(\xi,\eta) = \int e^{-i\langle z,\eta\rangle + i\langle k(z),\xi\rangle}\phi_2(z)dz$:
the differential of the phase is $Dk^T(z)\xi - \eta$,if $\|Dk(z)\|, \|(Dk(z))^{-1})\| \le C$ in the support of ϕ then if $\frac{|\eta|}{|\xi|}$ is either larger than say $2C$ or smaller than $\frac{1}{2C}$ then $\Phi(\xi,\eta)$ is dominated by arbitrarily large negative powers of $(1+|\xi|+|\eta|)$,we break the integral into two parts using a bump function $\beta(x)$ that is 1 when $\frac{1}{2C} \le |x| \le 2C$ and equal to zero if $|x|$ is outside the interval $[\frac{1}{4C}, 4C]$
we have that

$$I = \int a(x,\eta)e^{i\langle x,\eta\rangle}\left(\int e^{-i\langle z,\eta\rangle}\phi_2(z)e^{i\langle k(z),\xi\rangle}dz\right)d\eta = I_1 + I_2 =$$

$$\int a(x,\xi)e^{i\langle x,\xi\rangle}\Phi(\xi,\eta)(1-\beta(\frac{\eta}{|\xi|}))d\eta + \quad (I_1)$$

$$\int a(x,\xi)e^{i\langle x,\xi\rangle}\Phi(\xi,\eta)\beta(\frac{\eta}{|\xi|})d\eta = \omega^n\int e^{i\omega(\langle x-z,\eta\rangle + \langle k(z),\xi/\omega\rangle)}a(x,\omega\eta)\beta(\eta)\phi_2(z)dzd\eta \quad (I_2)$$

Where $\omega = |\xi|$. I_1 is rapidly decaying with respect to $|\xi|$ and I_2 can be estimated by the stationary phase method (we are applying it to the integral over $\mathbb{R}^{2n}$) ,the critical point is $(z,\eta) = (x, Dk^T(x)\xi/\omega)$ and the approximation yields (notice that ω^n is going to disappear and $\beta(Dk^T(x)\xi/\omega) = 1, \phi_2(x) = 1$)

$$e^{i\langle k(x),\xi\rangle} a(x, Dk^T(x)\xi) + o(1)$$

as $\omega \to \infty$. It turns out that using an asymptotic expansion of the stationary phase approximation given in [hormander] ,for quadratic phase functions it reads

$$\int_{\mathbb{R}^n} e^{i\lambda\langle Q(x-x_0),x-x_0\rangle/2} a(x)\mathrm{d}x. = \left(\frac{1}{\lambda}\right)^{n/2} \frac{e^{i\frac{\pi}{4}\operatorname{sgn} Q}}{|\det Q|^{1/2}} \sum_{0\le j\le N} \frac{\lambda^{-j}}{j!} \left(\frac{\langle Q^{-1}D, D\rangle}{2i}\right)^j (a(x))|_{x=x_0}$$

$$+O\left(\lambda^{-\frac{n}{2}-N-1} \sum_{|\alpha|\le n+2N+3} \sup_{x\in\mathbb{R}^n} \frac{|\partial^\alpha a(x-x_0;\lambda)|}{\langle x\rangle^{n+4N+5-|\alpha|}}\right).$$

(using morse lemma we can get an expansion of this sort with general phase functions with nondegenerate critical points) The pushforward has an asymptotic expansion:

$$k_* a(k(x),\eta) \sim \sum \frac{1}{\alpha!} \partial_\xi^\alpha a(x, Dk^T(x)\eta)[\partial_y^\alpha e^{i\langle k(y)-k(x)-Dk(x)(y-x),\eta\rangle}|_{y=x}]$$

Observe that this is a valid asymptotic expansion because the term in square brackets is a polynomial in η of degree $\le |\alpha|/2$ and the terms of the series are in $S^{m-|a|/2}$. The principal symbol is represented by $a(k^{-1}(x), Dk^T(k^{-1}(x))\eta)$ which suggests that the symbol should be viewed as a function on the cotagent bundle . The previous discussion leads to defining pseudodifferential operators on manifolds as operators whose restriction to coordinate charts are pseudodifferential operators and they also have a well defined principal symbol as a function on the cotagent bundle that transforms canonically. Concretely a pseudodifferential operator of order M on a smooth manifold M is an operator $P : C^\infty(M) \to C^\infty(M)$ that in any coordinate chart and compactly supported functions ϕ_1, ϕ_2 inside that chart $\phi_1 P \phi_2$ is a pseudodifferential operator of order m on flat space.The above discussion shows that this is independent of the chart used. A quantization procedure for defining pseudodifferential operators with a given principal symbol is to take a quadratic partition of unity ϕ_j subordinate to a coordinate cover of M and then take $P = \sum \phi_j P_j \phi_j$,where $\phi_j P_j \phi_j$ are locally defined pseudodifferential operators according the symbol. This is easily seen to be a pseudodifferential operator on M. Observe that by this procedure we can get pseudodifferential operators whose kernels are supported in arbitrary small neighborhoods of the diagonal.

They can also be defined as operators between vector bundles $P : C^\infty(M;E) \to C^\infty(M;F)$ where now in each trivializing chart are given by a matrix of pseudodiffs.

We mention that a similar reasoning shows that the wavefront set is also invariant under diffeomorphisms and therefore is an invariantly defined subset of the cosphere bundle ST^*X it is invariant under the action of pseudodifferential operators .

We sometimes prefer to define pseudodifferential operators acting on the bundle of half densities so that we don't have to introduce a measure on M to define adjoints.

More generally we consider pseudodifferential operators acting between vector bundles tensored with half densities $P : C^\infty(E \otimes \Omega^{1/2}) \to C^\infty(F \otimes \Omega^{1/2})$.Furthermore ,if we use half densities we get a well defined subprincipal symbol ,see [hormander].

1.1.7 Principal symbol, parametrix

We saw in the last section that the symbol is well defined modulo S^{m-1} on the cotangent bundle.Therefore for any pseudodifferential operator on a compact manifold we have a well defined principal symbol in $S^m(M)/S^{m-1}(M)$, In case vector bundles E, F are involved the symbol is a section of $\mathrm{Hom}(\pi^*E, \pi^*F)$ where $\pi : T^*M \to M$ is the projection, the space of order m such symbols is denoted by $S^m(M;E,F)$)

ellipticity

Ellipticity for pseudodifferential operators means that the symbol $\sigma(x,\xi)$ is "uniformly" injective when $\xi \to \infty$. A pseudodifferential operator $P = \mathrm{Op}_\sigma$ of degree m is called elliptic if it can be defined by a symbol $\sigma \in S^m(M,E,F)$ such that

$$|\sigma(x,\xi)\cdot u| \geqslant c|\xi|^m|u|, \quad \forall (x,\xi) \in T_M^\star, \quad \forall u \in E_x$$

for $|\xi|$ large enough, the estimation being uniform for $x \in M$.(Note that ellipticity of an operator implies ellipticity of the adjoint) If E and F have the same rank, the ellipticity condition implies that $\sigma(x,\xi)$ is invertible for ξ large. By taking a suitable C^{∞} truncating function $\theta(\xi) \geqslant 0$ equal to 0 for $|\xi| \leqslant R$ large and to 1 for $|\xi| \geqslant 2R$, one sees that the function $\sigma'(x, \xi) = \theta(\xi)\sigma(x, \xi)^{-1}$ defines a symbol in the space $S^m(M; F, E)$. Also, since

$$1 - \sigma'(x, \xi)\sigma(x, \xi) = 1 - \theta(\xi) \in \mathcal{D}(\mathbb{R}^m) \subset \mathcal{S}(\mathbb{R}^m),$$

this difference is a regularizing operator.From the composition formula we have $\mathrm{Id} - \mathrm{Op}_{\sigma'} \circ \mathrm{Op}_{\sigma} = \mathrm{Op}_{\rho}$, where $\rho \in S^{-1}(M; E, E)$, and thus $\rho^{\diamond j} \in S^{-j}(M; E, E)$. Choose a symbol τ equivalent to the asymptotic expansion $\mathrm{Id} + \rho + \rho^{\diamond 2} + \cdots + \rho^{\diamond j} + \cdots$. Then Op_{τ} is an inverse of $\mathrm{Op}_{\sigma'} \circ \mathrm{Op}_{\sigma} = \mathrm{Id} - \mathrm{Op}_{\rho}$ modulo $\mathcal{R}$. It is then clear that one obtains an inverse $\mathrm{Op}_{\tau \diamond \sigma'}$ of Op_{σ} modulo $\mathcal{R}$.similarly we can get a left inverse and then it is easy to show that the two inverses coincide modulo $\mathcal{R}$,obtaining a parametrix : Immediate consequence of the existence of parametrix are the following :

- **(hypoellipticity)**If $P\phi$ is smooth then ϕ is smooth.
- **(Garding inequality)** If $Pu \in H^s$ then $u \in H^{s+m}$ and $\|u\|_{H^{s+m}} \leq C_s \left(\|Pu\|_{H^s} + \|u\|_{L_2}\right)$
- **(Fredholmness)** Elliptic operators have finite dimensional kernel and cokernel

1.1.8 Fredholm operators

Let's review a few facts about fredholm operators .An operator $T : \mathcal{H}_1 \to \mathcal{H}_2$ is fredholm if it has finite kernel and cokernel(their class is denoted by $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$.The index function ind : $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2) \to \mathbb{Z}$ is defined as $\mathrm{ind}(T) = \dim(\ker T) - \dim(\ker T^*)$ In the previous section we showed that an elliptic operator is invertible modulo regulatizing operators that are compact (if we fix sobolev spaces). An invertible operator T modulo compacts is fredholm (T^* is also invertible modulo compacts.)
This is the content of Atkinson theorem which also implies that the converse holds:

Atkinson theorem

If $T \in \mathcal{F}(H_1, H_2)$, then there exists an $S \in \mathcal{B}(H_2, H_1)$ such that $ST - \mathrm{Id} \in \mathcal{K}(H_1, H_1)$ and $TS - \mathrm{Id} \in \mathcal{K}(H_2, H_2)$. Conversely, if $T \in \mathcal{B}(H_1, H_2)$ such that ST - Id and TS S' - Id are compact operators for some $S, S' \in \mathcal{B}(H_2, H_1)$, then $T \in \mathcal{F}(H_1, H_2)$.

Proof.suppose that $K = \mathrm{Id} - ST \in \mathcal{K}(H_1, H_1)$. Let $x \in \operatorname{Ker} T$, and B a bounded neighbourhood of x in $\operatorname{Ker} T$. Then $K(B) = B$ is relatively compact. Therefore $\operatorname{Ker} T$ is a locally compact topological vector space, and hence $\operatorname{Ker} T$ is finite dimensional. Next, consider Id $-TS' \in \mathcal{K}(H_2, H_2)$. Then $\mathrm{Id} - S'^*T^*$ is a compact operator, and proceeding as before we get $\operatorname{Ker} T^* = T(\mathcal{H}_1)^{\perp}$ is finite dimensional.
Conversely suppose T is Fredholm. Then, since $\dim \operatorname{Ker} T$ and $\operatorname{Codim} T(H_1)$ are finite, we can find closed subspaces V and W such that $H_1 = \operatorname{Ker} T \oplus V$, and $H_2 = T(H_1) \oplus W$. Then T maps V bijectively onto $T(H_1)$, and so $T \mid V$ is a topological isomorphism, by the open mapping theorem. We extend $(T \mid V)^{-1}$: $T(H_1) \longrightarrow H_1$ to an operator $S : H_2 \longrightarrow H_1$ by taking $S \mid W = 0$. Then Id $-ST$ is the projection of H_1 onto Ker T along V, and $\mathrm{Id} - TS$ is the projection of H_2 onto W along $T(H_1)$. Since Ker T and W are finite dimensional, Id $-ST$ and Id - TS are finite rank operators, and hence compact operators.
For operators on a single hilbert space the statement is.
A bounded operator T $\in \mathfrak{B}$(H) is Fredholm if and only if its image in the Calkin algebra is invertible.
Immediate consequences are:

- $\mathcal{F}(\mathcal{H})$ is an open subset of $\mathfrak{B}$(H)
- If T $\in \mathcal{F}(\mathcal{H})$ and K is compact then T + K $\in \mathcal{F}(\mathcal{H})$

Moreover the index is locally constant and constant under compact perturbation:

Additivity of the index

We can deduce from Atkinson theorem that if $T \in \mathcal{F}(H_1, H_2)$ and $S \in \mathcal{F}(H_2, H_3)$ then $ST \in \mathcal{F}(H_1, H_3)$. Moreover it holds that: $\mathrm{ind}(ST) = \mathrm{ind}(S) + \mathrm{ind}(T)$.
A quick proof follows from the kernel-cokernel exact sequence for the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_1 & \xrightarrow{\mathrm{Id}\oplus T} & \mathcal{H}_1\oplus\mathcal{H}_2 & \xrightarrow{(T,-\mathrm{Id})} & \mathcal{H}_2 & \longrightarrow & 0 \\ & & \downarrow T & & \downarrow ST\oplus\mathrm{Id} & & \downarrow S & & \\ 0 & \longrightarrow & \mathcal{H}_2 & \xrightarrow{S\oplus\mathrm{Id}} & \mathcal{H}_3\oplus\mathcal{H}_2 & \xrightarrow{(\mathrm{Id},-S)} & \mathcal{H}_3 & \longrightarrow & 0 \end{array}$$

Homotopy invariance of the index

It is in general true that the index gives a bijective map from the connected components of the fredholm operators to the integers.

$$\mathrm{ind}:\pi_0(\mathcal{F}(\mathcal{H}))\xrightarrow{\cong}\mathbb{Z}$$

First let us show that it is locally constant:
The index function ind : $\mathcal{F}(H_1,H_2)\longrightarrow\mathbb{Z}$ is locally constant, and therefore it is continuous, and homotopy invariant.

Proof. Let $T\in\mathcal{F}(H_1,H_2)$. Since Ker T and Coker T are finite dimensional, Ker T and $T(H_1)$ admit orthogonal complements V and W so that $H_1=\operatorname{Ker}T\oplus V$ and $H_2=T(H_1)\oplus W$. Let $\alpha:V\longrightarrow H_1$ be the inclusion, and $\beta:H_2\longrightarrow T(H_1)$ be the orthogonal projection onto $T(H_1)$ along W. Then α and β are Fredholm, and ind $\alpha=-\dim$ Ker T and ind $\beta=\dim$ Coker T. Then $\beta T\alpha$ is a Fredholm operator, and

$$\text{ind }\beta T\alpha=\text{ ind }\alpha+\text{ ind }T+\text{ ind }\beta=0.$$

Also $\beta T\alpha$ is an isomorphism in $\mathcal{B}(V,T(H_1)$. Therefore $\beta T'\alpha$ is an isomorphism if T' is sufficiently close to T, and then

$$\text{ind }\beta T'\alpha=\text{ ind }\alpha+\text{ ind }T'+\text{ ind }\beta=0,$$

or ind $T'=\operatorname{ind}T$. This means that the index function is continuous, and homotopy invariant.

Since multiplication by a fredholm operator of fixed index permutes connected components of $\mathcal{F}(\mathcal{H})$ (it does so modulo $\mathcal{K}$ but addition of compacts lands in the same connected component) and there exist shift operators with any given index to prove the first statement it suffices to prove that the set of fredholm operators of zero index $\mathcal{F}(\mathcal{H})_0$ is connected. For $T\in\mathcal{F}_0$ choose an isomorphism

$$\phi:\operatorname{Ker}T\to(\operatorname{Im}T)^{\perp}$$

(vector spaces of the same finite dimension!) and set

$$\Phi:=\begin{cases}\phi & \text{on }\operatorname{Ker}T\\ 0 & \text{on }(\operatorname{Ker}T)^{\perp}\end{cases}$$

By construction, we have $T+\Phi\in GL(\mathcal{H})$ which is connected and $T+t\Phi\in\mathcal{F}$ for $t\in[0,1]$.

Index and trace class operators

If for the inverse modulo compacts S of $T\in\mathcal{F}(H_1,H_2)$ we actually have that $I-ST$ and $I-TS$ are trace class and selfadjoint then: Index(T) = Tr($I-ST$) − Tr($I-TS$). Actually we shall prove something more general that: If $I-ST$,$TS-I$ are in some schatten class (which implies they are compact) so $(ST-I)^n,(TS-I)^n$ are trace class for some large n then.

$$\mathrm{Index}(T)=\mathrm{Tr}(I-ST)^n-\mathrm{Tr}(I-TS)^n$$

Proof: 1 is an isolated point in

$$\mathrm{K}=\{1\}\cup\mathrm{Spectrum}(I-TS)\cup\mathrm{Spectrum}(I-\mathrm{ST}).$$

(recall the essential spectrum)
Let γ be the boundary of a small closed disk D with center 1 such that $\mathrm{D}\cap\mathrm{K}=\{1\}$. Set

$$e=\frac{\mathrm{I}}{2\pi i}\int_\gamma\frac{d\lambda}{\lambda I-(I-ST)},\quad f=\frac{\mathrm{I}}{2\pi i}\int_\gamma\frac{d\lambda}{\lambda I-(I-TS)}.$$

From holomorphic functional calculus we have $e=e^2,f=f^2$.
E_1 = Range of e, F_1 = Range of f are finite dimensional, and admit respectively $\mathbf{E}_2=\operatorname{Ker}e,\mathbf{F}_2=\operatorname{Ker}f$ as supplements in H. For any $\lambda\in\mathbb{C}\backslash K$ one has,

$$(\lambda I-(I-ST))^{-1}S = S(\lambda I-(I-TS))^{-1} \Rightarrow eS = Sf$$
$$(\lambda I-(I-TS))^{-1}T = T(\lambda I-(I-ST))^{-1} \Rightarrow fT = Te$$

Thus,

$$T(E_1)\subset F_1,\quad T(E_2)\subset F_2,\quad S(F_1)\subset E_1,\quad S(F_2)\subset E_2.$$

Let T_j, S_j be the restrictions of T, S : $T: E_j \to F_j, j=1,2$ and $S: F_j \to E_j, j=1,2$. By construction the restrictions of ST to E_2 and of TS to F_2 are invertible operators, and hence:

1. $\operatorname{Index} T = \dim E_1 - \dim F_1$
2. $\operatorname{Trace}(\mathrm{I}_{\mathrm{E}_2} - S_2T_2)^n = \operatorname{Trace}(\mathrm{I}_{\mathrm{F}_2} - T_2S_2)^n$
   (proof: T_2 is a linear isomorphism ,$T_2(I_{E_2}-S_2T_2)^nT_2^{-1} = (I_{F_2}-T_2S_2)^n$)

The spectrum of $I_{E_1} - S_1T_1$ and of $I_{F_1} - T_1S_1$ contains only 1, thus:

$$\operatorname{Trace}(\mathrm{I}_{\mathrm{E}_1} - S_1T_1)^n - \operatorname{Trace}(\mathrm{I}_{\mathrm{F}_1} - T_1S_1)^n = \dim E_1 - \dim F_1$$

The result follows.

1.1.9 Conormal distributions

The distribution kernel of a pseudodifferential operator is given by $K(x,y)=\int e^{i\langle x-y,\xi\rangle}a(x,\xi)d\xi$ This suggests we look at distributions of the form $u(x)=\int e^{i\langle x',\xi\rangle}a(x'',\xi)d\xi$ where $x=(x',x''), x'\in R^{n-k}, x''\in R^k$ Interestingly a distribution of the form $u(x)=\int e^{i\langle x',\xi\rangle}a(x,\xi)d\xi$ with $a\in S^m(R^n\times R^k)$ can be converted to the above form . $u(x)$ is a smooth function outside $x'=0$,distributions of this sort are said to be conormal distributions for the submanifold $x'=0$ The fourier transform $\widehat{u}\in L^2_{loc}$ and for a function h on R^{n-k} that vanishes in a neighborhood of zero we have that $R^{-k-2m}\int|\widehat{u}|^2h(\xi/R)\le C$ which means that u belongs to the besov space $H_(-m-k/2)$.

This property is preserved under certain differentiations of u. $D_{j''}u(x) = \int e^{i\langle x',\xi\rangle}D_{j''}a(x'',\xi)d\xi$ $x'_iD'_j = \int e^{i\langle x',\xi\rangle}(x'_iD_{j'} - D_{\xi_i}\xi_j)a(x'',\xi)d\xi$ So $X_1X_2...X_Nu\in H_(-m-k/2)$ for any vector fields that are tangent to the plane $x'=0$. The converse is also true any such u can be expressed in the above form. The definition of conormal distributions on a submanifolds follows the last sentence. They possess a well defined principal symbol on the normal bundle which is the half density $a(x'',\xi)|dx''|^{1/2}|d\xi|^{1/2}$. Conormal distributions can be defined by an obvious modification the above expression on vector bundles (x' should be on the fiber , x'' on the base and ξ on the dual fiber) and with the aid of tubular neighborhoods written down in this way on any submanifold.

1.1.10 The pseudodifferential extension

Read first the part about K theory. Pseudodifferential operators of order 0 can be extended to bounded operators on $L_2(M)$. On classical pseudodifferential operators of order ≤ 0 the principal symbol map σ_0 defines thanks to homogeneity a function on the cosphere bundle S^*M. Negative order operators are compact .
Below is given a device for obtaining the principal symbol map for ≤ 0 operators.
Suppose on an open set around 0 in $\mathbb{R}^n$ we wan't to determine $\sigma_0(P)(0,\eta)$ for $|\eta|=1$:
Let $\phi(x)$ be a real bump function around zero of L_2 norm 1. Let f_N denote the function

$$f_N(y) = N^{n/2}\phi(Ny)e^{i\langle y,N^2\eta\rangle}$$

that has L_2 norm 1 and it's fourier transform is given by:

$$\hat{f_N} = N^{-n/2}\hat{\phi}(\frac{\xi}{N}-N\eta)$$

Standard calculations show that f_N is weakly convergent to 0 in L_2.
Using this the principal symbol is recovered as :
Proposition :

$$\lim_{N\to\infty}\langle f_N, Pf_N\rangle = \sigma_0(P)(0,\eta)$$

Proof :P is locally given by $Pf=\int p(x,\xi)e^{i\langle x,\xi\rangle}\hat{f}(\xi)$. $\sigma_0(P)(\eta)$ is just $\lim_{t\to\infty}p(0,t\eta)$.

$$\langle f_N, Pf_N\rangle = \int \phi(Nx)e^{-i\langle x,N^2\eta\rangle}p(x,\xi)e^{i\langle x,\xi\rangle}\hat{\phi}(\frac{\xi}{N}-N\eta)d\xi dx =$$

$$\int \phi(x)p(\frac{x}{N}, N\xi + N^2\eta)e^{i\langle x,\xi\rangle}\hat{\phi}(\xi)d\xi dx =$$

$$\int \phi(x)(p(\frac{x}{N}, N(\xi + N\eta)) - p(\frac{x}{N}, N^2\eta))e^{i\langle x,\xi\rangle}\hat{\phi}(\xi)d\xi dx + \int \phi(x)^2 p(\frac{x}{N}, N^2\eta)dx$$

The proposition should by now be clear.

By the construction of the principal symbol map it follows that σ_0 extends to $\Psi^*(M)$ the norm closure of $\Psi^0(M)$ in $\mathcal{B}(L_2(M))$,as a (norm bounded by 1) $*$ homomorphism $\sigma_0 : \Psi^*(M) \to C(S^*M)$,note that $\Psi^*(M)$ contains all compact operators. Since f_N is weakly convergent to 0 it is clear that σ_0 takes the compact operators to 0. Conversely if $P \in \Psi^*(M)$ and $\sigma_0(P) = 0$ then P should be compact. This follows at once because (1) one can construct as before a quantization map $C(S^*M) \to \Psi^0(M)$ that is a section of σ_0 (it's not a $*-$homomorphism!) this shows that σ_0 is surjective and (2) one then get's that P is the norm limit of a sequence of ≤ 0 pseudodifferential operators whose principal symbol vanishes and are negative order and compact. Alternatively one can use an argument of the form that P maps a dense span,weakly convergent sequence in $L_2(M)$ to a strongly convergent sequence.
Therefore we get the pseudodifferential extension short exact sequence:

$$0 \to \mathcal{K}(L_2(M)) \to \Psi^*(M) \to C(S^*M) \to 0$$

For future reference note that $\mathcal{K}(L_2(M))$ is identified with $C^*(M \times M)$ (the pair groupoid).
One can use this short exact sequence to get the index map:(the reader should read first the section about C^* algebraic K theory.)
We have a six term exact sequence:

$$\begin{array}{ccccc} \mathbb{Z} \cong K_0(\mathcal{K}(L_2(M))) & \longrightarrow & K_0(\Psi^*(M)) & \longrightarrow & K_0(C(S^*M)) \\ \text{ind}\uparrow & & & & \downarrow \\ K_1(C(S^*M)) & \longleftarrow & K_1(\Psi^*(M)) & \longleftarrow & K_1(\mathcal{K}(L_2(M))) = 0 \end{array}$$

The indicated index map is the same as the atiyah singer index map.
Proof:[10]
Since $K_1(\mathcal{K}) = K_1(\mathbb{C}) = 0$, one of the two connecting maps is trivial. The general algebraic definition of the connecting map in K-theory leads to the following recipe for constructing the analytic index of an elliptic symbol $a \in Psy^0(M; E, F)^{-1}$. The symbol $\tilde{a} = \begin{pmatrix} 0 & -a^{-1} \\ a & 0 \end{pmatrix}$ defines, after embedding E in a trivial bundle, an element of $GL_N^0(C^\infty(S^*M))$ (where the superscript 0 indicates the connected component of identity) for N sufficiently large. It can, therefore, be lifted to $GL_N^0(\Psi^0(M))$. For example, if one chooses $A \in \Psi^0(M; E, F)$ with $\sigma_{pr}(A) = a$ and $B \in \Psi^0(M; F, E)$ with $\sigma_{pr}(B) = a^{-1}$, then $S_0 = I - BA \in \Psi^{-1}(M; E)$ and $S_1 = I - AB \in \Psi^{-1}(M; F)$, hence

$$L = \begin{pmatrix} S_0 & -B - S_0 B \\ A & S_1 \end{pmatrix} = \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ A & I \end{pmatrix} \begin{pmatrix} I & -B \\ 0 & I \end{pmatrix} \in \Psi^0(M; E \oplus F)$$

defines such a lift. By definition,

$$\partial([a]) = [P] - [e],$$

where P and e are idempotents defined as follows:

$$P = L \begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix} L^{-1}, \quad e = \begin{pmatrix} 0 & 0 \\ 0 & I_F \end{pmatrix}.$$

The definition of $\partial([a])$ can be shown to be independent of the lift L. In particular, one may improve, at no extra cost, the choice of the parametrix B such that S_0 and S_1 are smoothing operators. Then

$$R = P - e = \begin{pmatrix} S_0^2 & S_0(I + S_0)B \\ S_1 A & -S_1^2 \end{pmatrix} \in \Psi^{-\infty}(M; E \oplus F)$$

and one has

$$\operatorname{Tr} R = \operatorname{Tr} S_0^2 - \operatorname{Tr} S_1^2 = \operatorname{Index} A,$$

1.1.11 Pseudodifferential operators on groupoids

Read first the section on groupoids.The following is taken from [35] Often it is interesting to consider not just pseudodifferential operators acting on a single manifold but families of pseudodifferential operators parametrized by another manifold or , on the fibers of a fibre bundle or more generally along the leaves of a foliation. These considerations are relevant to the family atiyah singer index theorem, and families of operators are used in the atiyah-bott proof of bott periodicity.
One can use pseudodifferential operators on covering spaces (that are invariant under deck transformations) to define pseudodifferential operators on the base space.
We present here a construction of which these are special cases.We need the following

Pseudodifferential operators along the leaves of a regular foliation

These are operators $P : C^\infty(M) \to C^\infty(M)$ such that for local trivializing charts of the foliation $U \subset M \to X \times T$ (X are coordinates on the leaves and T are transversal). The operator $\phi P \psi$ takes the form :

$$\int_X a(x,t,\xi) e^{i\langle x-y,\xi\rangle} g(y,t) dy d\xi$$

and the symbol $a(x,t,\xi) \in C^\infty(X \times T \times (\mathbb{R}^k)^*)$.The order is defined along the leaves.We shall also refer to these as differentiable families.
Suppose the groupoids we use are equiped with right haar systems.
Pseudodifferential operators on grouoids are differentiable families of classical pseudodifferential operators on $C^\infty(G_x)$ that are equivariant under the natural maps $R_\gamma : C^\infty(G_{s(\gamma)}) \to C^\infty(G_{r(\gamma)})$.Furthemore one can take a vector bundle on the base space and consider families of pseudodifferential operators on the pullbacks by the range map that are invariant under right multiplication. This works as follows:
Right translation is:

$$U_g : \mathcal{C}^\infty\left(\mathcal{G}_{s(g)}, r^*(E)\right) \to \mathcal{C}^\infty\left(\mathcal{G}_{r(g)}, r^*(E)\right)$$
$$\left(U_g f\right)(g') = f(g'g) \in \left(r^*E\right)_{g'} = E_{r(g')} = E_{r(g'g)}$$

Pseudodifferential operators of order m on a groupoid G (the set of which is denoted $\Psi^m(G)$) are differentiable families of pseudodifferential operators on the fibers of the source map that are invariant under the action of the groupoid.
This means we have a family $P_x : C^\infty(G_x) \to C^\infty(G_x)$ such that $P_{r(g)} U_g = U_g P_{s(g)}$
If we are not given a haar system it is more convenient to define pseudodifferential operators on density bundles. We use the bundles $\Omega^{1/2}(\ker ds)$ note that there is a naturall pullback map $U_g : C^\infty(G_{s(g)}, \Omega^{1/2}) \to C^\infty(G_{r(g)}, \Omega^{1/2})$ we mean pseudodifferential operators that commute with this action of G. We also consider operators on densities valued in a vector bundle . $P_x : C^\infty(G_x, \Omega^{1/2} \otimes r^*(E)) \to C^\infty(G_x, \Omega^{1/2} \otimes r^*(E))$
It is easy to see that composition and adjoints of pseudodifferential operators on groupoids are still invariant therefore they form a filtered (by order m) $*$-algebra denoted $\Psi^\infty(G) = \bigcup \Psi^m(G)$,(if we use coefficient bundles we denote it by $\Psi^\infty(G,E)$.)
The symbols of such operators should intuitively be functions on the algebroid dual given that the "symbols" of $P_{r(g)}$ and $P_{s(g)}$ as functions on $T^*G_{r(g)}$ and $T^*G_{s(g)}$ are related by $R_g^* : T^*G_{s(g)} \to T^*G_{r(g)}$ (pushforwards of forms) so they are determined by their values on $A^* = \bigcup T_x^*(G_x)$ by $p_{s(g)}(\xi_g) = p_{A^*}(R_g^*(\xi_g))$
On the one hand we have a quantization map $\Phi : C^\infty(A^*) \to \Psi(G)$
To describe this as well as the inverse (symbol map) we need an exponential map $\theta : A \to G$ that maps a neighborhood V_0 of the zero section of A to a section of the zero section of M in G. We get this as a generalization of the classical exponential map on manifolds with connection by introducing an invariant family of connections on G_x .Since $TG_x \cong r^*A$ we can use pullback connections (by r) of a connection on A.Note that θ sends A_x to G_x
Given θ (also choose a cutoff function β that is one on the unit space M) we define the quantization map by

$$p \in S^m(A^*) \to P_{s(\gamma_1)} f(\gamma_1) = \int_{\xi \in A^*_{r(\gamma_1)}} \int_{\gamma \in G_{s(\gamma_1)}} p(r(\gamma_1), \xi) e^{-i\langle \theta^{-1}(\gamma\gamma_1^{-1}), \xi\rangle} \beta(\gamma\gamma_1^{-1}) f(\gamma) d\gamma d\xi$$

This is easily seen to be $G-$ invariant. Conversely define the symbol map $\sigma_{\nabla,\beta} : \Psi^m(G) \to S^m_{cl}(A^*)$ using a generalization of the symbol map used on $\mathbb{R}^n$: Let for $\xi \in A_x^*$: $e_\xi \in C_c^\infty(G_x)$ be the function supported around the unit x $e_\xi(\gamma) = \beta(\gamma) e^{i\langle \theta^{-1}(\gamma), \xi\rangle}$. The symbol map is defined as

$$\sigma_{\nabla,\beta}(P)(\xi) = (P e_\xi)(x)$$

The principal symbol thus defined turns out to be independent of the connection and the cutoff function used.Moreover the quantization map descends to a map $\Phi : S^m_{cl}(A^*)/S^{m-1}_{cl}(A^*) \to \Psi^m(G)/\Psi^{m-1}(G)$ that is

independent of the connection and cutoff function used and is inverse to the symbol map. These can be generalized in the presence of coefficient bundles.The symbol in this case is given as a section of $S^m_{cl}(A^*, \pi^*(\mathrm{End}(E)$. For details see [NWX].
For any $x \in \mathcal{G}^{(0)}, T^*\mathcal{G}_x$ is a symplectic manifold, so $T^*_s\mathcal{G}= \cup_{x\in\mathcal{G}^{(0)}} T^*\mathcal{G}_x$ is a regular Poisson manifold with the leafwise symplectic structures. Now the Poisson structure on A^* can be considered as being induced from that on $T^*_s\mathcal{G}$. More precisely, let $\mathcal{R}^* : T^*_s\mathcal{G} \to A^*$ be the natural projection induced by the right translation, used to define a map $\mathcal{R}^* : \mathcal{C}^\infty(A^*(\mathcal{G})) \to \mathcal{C}^\infty(T^*_d\mathcal{G})$. We then have that the map $\mathcal{R}$ is a Poisson map.
The poisson structure on A^* is given by
$\{f, g\} = 0, \quad \{f, \hat{X}\} = -\rho(X)\cdot f, \quad \{\hat{X}, \hat{Y}\} = -\{\widehat{[X,Y]}\}$, with $f, g \in \mathcal{C}^\infty(M)$, and $X, Y \in \Gamma^\infty(M, A)$, viewed as functions $\hat{X}, \hat{Y}$ on A^*.
It holds that for the homogeneous parts of the principal symbols of order m and m' operators P, P' that

$$\sigma_{m+m'-1}([P, P']) = \{\sigma_m(P), \sigma_{m'}(P')\}$$

The pseudodifferential extension for groupoids

Pseudodifferential operators on groupoids can also be viewed as multipliers of the dense $*$ subalgebra $C_c(G, \Omega^{1/2})$ of $C^*(G)$ consisting of smooth sections of the bundle $\Omega^{1/2}(\ker(ds) \oplus \ker(dt))$ Pseudodifferential operators of order ≤ 0 are multipliers of $C^*(G)$ moreover negative order operators are actually elements of $C^*(G)$.

The first statement means that if P is a pseudodifferential operator with compact support in G and of order ≤ 0, then there exists a constant c such that, for all $f \in C^\infty_c(G)$, we have $\|P * f\| \leq c\|f\|$ and $\|f * P\| \leq c\|f\|$ (this is true for both the maximal and the reduced C^*-norm of G).

Proof. To establish this statement, first assume that P is of order $< -p$ where $p = \dim G - \dim M$ is the dimension of the algebroid. Note that if a is a symbol of order $< -p$, then P_a is a continuous function. Therefore P is a continuous function with compact support on G, and thus an element of $C^*(G)$. If P is of order $< -p/2$, then $\|P * f\|^2 = \|f^* * P^* * P * f\|$ (and $\|f * P\|^2 = \|f * P * P^* * f^*\|$) and as $P^* * P$ is of order $< -p$, it is in $C^*(G)$ and thus $\|P * f\|^2 \leq \|P^* * P\|\,\|f\|^2$. It follows that P is a multiplier, and as $P^* * P \in C^*(G)$ we find $P \in C^*(G)$. If P is of negative order, $(P^*P)^{2^k} \in C^*(G)$ for some $k \in \mathbb{N}$, and by induction in $k, P \in C^*(G)$. Let P be a pseudodifferential operator of order 0 . Note first that every smooth function $q \in C^\infty_c(M)$ is a pseudodifferential operator of order 0 with principal symbol $\sigma_q : (x, \xi) \mapsto q(x)$ - and of course a bounded multiplier: we have $(q * f)(\gamma) = q(r(\gamma))f(\gamma)$ and $(f * q)(\gamma) = f(\gamma)q(s(\gamma))$ Let $q \in C_c(M)$ which is equal to 1 on the support of $\sigma_P - i.e.$ the projection on M of the closure of $\{(x, \xi); \sigma_q(x, \xi) \neq 0\}$ (which is assumed to be compact in the space of half lines of the bundle $\mathscr{A}^*$). Let $c \in \mathbb{R}_+$with $c > \sigma_P(x, \xi)$ for all (x, ξ). Put $b(x, \xi) = q(x)\sqrt{c^2 + 1 - |\sigma_q(x, \xi)|^2}$, and let Q be a pseudodifferential operator with principal symbol b. Then $P^*P + Q^*Q$ which has symbol $\left(1 + c^2\right)|q|^2$ is of the form $\left(1 + c^2\right)|q|^2 + R$ where R is of negative order and therefore $P^*P + Q^*Q$ is bounded. For all $f \in C_c(G), \|Pf\|^2 = \|f^*P^*Pf\| \leq \|f^*P^*Pf + f^*Q^*Qf\| \leq \|P^*P + Q^*Q\|\,\|f\|^2$, and thus $f \mapsto Pf$ is bounded. In the same way $f \mapsto fP$ is bounded.
The symbol map sends pseudodifferntial operators of order ≤ 0 to functions on the cosphere bundle SA^* (by the rule) $\sigma_0(P)(\xi) = \lim_{t\to\infty} \sigma_P(t\xi)$ and it extends to a map

$$\sigma_0 : \Psi^*(G) \to C(SA^*)$$

on the closure of ≤ 0 order operators in the multiplier algebra. Whose kernel is $C^*(G)$ so we have a short exact sequence ,the generalization of the pseudodifferential extension on manifolds.

$$0 \to C^*(G) \to \Psi^*(G) \xrightarrow{\sigma} C_0(SA^*) \to 0$$

Exactness is proved in exactly the same way as in the manifold case and it also gives an analytical index map $K_1(C_0(SA^*)) \to K_0(C^*(G))$

1.2 K theory,K homology,KK theory

1.2.1 Topological K theory

Topological K theory is a generalized cohomology theory.It assigns to every compact space the grothendieck ring con- structed out of isomorphism classes of vector bundles over X under direct sum and tensor product. Every element of $K(X)$ can be represented as a formal difference of 2 vector bundles: $[E] - [F]$.It is a very basic fact that to any vector bundle one can add another to make it trivial. From this it is easy to see that any element in $K(X)$ can be represented by $[E] - [\mathbb{C}^n_X]$ and that $[E] = [F]$ if the vector bundles are stably isomorphic $E \oplus \mathbb{C}^N_X \cong F \oplus \mathbb{C}^N_X$ where $\mathbb{C}^N_X$ represents a trivial vector bundle.

Classifying space

Recall that the infinite grassmanian Gr_n of n-dimensional subspaces of $\mathbb{C}^\infty$ classifies n dimensional vector bundles.
Using the equivalence of categories of principal $U(n)$ bundles and complex vector bundles (that have a hermitian structre) we can also use $BU(n)$ as classifying space moreover a map $X \to BU(n)$ corresponds to the pullback bundle over X of the universal n-dimensional vector bundle over $BU(n)$. The stable unitary group $U = \lim U(n)$ is the limit of the sequence of inclusions $U(n) \to U(n+1)$ .It therefore holds that BU classifies stable isomorphism classes of vector bundles over X ,therefore by the above remarks (that any element in $K(X)$ can be represented by $[E] - [\mathbb{C}^n_X]$) $BU \times \mathbb{Z}$ is a classifying space for $K(X)$: $[X, BU \times \mathbb{Z}] \cong K(X)$(The second factor $\mathbb{Z}$ should represent the image of $K(X) \to K(\cdot)$.From the atiyah janich theorem we also get that $\mathfrak{F}(\mathcal{H})$ the space of fredholm operators on a separable hilbert space is also a classifying space for K theory.

$X \to K(X)$ is a contravariant functor if we use the pullback of vector bundles (if $f : X \to Y$ is a map $f^* : K(Y) \to K(X)$ sends the class of a vector bundle over Y to the class of the pullback bundle f^*E over X). The induced map remains the same under homotopy.To see this a vector bundle E over $X \times [0,1]$ can be trivialised in the over the t direction. To see this choose a partition of unity $\phi_1, ..., \phi_N$ on X such that $\phi_i \leq \epsilon$ and subordinate to a cover U_i such that E is trivial on sets of the form $U_i \times [a, a+\epsilon]$ and then gradually trivialise E in the t direction over $\{(x,t) | t \leq \phi_1(x) + ... + \phi_i(x)\}$.

K theory for C^* algebras (which we are going to discuss shortly) is motivated by topological K theory

The relation is quite clear from the fundamental
serre swan theorem: the category of vector bundles over X is the same as the category of projective modules over C(X)).
To a vector bundle one can add another vector bundle such that the sum is trivial.Therefore a vector bundle gives rise to it's module of sections which is a projective $C(X)$ module.Conversely given a projective module P one get's a vector bundle whose fiber over x is P/I_xP where I_x is the ideal of functions that vanish at x. One uses projectivity ($P \oplus P'$ is free) to show that if $s_1, s_2, .., s_k$ give a basis of P/I_xP they are linearly independent in a neighborhood of x and give also a basis.

($\dim(P/I_xP), \dim(P'/I_xP')$ are lower semicontinuous and thus also upper semicontinuous) .
If $E \oplus E' \cong X \times \mathbb{C}^n$ is trivial then E can be given by the family of projections parametrized by X from $\mathbb{C}^n$ to the summand E_x.
Projections over $C(X)$ and projective modules over $C(X)$ represent the same thing ,the K theory of the C^* algebra $C(X)$. The basic results of K topological K theory, can be reformulated in C^* algebraic K theory.

1.2.2 Topological K theory

We need to review a few facts about topological K theory. If $(X, \cdot)$ is a pointed space the reduced K theory $\tilde{K}(X)$ is the kernel of the map $K(X) \to K(\cdot)$ (explicitly formal differences of vector bundles over X whose ranks in the connected component of X are equal).For a locally compact space X define $K(X) = \tilde{K}(X^+)$ (the one point compactification ,if X were compact then this just gives unreduced K theory.)Reduced K theory is functorial in the category of pointed spaces. The relative K theory for a compact pair $K(x, A)$ is the subring of $K(X)$ of elements that are trivial on A (for example $K(X, \cdot) = \tilde{K}(X)$). It is straightforward to prove (using tietze extension and partitions of unity) that if a vector bundle is trivial on A then it is the pullback of a vector bundle on the quotient $X \to X/A$.
We can state that $K(X, A) = \tilde{K}(X/A)$.
If $(X, A, \cdot)$ is a pointed pair then this just says that $\tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$ is exact Alternatively we could use mapping cones and define $K(X, A) = \tilde{K}(X \cup_i CA)$ since we can just collapse contractible subspaces given the above exact sequence.

long exact sequence

From the classical puppe coexact sequence in algebraic topology one get's by the above remarks the long exact sequence in (reduced) K theory. Recall also that there is a classifying space for K theory.

$$... \to \tilde{K}(\Sigma(X/A)) \to \tilde{K}(\Sigma X) \to \tilde{K}(\Sigma A) \to \tilde{K}(X/A) \to \tilde{K}(X) \to \tilde{K}(A)$$

$\tilde{K}(\Sigma^i X) = \tilde{K}(S^i \wedge X)$ is also denoted as $\tilde{K}^{-i}(X)$ The graded K groups.

operations in K theory

Using the long exact sequence one shows that for $A \to X$ retractions ($\Sigma A \to \Sigma X$ is also a retraction and the retraction give splitting morphisms in K theory) there is a splitting of the K-theory group of X as a direct sum of the K-theory of A and the relative K-theory of the quotient space X/A :

$$K(X) \simeq K(A) \oplus K(X, A)$$

and in the pointed case a splitting of the reduced K-theory groups

$$\tilde{K}(X) \simeq \tilde{K}(A) \oplus K(X, A).$$

Using this one finds that the reduced K theory of the joint of 2 pointed spaces is:

$$\tilde{K}(X \vee Y) \simeq \tilde{K}(X) \oplus \tilde{K}(Y)$$

($X \to X \vee Y$ and $Y \to X \vee Y$ are retracts and $X \vee Y/X \cong Y$)
external product
Let X and Y be topological spaces. Then the external tensor product of topological vector bundles $E \to X$ and $F \to Y$:
$pr_1^* E \otimes pr_2^* F \to X \times Y$
induces on K-groups an external product

$$\boxtimes : K(X) \oplus K(Y) \to K(X \times Y)$$

We want to see that this restricts to an operation on reduced K-theory.We have that
Let (X, x_0) (Y, y_0) be two pointed compact Hausdorff spaces with $X \wedge Y$ their smash product. Then there is an isomorphism of reduced K-theory groups

$$\tilde{K}(X \times Y) \simeq \tilde{K}(X \wedge Y) \oplus \tilde{K}(X) \oplus \tilde{K}(Y).$$

Proof. Be definition, the smash product is the quotient topological space of the product topological space by the wedge sum:

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

Hence the long exact sequence takes the form

$$\tilde{K}(\Sigma(X \times Y)) \xrightarrow{\Sigma i^*} \tilde{K}((\Sigma X) \vee (\Sigma Y)) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \xrightarrow{i^*} \tilde{K}(X \vee Y).$$

By the above the two terms involving reduced topological K-theory of joint sum are direct sums of the reduced K-theory of the wedge summands:

$$\tilde{K}(\Sigma(X \times Y)) \xrightarrow{\Sigma i^*} \tilde{K}(\Sigma X) \oplus \tilde{K}(\Sigma Y) \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \xrightarrow{i^*} \tilde{K}(X) \oplus \tilde{K}(Y).$$

Now observe that , the morphisms i^* and Σi^* are split epimorphisms, with section given by "external direct sum"

$$\begin{array}{ccc} \tilde{K}(X) \oplus \tilde{K}(Y) & \to & \tilde{K}(X \times Y) \\ (E_X, E_Y) & \mapsto & p_X^* (E_X) + p_Y^* (E_Y) \end{array}.$$

Thus we get split short exact sequence

$$0 \to \tilde{K}(X \wedge Y) \to \tilde{K}(X \times Y) \to \tilde{K}(X) \oplus \tilde{K}(Y) \to 0$$

Because the external product between elements in reduced K theory vanishes by definition on $X \times y_0$ and $x_0 \times Y$ we get that it restricts to an external product $\boxtimes : \tilde{K}(X) \oplus \tilde{K}(Y) \to \tilde{K}(X \wedge Y)$. Note that external products are natural with respect to functorial homomorphisms . Pulling back the externel product $K(X) \times K(X) \to K(X \times X) \xrightarrow{\Delta^*} K(X)$ by the diagongal we get the internal product and the ring structure of $K(X)$.
Using the external product of reduced K theory,we can get an internal (graded) product of graded K groups

$$\tilde{K}^{-i}(X) \times \tilde{K}^{-j}(X) \to \tilde{K}^{-i-j}(X)$$

As $\tilde{K}(S^i \wedge X) \times \tilde{K}(S^j \wedge X) \to \tilde{K}(S^i \wedge X \wedge S^j \wedge Y) = \tilde{K}(S^{i+j} \wedge X \wedge X) \xrightarrow{\Delta^*} (S^{i+j} \wedge X)$

bott periodicity

For S^2 the Euclidean 2 -sphere, write

$$h \in K\left(S^2\right)$$

for the complex topological K-theory class of the basic complex line bundle on the 2-sphere(which has the clutching function on the meridian S^1 $z \to$ multiplication by$z$.

$$\beta := h - 1 \in \tilde{K}\left(S^2\right) = K(\mathbb{C}).$$

This is known as the Bott element external product with which gives the bott periodicity isomorphism $\tilde{K}(X) \to \tilde{K}(S^2 \wedge X)$

Iterating the bott periodicity external product we get that if X is a space $K(X) \to K(X \times V)$ (where V) is represented by external product by the element $\lambda_V \in K(V)$ represented by the complex $... \to \bigwedge^i V \xrightarrow{v} \bigwedge^{i+1} V \to ...$

A special case of this is that $\tilde{K}(S^2) \cong \mathbb{Z}$. We can also determine the ring structure of $\tilde{K}(S^2)$ if we note that $(h-1)^2 = h^2 + 1 - 2h = 0$ since $h^2 + 1$ and $2h$ can be easily shown to have homotopic clutching functions.From bott periodicity we have that $\tilde{K}^{-i-2}(X) \cong \tilde{K}^{-i}(X)$ Therefore $i (\mathrm{mod}\, 2)$ determines $\tilde{K}^{-i}(X)$ and we denote $\tilde{K}^0, \tilde{K}^1$ the even and odd ones respectively.Using that the bott map is natural with respect to functorial homomorphisms ,the long exact sequence in K theory becomes periodic and is really a six term exact sequence:

$$\begin{array}{ccccc}
\tilde{K}^0(X,A) & \longrightarrow & \tilde{K}^0(X) & \longrightarrow & \tilde{K}^0(A) \\
\uparrow & & & & \downarrow \\
\tilde{K}^1(A) & \longleftarrow & \tilde{K}^1(X) & \longleftarrow & \tilde{K}^1(X,A)
\end{array}$$

equivariant K theory

Similarly equivariant K theory is defined in the presence of a group action on a space . with G vector bundles.(meaning there is an action of G on the vector bundle it'self of the form $E_x \to E_{gx}$ is linear.The basic results of K theory pass over to equivariant K theory,(write later).

We also have external products , pullback by G maps and equivariant bott periodicity as we shall see. Equivariant K theory is related to standard K theory if the action is free in which case we have: $K_G(X) \cong K(X/G)$

It is easy to see that if V is a complex representation space of G then λ_V also naturally defines an element in $K_G(V)$ (use equivariant K theory with compact supports,λ_V is the bott element)

We have the equivariant bott periodicity that external product with λ_V induces an isomorphism $K_G(X) \to K_G(X \times V)$

The proof is an equivariant version of the construction of the inverse map with families of dolbeaut operators.(see atiyah bott)

From the above we deduce the thom isomorphism theorem: Let $E \to X$ be a hermitian vector bundle and P the principal $U(n)$ bundle of orthonormal frames. Then E is the associated bundle $P \times \mathbb{C}^n/G$. Then under the isomorphisms $K_{U(n)}(P) = K(X)$,$K_{U(n)}(P \times \mathbb{C}^n) = K(E)$ it is easy to see that the bott periodicity isomorphisms corresponds to external product with the element in $K(E)$ represented by the complex

$... \to \bigwedge^i E \xrightarrow{e} \bigwedge^{i+1} E \to ...$

hence we get the thom isomorphism $K(X) \to K(E)$

1.2.3 K theory with compact supports

There is another definition of K theory (and equivariant K theory) using complexes of vector bundles .This goes as follows :we take homotopy complexes of vector bundles that are exact outside a compact set and factor out complexes that are everywhere exact. This is the form of K theory used in index theorems that it is the same as the K theory we defined before can be found in [Lawson Michelson]

1.2.4 K theory for C^* algebras

K_0

The K theory group of a C^* algebra is constructed out of projections under direct sum and stable equivalence.We immediately give the relevant definitions. By equivalence of projections we shall refer to:

Denote the set of projections in $M_n(A)$ as $\mathcal{P}_n(A)$. and denote $\mathcal{P}_\infty(A) = \bigcup \mathcal{P}_n(A)$.

If $p, q \in \mathcal{P}_n(A)$.

- $p \sim q$ if there exists v in A with $p = v^*v$ and $q = vv^*$ (Murrayvon Neumann equivalence),

- $p \sim_u q$ if there exists a unitary element u in $\mathcal{U}(\widetilde{A})$ with $q = upu^*$ (unitary equivalence).
- $p \sim_h q$ if there exists a homotopy of projections p_t such that $p_0 = p, p_1 = q$

We are going to show that these three notions of equivalence are essentially equivalent :
Proposition Let p, q be projections in a C^*-algebra A.

1. If $p \sim q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_u \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.
2. If $p \sim_u q$, then $\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \sim_h \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix}$ in $M_2(A)$.
3. If $p \sim_h q$, then $p \sim_u q$.
4. If $p \sim_u q$, then $p \sim q$

Similarly to topological K theory $K_0(A)$ for unital A. is defined as the grothendieck group (ring) made out of equivalence classes of projections under direct sum (it is an abelian group). The zero projection in any matrix ring over A should represent the zero element.
so we have the standard picture of $K_0(A)$:
(The standard picture of K_0 - the unital case). Let A be a unital C^*-algebra. Then

$$\begin{aligned} K_0(A) &= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_\infty(A)\} \\ &= \{[p]_0 - [q]_0 : p, q \in \mathcal{P}_n(A), n \in \mathbb{N}\}. \end{aligned}$$

Moreover,

1. $[p \oplus q]_0 = [p]_0 + [q]_0$ for all projections p, q in $\mathcal{P}_\infty(A)$
2. $[0_A]_0 = 0$, where 0_A is the zero projection in A.
3. if p, q are mutually orthogonal projections in $\mathcal{P}_n(A)$, then $[p+q]_0 = [p]_0 + [q]_0$
4. for all p, q in $\mathcal{P}_\infty(A)$, $[p]_0 = [q]_0$ if and only if p, q are stably equivalent ,this means that we can add another projection r and possibly add 0 projections such that $p \oplus r \sim q \oplus r$, this is denoted $p \sim_s q$.

K_0 is turned into a functor on the category of C^* algebras and $*$ homomorphisms in an obvious way.Given proposition 1.2.1 it's immediate to see that K_0 is homotopy invariant.It is easy to see that the rank of projections gives an isomorphism $K_0(\mathbb{C}) \cong \mathbb{Z}$,in other cases the computation of K_0 is not that straightforward.
To prove the proposition let's review a few lemmas that are of much use when one deals with C^* algebraic K theory:

1. If p, q are projections in a C^*-algebra A and $\|p - q\| < 1$, then $p \sim_h q$.
2. If $\|p - q\| < 1/2$ then $p \sim_u q$
3. If a is a self-adjoint element in A with $\delta = \left\|a - a^2\right\| < 1/4$, then there is a projection p in A with $\|a - p\| \leqslant 2\delta$.
4. Let p, q be projections in A. If there exists an element x in A with $\|x^*x - p\| < 1/2$ and $\|xx^* - q\| < 1/2$, then $p \sim q$.

(1) is a direct consequence of the following lemma:
Let p be a projection in a C^*-algebra A, and let a be a selfadjoint element in A. Put $\delta = \|p - a\|$. Then

$$\operatorname{sp}(a) \subseteq [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$$

(1) given this use continuous functional calculus to deform the path $tp + (1-t)q$ into a path of projections(the lemma itself is an easy exercise).
For Item (2) consider the polar decomposition of the invertible element $pq + (1-p)(1-q) = z = |z|u$ it holds that $p = uqu^*$
Items (3),(4) follow from functional calculus arguments.

Proof of proposition:
For Item (1) of proposition 1.2.1 ($p = v^*v$ and $q = vv^*$). Use the unitary wu in $M_2(A)$ where

$$u = \begin{pmatrix} v & 1-q \\ 1-p & v^* \end{pmatrix}, \quad w = \begin{pmatrix} q & 1-q \\ 1-q & q \end{pmatrix}$$

Item (2) follows from an easy rotation argument.
Item (3) is a direct consequence of lemma (2) above and
Item (4) is trivial (take $v = qu^*$)

To define K_0 for nonunital C^* algebras consider the unitization $\tilde{A}$ one has the $*$ homomorphism $s : \tilde{A} \to \mathbb{C}$ that sends $\lambda 1 + a$ to λ. $K_0(A)$ is defined as the kernel of $K_0(s) : K_0(\tilde{A}) \to K_0(\mathbb{C}) = \mathbb{Z}$. This is entirely analogous to the definition of topological K theory of non compact spaces . Note that for algebras that are already unital this construction gives the same as the initial definition of K_0

One get's similar results as above in the non-unital case and K_0 is turned in an obvious way into a homotopy invariant functor.Some manipulations allow the following description of $K_0(A)$:
(The standard picture of $K_0(A)$). One has for each C^*-algebra A that

$$K_0(A) = \left\{ [p]_0 - [s(p)]_0 : p \in \mathcal{P}_\propto(\widetilde{A}) \right\}.$$

Moreover, the following hold. (i) For each pair of projections p, q in $\mathcal{P}_\infty(\widetilde{A})$, the following conditions are equivalent:
(a) $[p]_0 - [s(p)]_0 = [q]_0 - [s(q)]_0$,
(b) there exist natural numbers k and l such that $p \oplus 1_k \sim_0 q \oplus 1_l$ in $\mathcal{P}_\infty(\widetilde{A})$
(c) there exist scalar projections r_1 and r_2 such that $p \oplus r_1 \sim_0 q \oplus r_2$.
(ii) If p in $\mathcal{P}_\infty(\widetilde{A})$ satisfies $[p]_0 - [s(p)]_0 = 0$, then there is a natural number m with $p \oplus 1_m \sim s(p) \oplus 1_m$.
(iii) If $\varphi : A \to B$ is a *-homomorphism, then

$$K_0(\varphi)\left([p]_0 - [s(p)]_0\right) = [\widetilde{\varphi}(p)]_0 - [s(\tilde{\varphi}(p))]_0$$

for each p in $\mathcal{P}_\infty(\widetilde{A})$.
Note also that if A_1, A_2 are unital C^* algebras then the tensor product gives a
If A_1, A_2 are unital C^* algebras then the tensor product of projections gives an external product map $K_0(A_1) \times K_0(A_2) \to K_0(A_1 \otimes A_2)$.

stability,continuity

Now it is a trivial matter to see that taking matrix algebras one get's the same K_0 groups moreover the inclusion in the upper left corner $A \to M_n(A)$ induces an isomorphism $K_0(A) \cong K_0(M_n(A))$ Now we are going to discuss continouity properties of K_0. our purpose is to be able to compute the K theory of the compact operators on a separable hilbert space ,we are going to show that $K_0(\mathcal{K}) \cong \mathbb{Z}$.
Let $A_n \underset{\mu_n}{\longrightarrow} A$ be an inductive limit of a sequence of C^* algebras $A_n \underset{\phi_n}{\longrightarrow} A_{n+1} \underset{\phi_{n+1}}{\longrightarrow}$ ....(Inductive limits of C^* algebras of this sort always exist ,the limit is constructed as $A = \prod_{n=1}^{\infty} A_n / \sum_{n=1}^{\infty} A_n$,if all $A_n = \mathbb{C}$ then this is just $\updownarrow_\infty / c_0$)
By universal properties we get a morphism $\lim(K_0(A_n)) \to K_0(A)$.Where $\lim(K_0(A_n))$ denotes the inductive limit in the category of abelian groups. The result is that this is an isomorphism. For example if we take the inductive limit $\mathbb{C} \to M_2(\mathbb{C}) \to ... \to M_n(\mathbb{C}) \to ...\mathcal{K}$ we get that $K_0(\mathcal{K}) = \mathbb{Z}$. A similar result holds if we replace $\mathbb{C}$ with an arbitrary C^* algebra A then we get $K_0(A) \cong K_0(\mathcal{K}A)$.
The proof of this proceeds as follows ,first consider the unitizations $\tilde{A_n}$ use a combinations of lemmas 1234 above to prove that $\lim(K_0(A_n)) \to K_0(A)$ is surjective (each $[p] \in K_0(\tilde{A})$ is the image of some $K_0(\mu_n)$) use a similar argument to show injectivity.
Before we end this section let's review the relatiion of this sort of K theory for C^* algebras with the ring theoretic counterpart constructed out of projective modules or equivalently idempotent elements over A (denoted $\mathcal{I}_\infty(A)$)
The equivalence relation used in the context of idempotents is:$\approx_0$ on $\mathcal{I}_\infty(A)$. Suppose that e belongs to $\mathcal{I}_n(A)$ and f to $\mathcal{I}_m(A)$. Then $e \approx_0 f$ if $e = ab$ and $f = ba$ for some elements a in $M_{n,m}(A)$ and b in $M_{m,n}(A)$. Addition is direct sum.
The resulting K theories are equivalent because of the following :
(i) for every idempotent element e in A, there is a projection p in A with $e \approx_0 p$.:
(ii) For projections p and q in A, show that $p \sim_0 q$ if and only if $p \approx_0 q$. the reader is refered to [rordam]

K_1

Now we are going to define another K theory group which we are going to build out of unitaries ,the development is similar to the previous section so we just state the relevant results.
Let A be a unital C^*-algebra, and let as usual $\mathcal{U}(A)$ denote its group of unitary elements. Set

$$\mathcal{U}_n(A) = \mathcal{U}\left(M_n(A)\right), \quad \mathcal{U}_\infty(A) = \bigcup_{n=1}^{\infty} \mathcal{U}_n(A)$$

Define a relation $\sim_1$ on $\mathcal{U}_\infty(A)$ as follows. For u in $\mathcal{U}_n(A)$ and v in $\mathcal{U}_m(A)$, write $u \sim_1 v$ if there exists a natural number $k \geqslant \max\{m, n\}$ such that
$u \oplus 1_{k-n} \sim_h v \oplus 1_{k-m}$ in $\mathcal{U}_k(A)$, where 1_r is the unit in $M_r(A)$ (and with the convention that $w \oplus 1_0 = w$ for all w in $\mathcal{U}_\infty(A)$).
(The K_1-group). For each C^*-algebra A define

$$K_1(A) = \mathcal{U}_\infty(\widetilde{A}) / \sim_1 .$$

Let $[u]_1$ in $K_1(A)$ denote the equivalence class containing u in $\mathcal{U}_\infty(\widetilde{A})$. Define a binary operation $+$ on $K_1(A)$ by $[u]_1 + [v]_1 = [u \oplus v]_1$, where u, v belong to $\mathcal{U}_\infty(\widetilde{A})$.
The following hold:

- $[u \oplus v]_1 = [u]_1 + [v]_1$
- $[1]_1 = 0$
- if u, v belong to $\mathcal{U}_n(\widetilde{A})$, then $[uv]_1 = [vu]_1 = [u]_1 + [v]_1$ (proof by rotation argument)
- for u, v in $\mathcal{U}_\infty(\widetilde{A})$, $[u]_1 = [v]_1$ if and only if $u \sim_1 v$
- u^* is an additive inverse ,this follows from (3), therefore K_1 is an abelian group

For example. $K_1(\mathbb{C}) = K_1\left(M_n(\mathbb{C})\right) = 0$ As $U(n)$ is connected. . More generally, $K_1(B(H)) = 0$ for each Hilbert space H.(you may use borel functional calculus).Kuiper theorem moreover states that $GL(\mathcal{H})$ is contractible.
Remarks: One doesn't need to take a unitization if A is already unital and even if it is used the resulting groups would be the same . Since $U(A)$ is a deformation retract of $GL(A)$ (through polar decomposition) $K_1(A)$ can be defined with invertible elements instead.One has similarly to the case of projections the lemma
If u, v are unitary elements in A with $\|u - v\| < 2$, then $u \sim_h v$. (this follows from functional calculus)
The stability and continouity properties of K_0 also hold for K_1 the proofs follow the same pattern. For example this let's us state that with $\mathcal{K}$ being the algebra of all compact operators on a separable Hilbert space, we have $K_1(\mathcal{K}) = 0$.

The six term exact sequence

Given an extension of C^* algebras similarly to topological K theory there is a six term exact sequence of K_0, K_1 groups. So let $0 \to I \to A \to A/I|to0$ be an exact sequence of C^* algebras. Then there is a natural six term exact sequence:

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A) & \longrightarrow & K_0(A/I) \\ \delta_1 \uparrow & & & & \downarrow \delta_0 \\ K_1(A/I) & \longleftarrow & K_1(A) & \longleftarrow & K_1(I) \end{array}$$

δ_1 is the index map and δ_0 is the exponential map,the rest are functiorial homomorphisms.The exactness of this sequence at $K_0(A)$ and $K_1(A)$ is due to the fact that K_0, K_1 are half exact functors.

The index map

δ_1 is called the index map because it is exactly the index map in the case of the calkin algebra extension and it gives the index of fredholm operators. The image of $K_1(A) \to K_1(A/I)$ is equal to the kernel of the index map δ_1.Thus $K_1(A) \to K_1(A/I)$ may or may not be surjective and the index map therefore gives an obstruction to lifting unitaries over surjections.In the case of the calkin extension an essentially unitary operator (viewed as an element in the calkin algebra) with nonzero index,cannot be lifted to a unitary in $\mathcal{B}(\mathcal{H})$ because the latter has index 0, this is a topic we are going to discuss later in more detail.

$K_0(SA)$ **is isomorphic to** $K_1(A)$

SA denotes the suspension C^* algebra of continuous loops $f : S^1 \to A$ such that $f(1) = 0$ (or equivalently $f : [0,1] \to A$ such that $f(0) = f(1) = 0$. This follows at once from the short exact sequence $0 \to SA \to CA \to A \to 0$ and the fact that the cone CA ($f : [0,1] \to A$ with $f(0) = 0$) is contractible) the index map gives the isomorphism.We define also higher K groups via $K_n(A) = K_0(S^nA)$ and it is shown that any short exact sequence of C^*-algebras gives rise to a long exact sequence of K-groups.

Description of the index map:

let u in $\mathcal{U}_n(\widetilde{A/I})$ be given.

(i) There exist a unitary v in $\mathcal{U}_{2n}(\widetilde{A})$ and a projection p in $\mathcal{P}_{2n}(\widetilde{I})$ such that

$$\widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \widetilde{\varphi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad s(p) = \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) If v and p are as in (i), and if w in $\mathcal{U}_{2n}(\widetilde{A})$ and q in $\mathcal{P}_{2n}(\widetilde{I})$ satisfy

$$\widetilde{\psi}(w) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}, \quad \widetilde{\varphi}(q) = w \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} w^*$$

then $s(q) = \operatorname{diag}(1_n, 0_n)$ and $p \sim_u q$ in $\mathcal{P}_{2n}(\widetilde{I})$.

(First standard picture of the index map). Let

$$0 \longrightarrow I \xrightarrow{\varphi} A \xrightarrow{\psi} A/I \longrightarrow 0$$

be a short exact sequence of C^*-algebras. Let n be a natural number, and suppose that u in $\mathcal{U}_n(\widetilde{A/I}), v$ in $\mathcal{U}_{2n}(\widetilde{A})$, and p in $\mathcal{P}_{2n}(\widetilde{I})$ satisfy

$$\widetilde{\varphi}(p) = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*, \quad \widetilde{\psi}(v) = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix}.$$

Then $\delta_1([u]_1) = [p]_0 - [s(p)]_0$.

The isomorphism $\theta_A : K_1(A) \to K_0(SA)$ has the following concrete description. Let u in $\mathcal{U}_n(\widetilde{A})$ with $s(u) = 1_n$ be given. Let v in $C\left([0,1], \mathcal{U}_{2n}(\widetilde{A})\right)$ be such that $v(0) = 1_{2n}$, $v(1) = \operatorname{diag}(u, u^*)$, and $s(v(t)) = 1_{2n}$ for every t in $[0,1]$. Put

$$p = v \begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix} v^*.$$

Then p is a projection in $\mathcal{P}_{2n}(\widetilde{SA}), s(p) = \operatorname{diag}(1_n, 0_n)$, and

$$\theta_A([u]_1) = [p]_0 - [s(p)]_0.$$

Using the isomorphism $K_0(SA) \cong K_1(A)$,one can describe the six term exact sequence in terms of mapping cones construction resulting into a more algebraic topological description of the index map,This works as follows: The mapping cone $C(A, A/J)$ of the surjective $*$-homomorphism $\pi : A \to A/J$ is the algebra consisting of pairs (a, f), where $a \in A, f : [0,1] \to A/J$ is continuous, $f(0) = 0$, and $f(1) = \pi(a)$.

For example, the mapping cone of the identity map $A \to A$ is isomorphic to the cone $C(A)$ defined above. The mapping cone $C(A, A/J)$ is a C^*-algebra under the natural pointwise operations. Its K-theory is described by another excision result:

The $*$-homomorphism $\mathrm{J} \to \mathrm{C}(\mathrm{A}, \mathrm{A}/\mathrm{J})$ given by $\mathrm{a} \mapsto (\mathrm{a}, 0)$ induces an isomorphism from $\mathrm{K}_0(\mathrm{J})$ to $\mathrm{K}_0(\mathrm{C}(\mathrm{A}, \mathrm{A}/\mathrm{J}))$. Notice that the suspension and mapping cone constructions are related by the short exact sequence

$$0 \longrightarrow S(A/J) \longrightarrow C(A, A/J) \longrightarrow A \longrightarrow 0$$

where the first $*$-homomorphism is given by $\mathrm{f} \mapsto (0, f)$. This $*$-homomorphism induces a map

$$\mathrm{K}_1(A/J) = \mathrm{K}_0(\mathrm{S}(A/J)) \longrightarrow \mathrm{K}_0(\mathrm{C}(\mathrm{A}, \mathrm{A}/\mathrm{J})) = \mathrm{K}_0(\mathrm{J})$$

which is exactly the index map.

Bott periodicity

First we consider a unital C^*-algebra A. For every natural number n and every projection p in $\mathcal{P}_n(A)$, define the projection loop $f_p : \mathbb{T} \to \mathcal{U}_n(A)$ by

$$f_p(z) = zp + (1_n - p)\,, \quad z \in \mathbb{T}$$

This is the bott map ,it can be shown to coincide with the generator of $K_2(\mathbb{C}) = K_0(C_0(\mathbb{R}^2))$ (Bott periodicity).
The Bott map described above $\beta_A : K_0(A) \to K_1(SA)$ is an isomorphism for every C^*-algebra A.
For every C^*-algebra A and every integer $n \geqslant 0$,

$$K_{n+2}(A) \cong K_n(A).$$

1.2.5 K homology

We are going first to present the geometric intuition behind K homology.The first step towards this direction is understnanding atiyah ell.

Atiyah ell

The reader should go first to the section about pseudodifferential operators before reading this section.
Atiyah $ell(X)$ arises when trying to abstract the properties of pseudodifferential operators on the compact manifold X. (in particular of order 0) An elliptic pseudodifferential operator (of order zero,acting between vector bundles E, F) gives rise to a Fredholm operator between Hilbert spaces (some L_2 spaces $P : L_2(E) \to L_2(F)$). There is however more structure in the elliptic operator which has been ignored on passing to the Hilbert space. To reinstate this further structure we must make use of the fact that our Hilbert spaces are not just abstract vector spaces but are in fact function spaces. Thus they not only admit multiplication by complex scalars but also by continuous functions on X Of course P does not commute with multiplication by f but the commutator $Pf - fP$ turns out to be a compact operator.(approximate f by smooth functions)
We have now arrived at a property of pseudo-differential operators which can be abstracted out and applied to general topological spaces. Thus let X be any compact Hausdorff space and let H_1, H_2 be two Hilbert spaces equiped with representations of $C(X)$. A bounded linear operator

$$P : H_1 \to H_2$$

will be called an operator on X if for any $f \in C(X)$ the commutator $Pf - fP$ is a compact operator. P will be called an elliptic operator on X if, in addition, it is a Fredholm operator. There is then another operator Q on X such that $QP - I$ and $PQ - I$ are both compact (where I denotes the identity operator). The set of all elliptic operators on X will be denoted by $\mathrm{Ell}(X)$.
If X is a point then Ell(point) is given by fredholm operators and the index gives a map: Ell(point) $\to \mathbb{Z}$.
Consider next the dependence of $\mathrm{Ell}(X)$ on X. If $f : X \to Y$ is a continuous map of compact spaces, we get a homomorphism of rings $f^* : C(Y) \to C(X)$. If H_1, H_2 are Hilbert space modules for $C(X)$ they can then be viewed, using f^*, as $C(Y)$ representations. In this way an elliptic operator on $P : H_1 \to H_2$ on X can be viewed as an elliptic operator on Y, thus f induces

$$f_* : \mathrm{Ell}(X) \to \mathrm{Ell}(Y),$$

so that elliptic operators depend covariantly on the underlying space.
In particular, if Y is a point,

$$\mathrm{Ell}(X) \to \mathrm{Ell}\ \text{(point)}\ \to \mathbb{Z}$$

is given by $P \mapsto$ index P.
The main construction we need is one which defines a "cap-product" between $\mathrm{Ell}(X)$ and $K^0(X)$. More precisely, given $P \in \mathrm{Ell}(X)$ and a vector bundle V on X we shall define a new element $P \cap V \in \mathrm{Ell}(X)$. For fixed P and variable V the map $V \mapsto$ index $(P \cap V)$ will then extend by linearity to a homomorphism

$$K^0(X) \to Z.$$

In this way (varying P) we will obtain a map

$$\mathrm{Ell}(X) \to \mathrm{Hom}_{\boldsymbol{Z}}\left(K^0(X), \boldsymbol{Z}\right)$$

Given a (hermitian) vector bundle V over X we define the hilbert spaces $\mathcal{H}_j^V$ of sections of V as follows.Roughly these should be $\Gamma(V) \otimes_{C(X)} H_j$.Represent V by a projection T_1 over $C(X)$ (as in the previous section) By the representations T can be viewed as a projection acting on $\mathcal{H}_1^n$ and $\mathcal{H}_2^n$. Define P^V by $TP^nT : T\mathcal{H}_1^n \to T\mathcal{H}_2^n$ .Then it is easy to see that P^V commutes modulo compacts with the action of $C(X)$ and $Q^V = TQ^nT$ is a parametrix for P^V.
Thus we get an element P^V of $\mathrm{Ell}(X)$ .This is independent of the projection used in an appropriate sense which is the subject of later sections.Certainly though $\mathrm{Index}(P^V)$ depends only on V and it extends to a homomorphism $K^0(X) \to \mathbb{Z}$. Remark: if P is given by an elliptic pseudodifferential operator of order zero acting between vector bundles $E \to F$ and $\mathcal{H}_1 = L_2(E)$, $\mathcal{H}_2 = L_2(F)$
Then P^V can be seen as an ellitpic operator acting between $P^V : E \otimes V \to F \otimes V$ with principal symbol given by $\sigma_P \otimes \mathrm{Id}_V : E \otimes V \to F \otimes V$.
P^V can equivalently be constructed by defining locally where the vector bundle V is trivial and patching together using parititions of unity.
This construction can be extended to positive order operators and the principal symbol still is going to be given by $\sigma_{P^V} = \sigma_P \otimes \mathrm{Id}_V$.Specifying this symbol (and since we will be mainly interested in $\mathrm{Index}(P^V)$)can be taken as the definition. In the level of symbols this is a product of (graded) K-theory classes.we will be seeing this many times from now on.

Atiyah janich theorem

In this section we will prove a theorem that will let us generalize our previous constructions .This and the atiyah-bott proof of bott periodicity is where KK theory manifests it'self.
In the section about fredholm operators we proved that $\mathrm{Index} : [\mathrm{point}, \mathcal{H}] \to \mathbb{Z}$ is a bijection ,we intend to generalize this.
Index bundles:
Suppose that X is a compact space and that $T : X \to \mathscr{F}(\mathcal{H})$ is a continuous map, so that T_x is a family of Fredholm operators depending continuously on the parameter $x \in X$. If $\dim\mathrm{Ker}\, T_x$ is independent of x the family of vector spaces $\mathrm{Ker}\, T_x$ forms a vector bundle $\mathrm{Ker}\, T$ over X and similarly for $\mathrm{Ker}\, T^*$. We can then define the index of the family by

$$\text{index } T = [\mathrm{Ker}\, T] - [\mathrm{Ker}\, T^*] \in K(X)$$

If $\dim\mathrm{Ker}\, T_x$ is not independent of X
Then to construct the index we need the following:
Proposition: Let X be a compact space, and $T : X \longrightarrow \mathcal{F}(\mathcal{H})$ be a continuous map (T is called a continuous family of Fredholm operators on X). Then (i) there exists a closed subspace $V \subset \mathcal{H}$ of finite codimension such that for any $x \in X$,

$$V \cap \mathrm{Ker}\, T_x = \{0\}.$$

(ii) the family vector spaces $\cup_{x\in X} \mathcal{H}/T_x(V)$ (topologized as a quotient space of $X \times \mathcal{H}$) is a vector bundle over X. The vector bundle is denoted by $\mathcal{H}/T(V)$.
Proof. For each $x \in X$, take $V_x = (\mathrm{Ker}\, T_x)^\perp$. Then T_x maps V_x isomorphically onto $T_x(H)$. There is a neighbourhood $\mathcal{U}_x$ of T_x in $\mathcal{B}$ such that for each $S \in \mathcal{U}_x, V_x \cap \mathrm{Ker}\, S = \{0\}$. Let $U_x \subset X$ be the inverse image under T of the open set $\mathcal{U}_x \cap \mathcal{F}(\mathcal{H})$. If $y \in U_x$, then $V_x \cap \mathrm{Ker}\, T_y = \{0\}$. Using the compactness of X, choose a finite covering $U_{x_1}, U_{x_2}, \cdots, U_{x_k}$ of X. Then $V = \cap_{j=1}^k V_{x_j}$ satisfies (i). To get (ii) one can deduce that $\cup_y H/T_y(V)$ is locally trivial when y varies in a neighbourhood of x, and so it is a vector bundle over X.

The index of a continuous family $T : X \longrightarrow \mathcal{F}$ is defined by

$$\text{ind } T = [H/V] - [H/T(V)] \in K(X),$$

where H/V denotes the trivial bundle $X \times (H/V)$. The virtual bundle $[H/V] - [H/T(V)]$ defining ind T is called the index bundle. Thus we get the family index map.It is easy to show that it is independent of the subspace V used.
At the level of maps $X \to \mathcal{F}(\mathcal{H})$ we have naturality:If $f : X' \to X$ is continuous then $f^*(\text{ ind } T) = \text{ ind } (T \circ f)$

Moreover If we have a homotopy of maps $S_t : X \to \mathcal{F}(\mathcal{H})$ this defines an index bundle over $X \times [0, 1]$ and we immediately get homotopy invariance.
In short Index gives a natural transformation between the contravariant functors: $X \to [X, \mathcal{F}(\mathcal{H})]$ and $X \to K(X)$.
$[x, \mathcal{F}(\mathcal{H})]$ can be shown to have the structure of an abelian group under pointwise composition of fredholm operators, the constant identity valued function as 0 element and pointwise adjoint as an inverse.

The above naturall transformation can be shown to be a homomorphism of abelian groups.
The theorem of Atiyah janich states that

$$[X, \mathcal{F}(\mathcal{H})] \xrightarrow{\text{Index}} K(X)$$

is an isomorphism. For the proof see [amiyah mukherjee]

Atiyah bott

Bott periodicity can be proved using the above ideas. First given an elliptic operator on X one gets a map $K(X) \to \mathbb{Z}$ as before. We generalize this , assume we are given a family of elliptic operators over X parametrized by a another space Y. This is equivalently given by an operator P acting between vector bundles over $X \times Y$ that is elliptic on every $X \times \{y\}$.
As shown in the previous section this family has an index $\text{ind}(P) \in K(Y)$. For vector bundles over $X \times Y$ $V \to \text{ind}(P^V)$ gives a map $K(X \times Y) \xrightarrow{\text{ind}_P} K(Y)$.
If a vector bundle is trivial in the X direction (namely) it is a pullback bundle $\text{pr}_2^*(W)$ on Y then it is easy to see that $\text{ind}_P(\text{pr}_2^*(W)) = \text{ind}(P) \cdot W \in K(Y)$.
Moreover it is easy to see that we have a naturall (with respect to Y) transformation $K(Y)$ module homomorphism .

$$\text{ind}_P : K(X \times Y) \to K(Y)$$

(It is here that the product in KK is starting to manifest itself.)
Starting from this bott periodicity can be proved as follows.
Identify the sphere S^2 with the complex projective line $\mathbb{C}P^1$ and consider the dolbeaut operator $\overline{\partial}$ from functions to forms of type $(0, 1)$.
It is shown that $\text{ind}_{\overline{\partial}}(\beta) = 1$ where β is the bott elemen.
Using $\text{ind}_{\overline{\partial}}$ we get a left inverse to bott periodicity as follows:

$$\tilde{K}(S^2 \wedge X) \to \tilde{K}(S^2 \times X) \xrightarrow{\text{ind}_{\overline{\partial}}} \tilde{K}(X)$$

(Use naturallity for the map $\cdot \to X$ to see that the above is a well defined sequence of maps.It is in general true that ind_P descends to a natural $\tilde{K}(Y)$ module map $\tilde{K}(X \wedge Y) \to \tilde{K}(Y)$)
That this defines a left inverse to the bott map follows from the $K(X)$ module property of $\text{ind}_{\overline{\partial}}$.
Consider the naturallity square:

$$\begin{array}{ccc} \tilde{K}(S^2 \wedge (S^2 \wedge X)) & \xrightarrow{\text{ind}_{\overline{\partial}}} & \tilde{K}(S^2 \wedge X) \\ \uparrow & & \uparrow \\ \tilde{K}(S^2 \wedge X) & \xrightarrow{\text{ind}_{\overline{\partial}}} & \tilde{K}(X) \end{array}$$

Use this ,the $K(S^2 \wedge X)$ module property of $\text{ind}_{\overline{\partial}}$ .multiplication with the pullback of the bott element $\text{pr}_1^*(\beta) \in$ and the map $S^2 \wedge S^2 \wedge X \to S^2 \wedge S^2 \wedge X$ that interchanges the factors (and is homotopic to the identity.) To get that $(\text{bott}) \circ \text{ind}_{\overline{\partial}} = \text{ind}_{\overline{\partial}} \circ (\text{bott})$ coincide as maps on $K(S^2 \wedge X)$ and conclude that $\text{ind}_{\overline{\partial}}$ is also a right inverse to the bott map.
This proof can be compared to the classical proof that one analyzes the clutching function along $S^1 \wedge X$, the reader is refered to [atiyah bott].
The thom isomorphism is proved using the higher dimensional case (iteration) of this bott isomorphism in $U(n)$ equivariant K theory over a $U(n)$ principal bundle over X,(write later)

1.3 Spectral triples

When one is doing index theory he encounters a very specific structure which essentially gives a representative of a K-homology class as atiyah ell does, we are going to straight give the definition and then some examples.

Definition: A **spectral triple** $(A, \mathcal{H}, D)$ consists of a hilbert space $H = H^+ \oplus H^-$ represented on by a C^* algebra A together with a (possibly unbounded) densely defined self adjoint operator D such that $[D, a]$ is bounded, and D has a compact resolvent. If the hilbert space $H = H^+ \oplus H^-$ is graded together with a grading operator γ such that D is odd ($D\gamma + \gamma D = 0$) and A is represented as even operators ($\gamma a = a\gamma$) then the spectral triple is graded ,otherwise it is ungraded.
Examples include

- The hodge de rham triple $(C^\infty(M), L^2(\bigwedge(T^*M)), d+d^*)$ and the grading is mod 2 the degree of forms.
- The signature operator triple is the same as above except that the grading operator is given by $i^{p(p-1)+1}\star$ on forms of degree p.
- The dirac operator on a spinor bundle
- Basically any first order pseudodifferential operator on sections of a graded vector bundle (which is odd with respect to the grading) and multiplication by functions gives a spectral triple.
- Here is a non commutative example: The noncommutative torus.
  Consider the universal C^*-algebra generated by 2 unitaries U, V subject to the relation $VU = e^{i\theta}UV$. For $\theta = 0$ this is just the continuous functions on the torus, it can be seen as the crossed product $C(S^1) \times_{R_\theta} \mathbb{Z}$ with the automorphism $R_\theta g(z) = g(e^{i\theta}z)$ and U will be given by the identity function z whereas V will be the image of 1 in the crossed product ,by definition $VUV^* = e^{i\theta}U$. Consider the following subalgebra

  $$\mathcal{A}_\theta = \sum c_{nm}U^nV^m \ \ , c_{nm} \text{ is rapidly decaying}$$

  If $\theta = 0$ is 0 this would be just the smooth functions on the torus and their integral would be given by c_{00} .Motivated by this we define a trace ϕ by $\phi(\sum c_{nm}U^nV^m) = c_{00}$ and an inner product $\langle a, b\rangle = \phi(a^*b)$ Completing we get a hilbert space $\mathcal{H}_\theta$ on which $\mathcal{A}_\theta$ acts by multiplication.
  Now recall that the dirac operator on the plane $(\mathbb{R}/2\pi\mathbb{Z})^2$ is given by $\begin{pmatrix} 0 & \partial_x + i\partial_y \\ \partial_x - i\partial_y & 0 \end{pmatrix}$ on $C^\infty((\mathbb{R}/2\pi\mathbb{Z})^2) \oplus C^\infty((\mathbb{R}/2\pi\mathbb{Z})^2)$. Motivated by this we define the derivations on $\mathcal{A}_\theta$,corresponding to ∂_x, ∂_y: $\delta_1(U) = iU, \delta(V) = 0$ and $\delta_2(U) = 0, \delta_2(V) = iV$ and the "dirac operator" densely defined on $\mathcal{H}_\theta \oplus \mathcal{H}_\theta$: $\mathcal{D} = \begin{pmatrix} 0 & \delta_1 + i\delta_2 \\ \delta_1 - i\delta_2 & 0 \end{pmatrix}$ So we get a spectral triple $(\mathcal{A}_\theta, \mathcal{H}_\theta \oplus \mathcal{H}_\theta, \mathcal{D})$ leaving the verification to the reader.

If one defines D acting on suitably chosen spaces as in the analysis parat then D is a fredholm operator : $D(I + D^2)^{-1}$ is a parametrix.
The index of D is non-interesting because it's 0 ,D being a selfadjoint operator.However if we write D as

$$\begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix} : H^+ \oplus H^- \to H^+ \oplus H^-$$

Then D^+ is also fredholm $D^-(I + D^2)^{-1}$ is a paramterix and $\text{ind}(D^+) = -\text{ind}(D^-)$ is of interest.
This is the trivial instance of the index pairing which we are now going to define after a short geometrical discussion. Before that a short digression when A is $C(X)$ then we have a notion of derivation of functions (elements of A) on that space using just the data of the spectral triple . This could be represented as $[D, a]$ so we can actually obtain obtain a metric on, the space X (which is spec(A) or the character space) by defining for two points ϕ, ψ :

$$d(\phi, \psi) = \sup(|\phi(a) - \psi(a)| \, , ||[D, a]|| \leq 1)$$

The noncommutative analog of a the gelfand spectrum is the pure space so in a manner analogous to this one obtains a metric there.

1.3.1 The index pairing

We shall return to this later but first let's give some motivation for this coming from geometry. Recall that a spinor bundle $S \to M$ is a hermitian graded bundle $S = S^+ \oplus S^-$ over M together with a clifford multiplication $T^*_xM \otimes S_x \to S_x$ such that the square of multiplication by ξ_x is given by the scalar $-||\xi_x||^2$ in other words it is a rperesentation of $\text{Cliff}(T^*_xM) \to \text{End}(S_x)$(also multiplication by ξ_x is odd with respect to the grading).For example $\bigwedge T^*M$ is a spinor bundle with the clifford multiplication $\xi_x \to \xi_x \wedge +i_{\xi_x}$ (i_{ξ_x} denotes metric contraction). Recall that a dirac operator on a vector bundle S is one who's symbol has the property that $\sigma(\xi_x)^2 s_x = -|\xi_x|^2 s_x$.
Recall that a spinor connection ∇ on s is a compatible connection with the grading, the hermitian structure and also compatible with the clifford multiplication in the sense that for $a \in \Gamma(T^*M), s \in \Gamma(S)$ we have:
$\nabla_X(a \cdot s) = (\nabla^{\text{LC}}_X a) \cdot s + a \cdot \nabla_X s$ (we use the levi civita connection). In this case the S comes equiped with a dirac operator given by $D : S \xrightarrow{\nabla} T^*M \otimes S \xrightarrow{\text{cliff}} S$. Explicitly $Ds = \sum e^*_i \cdot \nabla_{e_i} s$ where e_i is a basis of T_xM and e^*_i is the dual basis on the cotangent space. Obviously D is odd with respect to the grading of S therefore we get a spectral triple.
Now we come to the notion of twisting with a hermitian bundle E equiped with a compatible connection ∇^E. The vector bundle $S \otimes E$ is also a spinor bundle with the clifford multiplication acting on the first factor S and

also has a spinor connection $\nabla^{S\otimes E} = \nabla^S \otimes \text{id} + \text{id} \otimes \nabla^E$. Therefore we can form as before the twisted dirac operator $D_E : S^+ \otimes E \oplus S^- \otimes E \to S^+ \otimes E \oplus S^- \otimes E$ It is given explicitly by $D_E(s \otimes e) = Ds \otimes e + \sum (e_i^* \cdot s) \otimes \nabla_{e_i} e$ we will return to the twisting of operators later.

The index of the operator $D_E^+ : S^+ \otimes \to S^- \otimes$ is given by the atiyah singer index theorem :

$$\text{ind}(D_E^+) = \int_M \hat{A}(TM) \cdot ch(E)$$

where $\hat{A}$ is the A-genus class and ch is the total chern class.
This is a special instance of the index pairing between the spectral triple given by the dirac operator (which as we will see later represents a K-homology class) and the K theory class given by the vector bundle E and the result is the index of an elliptic operator (an integer).

1.3.2 Brown Douglas Fillmore theory

K-homology is relevant both to geometry and operator algebras. In geometry an elliptic operator gives canonically a K homology cycle and in operator algebras it appears in the theory of extensions as we shall see which are classified by the brown douglas fillmore theory. The pairing of K homology and K theory is interpreted as an index pairing and is relevant of course to index theory.All of the interactions we are going to encounter between K homology and K theory are special cases of a vast generalization due to Kasparov ,that of KK teory as we shall see in the next section. Bdf theory began from a question about essentially unitary operators. An essentially unitary operator T on a hilbert space is such that $TT^* \sim T^*T$ are equal up to compact operators (denote this by $\sim$).
The question is whether two such operators T_1, T_2 are essentially unitarily equivalent which means that there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ which intertwines them up to compacts: $T_2 \sim UT_1U^*$. The answer to this question is considerably easier when we deal with essentially self adjointness.
The **essential spectrum** of an operator T is the spectrum of it's image in the calkin algebra . By atkinson theorem this just means the set of $\lambda \in \mathbb{C}$ such that $\lambda I - T$ fails to be invertible modulo compact operators i.e. fails to be fredholm.
Obviously essential unitary equivalence implies that two operators have the same essential spectrum. For essentially self adjoint operators this goes the other way around so let's restrict our attention to these for the moment.
We can also just consider self adjoint operators by taking $\frac{T+T^*}{2}$.
First we have the following :
Lemma :The essential spectrum of a self-adjoint operator is comprised of the limit points of the spectrum plus the isolated points (that are eigenvalues) of infinite multiplicity.
Proof:Isolated eigenvalues of finite multiplicity are obviously not in the essential spectrum so it suffices to prove that the rest are.
For $\lambda \in \sigma(T)$ take an approximate eigenvector (a sequence of unit vectors v_k such that $||Tv_k - \lambda v_k||$) as in the finite dimensional case it is easy to see that two different approximate eigenvectors for different λ, λ' are approximately orthogonal $\langle v_k, v_k' \rangle \to 0$. Then if λ is a limit point of $\sigma(T)$ then obtain an orthonormal sequence w_k such that $(T - \lambda I)w_k \to 0$ and conclude that $T - \lambda I$ is not a fredholm operator.
Then we have the following well known result :
Weyl Von Neumann Theorem :Every bounded self adjoint operator on a separable hilbert space is an arbitrarily small compact pertrubation of a diagonal operator.
Proof: It suffices to prove this for the multiplication operator $T : \phi(\lambda) \to \lambda\phi(\lambda)$ on $\mathcal{H} = L_2(\sigma(T), \mu)$ by the spectral theorem.Take a sequence of refinements of $\sigma(T)$ in sets of diameter at most $\epsilon/2^n$ in each step and denote $\mathcal{H}_n \subset \mathcal{H}_{n+1}$ the subspace of functions that are constant in the n-th refinement and deonte P_n the orthogonal projection on $\mathcal{H}_n$. T is within $\epsilon/2^n$ in norm to a multiplication operator that is constant on the n-th refinement, this shows $||P_nT - TP_n|| \leq 2\epsilon/2^n$. Denote the projection $Q_n = P_n - P_{n-1}$ and since $P_n = \sum Q_n$ converges strongly to the identity :$T = \sum TQ_n = \sum Q_nTQ_n + [T, Q_n]Q_n$ where the first term is a direct sum of finite rank selfadjoint (hence diagonal) operators and the second is bounded in norm by 2ϵ and is easy to see it's compact.
The diagonal operator from the last theorem has the same essential spectrum as T ,which is the set of limit points of it's eigenvalue sequence and it's easy to see that for 2 diagonal operators with the same essential spectrum we can arrange their eigenvalue sequences to differ by an element of c_0 establishing unitary equivalence.
So we proved our first point :
The essential spectrum classifies the essential unitary equivalence classes for essentially self adjoint operators.
Things are not so straightforward for essentially normal operators. We have another invariant however that of the index. Essentially unitary equivalent fredholm operators have the same index : index$(T - \lambda I)$ for $\lambda \notin \sigma_{\text{ess}}(T) = X$ is a well defined function on essentially unitary equivalence classes of operators with essential

spectrum X to be denoted by $\text{Ext}(X)$ from now on. $\text{Ext}(X)$ carries an addition operation given by direct sum of operators. Also note that by homotopy invariance the index function $\text{index}(T - \lambda I)$ is locally constant on $\mathbb{C} \backslash X$ and 0 on the unbounded component of $\mathbb{C} \backslash X$.
In case the essential spectrum is the circle ($|z| = 1$) then one can show that index $(T) : \text{Ext}(S^1) \to \mathbb{Z}$ is an isomorphism of semigroups.
The brown douglas fillmore theorem is just the obvious generalization of this , the index function is an isomorphism of groups $\text{Ext}(X)$ and the integer functions on the bounded components of $\mathbb{C} \backslash X$. Let's review how C^* algebra extensions get involved in the determination of $\text{Ext}(X)$.
Given an essentially unitary operator T with essential spectrum X ,the first thing we get is an injective unital (it maps the unit elements to each other) $*$-morphism $C(X) \to \mathcal{Q}(\mathcal{H})$.Now consider the C^* algebra E_T generated by T, T^* and compact operators. We get an exact sequence of C^* algebras

$$0 \to \mathcal{K}(\mathcal{H}) \to E_T \to C(X) \to 0$$

Where $E_T \to C(X)$ is given by projection to the calkin algebra composed with the functional calculus morphism.
If T, T' are essentially unitary equivalent then Ad_U intertwines the morphisms $C(X) \to \mathcal{Q}(\mathcal{H})$, $C(X) \to \mathcal{Q}(\mathcal{H}')$ we also naturally get an isomorphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{K}(\mathcal{H}) & \longrightarrow & E_T & \longrightarrow & C(X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle Ad_U} & & \downarrow{\scriptstyle Ad_U} & & \Downarrow & & \\ 0 & \longrightarrow & \mathcal{K}(\mathcal{H}') & \longrightarrow & E_{T'} & \longrightarrow & C(X) & \longrightarrow & 0 \end{array}$$

This suggests that $\text{Ext}(X)$ be equivalently given by isomorphism classes of injective morphisms $C(X) \to \mathcal{Q}(\mathcal{H})$ or extensions of $C(X)$ by $\mathcal{K}(\mathcal{H}) : 0 \to \mathcal{K}(\mathcal{H}) \to E_T \to C(X) \to 0$. This is indeed the case [HR]. And in general any extension of a C^* algebra A by $\mathcal{K}(\mathcal{H})$ can be equivalently given by an injective morphism $A \to \mathcal{Q}(\mathcal{H})$. To see this note that for an exact sequence of C^* algebras $0 \to J \to E \to A \to 0$ we naturally have a morphism from E to the multiplier algebra of $J : \mathcal{M}(J)$ and a morphism $A = E/J \to \mathcal{M}(J)/J = \mathcal{Q}(J)$ into the corona algebra,(known as the busby invariant). (in case $J = \mathcal{K}(\mathcal{H})$ we get a representation of E on $\mathcal{H}$ by $\rho(e)j \cdot v = ej \cdot v$ for $j \in \mathcal{K}(\mathcal{H}) \subset E$).
Conversely from a morphism $\phi : A \to \mathcal{Q}(\mathcal{H})$ we get an extension by setting taking E to be the pullback C^* algebra $\subset \mathcal{B}(\mathcal{H}) \oplus A : (T, a) | \pi(T) = \phi(a)$ and an exact sequence $0 \to \mathcal{K}(\mathcal{H}) \to E \to A \to 0$. Equivalence in the first case implies equivalence in the second case ,for details see so we can call $\text{Ext}(A)$ the set formed by the equivalence classes of injective ,unital extensions of A in any of the 2 above notions.
$\text{Ext}(A)$ carries a semigroup structure (easy to see if we view it as extensions $A \to \mathcal{Q}(\mathcal{H})$) . In short we have the following :
Ext(X) and Ext$(C(X))$ are isomorphic as semigroups
An extension $A \to \mathcal{Q}(\mathcal{H})$ is split if it admits a lift $A \to \mathcal{B}(\mathcal{H})$ these intiutively represent the zero element in the semigroup $\text{Ext}(A)$
Furthermore an extension $\phi : A \to \mathcal{Q}(\mathcal{H})$ is semisplit if the direct sum $\phi \oplus \phi'$ with another extension $\phi' : A \to \mathcal{Q}(\mathcal{H}')$ is split. ϕ' is supposed to represent an additive inverse.
Before we go on we have to review a very basic extension.
The toeplitz extension:On $L_2(S^1)$ denote P the projection on the subspace that is the closed span $\mathcal{H}$ of $\{e^{in\theta} | n \geq 0\}$ and denote by M_g the operator of multiplication by $g \in C(S^1)$.It is easy to prove that P, M_g commute up to compacts. Denote $T_g = PM_gP \in \mathcal{B}(\mathcal{H})$ then $T_{g_1 g_2} \sim T_{g_1} T_{g_2}$ and $T_g^* = T_{\overline{g}}$ so we get the toeplitz extension $C(S^1) \to \mathcal{Q}(\mathcal{H})$. A few facts about that:
It's straightforward to show that the essential spectrum of T_g is given by the range of g (think about the C^* algebra produced by T_g and the compacts , an inverse modulo compacts should belong there). If g doesn't have a zero the index of T_g is equal to the winding number of g one proves that by reducing to the case of z^n by homotopy.The toeplitz extension can be generalized in geometry if we take the C^* algebra to be the $C(M)$ and let P be a spectral projection of a selfadjoint pseudodifferential operator on a compact manifold M.
This suggests also a device for producing extensions : If we have a representation of A on $\mathcal{H}$ and P is a projection that commutes up to compacts with the action of A then we get an extension $A \to \mathcal{Q}(P\mathcal{H})$ by the rule $a \to P\rho(a)P$. This is an abstract toeplitz extension. The following fact is straightforward :
Any semisplit extension is the same as an abstract toeplitz extension.
At this stage it would be convenient that for a C^* algebra every extension is semisplit. We have the following .
Every extension $C(X) \to \mathcal{Q}(\mathcal{H})$ for a compact metric space is semisplit.
We are not going to cover every detail of this but certain arguments in the proof are of interest to us.
We need the notion of completly positive map between C^* algebras A linear map $\sigma : A \to B$ is completely positive if $\sigma(1) = 1$ and $\sum b_i^* \sigma(a_i^* a_j) b_j \geq 0$ for any $a_1, ..., a_n$,$b_1, ..., b_n$.
Maps $A \to \mathcal{B}(\mathcal{H})$ of the following short are completely positive : If $V : \mathcal{H} \to \mathcal{H}_1$ is an isometric embedding and $\rho : A \to \mathcal{B}(\mathcal{H})$ is a representation then $\sigma(a) = V^* \rho(a) V$ is completely positive. By Stinespring theorem these

maps (compression to subspaces) are all the completely positive maps (the proof is an adaptation of the GNS construction).
It is easy to show that an extension $A \to \mathcal{Q}(\mathcal{H})$ is semisplit iff it has a completely positive lifting $A \to \mathcal{B}(\mathcal{H})$. Now the above is proved via the following two results; There is a sequence completely positive maps $\phi_n : A \to \mathcal{Q}(\mathcal{H})$ that converge pointwise and lift to $\mathcal{B}(\mathcal{H})$.
These are constructed by the nuclearity of $C(X)$: approximate ϕ by finite rank completely positive operators that can be lifted to $\mathcal{B}(\mathcal{H})$.
If σ_n is a sequence of completely positive liftable maps $A \to \mathcal{Q}(\mathcal{H})$ converging pointwise to σ then σ is liftable: Let ρ_n be liftings to $\mathcal{B}(\mathcal{H})$ and take a countable dense set in $a_1, a_2, ... \in C(X)$. We can assume that the following holds $||\sigma_N(a_j) - \sigma_{N-1}(a_j)||_{\mathcal{Q}(\mathcal{H})} < 2^{-N}$ for $j < N$.

We alter the liftings ρ_N to ρ'_N such that $||\rho'_N(a_j) - \rho'_{N-1}(a_j)|| < 2^{-N}$ for $j < N$. We do this with the help of a quacicentral apporximate unity u_k in $\mathcal{K}(\mathcal{H})$ for the separable C_* algebra produced by $\mathcal{K}(\mathcal{H})$ and $\rho_n(A)$ for all n.
Inductively set

$$\rho'_{N+1}(a) = (1 - u_k)^{1/2} \rho_{N+1}(a)(1 - u_k)^{1/2} + u_k^{1/2} \rho'_N(a) u_k^{1/2}$$

these project to $\sigma_{N+1}(a)$ and for large enough k it can be arranged that $||\rho'_{N+1}(a_j) - \rho'_N(a_j)|| < 2^{-N-1}$. Therefore the completely positive ρ'_N converge pointwise to a completely positive lifting of σ.(this sort of constructions are encountered in Kasparov theory)
We are now in a position to say that every every extension of $C(X)$ is given by an abstract toeplitz extension ,or equivalently a representation of $C(X)$ on a hilbert space together with a projection that commutes up to compacts with this representation. As mentioned earlier a split extension represents the 0 element in Ext, this is not at all easy to prove and is a consequence of voiculescu theorem, [23] therefore we have also shown that **Ext**$(C(X))$ **is an abelian group.**(this holds for nuclear ,separable c^* algebras A)
The above suggest that we are really interested in a K_0 group of projections that commute up to compacts with a representation of A for which nonzero a don't act as compact operators(this is called an ample representation) .We define the relevant C^* algebras,the dual C^*algebra $D_\rho(A) =$ of an ample representation $\rho : A \to \mathcal{B}(\mathcal{H})$ as The subalgebra of $\mathcal{B}(\mathcal{H})$ of operators that commute up to compacts with $\rho(A)$.It is a consequence of voiculescu theorem that two ample representations are unitarily equivalent therefore we shall just write $D(A)$ for the dual algebra as the representation is irrelevant.
Projections in $M_n(D_\rho)$ is the same thing as projections in D_σ where σ is the direct sum representation $\rho \oplus \rho \oplus ... \oplus \rho$ therefore the K_0 group $K_0(D(A))$ is given by projections $P \in D(A)$.Furthermore Iis shown to represent the 0 element and an additive inverse is given by $I - P$
If we associate to each P the toeplitz extension it defines then it is easy to see that two murray von neumann equivalent projections define unitarily equivalent toeplitz extensions. Therefore combining all of the above we see that this gives an isomorphism of groups:

$$K_0(D(A)) \cong \text{Ext}(A)$$

We now come to the definition of K-homology groups .The K homology groups $K^1(A), K^0(A)$ are defined as $K^1(A) = K_0(D(\tilde{A}))$ and $K^0(A) = K_1(D(\tilde{A}))$ (where $\tilde{A}$ denotes the algebra with a unit adjoined).Note that K^0, K^1 are contravariant functors in an obvious way ,if $B \to A$ is an injective homomrphism we obtain an inclusion $D(A) \subset D(B)$ .To define this for not necessarily injective morphisms we refer the reader to [23] this will become much easier when we introduce another representation of $K-$homology that of fredholm modules
Now we define the index pairing between K-theory and K-homology.(recall the toeplitz extension and what all of that means there)

- Pairing of $K^1(A)$ and $K_1(A)$
  An element of $K^1(A)$ is given by a projection $P \in \mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is an ample representation space of $\tilde{A}$ and an element of $K_1(A)$ is given by a unitary u over some matrix ring $M_n(\tilde{A})$ which can be made to act on $\mathcal{H}^n$. Let P_n denote the n fold direct sum of P
  $\langle K^1(A), K_1(A)\rangle$ is defined by $\langle [P], [u]\rangle = \text{index}(P_n u P_n)$ on $P_n \mathcal{H}^n$.$p_n u P_n$ is fredholm because $P_n u^* P_n$ is a two sided inverse modulo compacts.(for example in the toeplitz extension the pairing of the toeplitz projection and the unitary z is 1)

- Pairing of $K^0(A)$ and $K_0(A)$
  Similarly a pairing of a class in $K^0(A)$ given by a unitary $U \in \mathcal{B}(\mathcal{H})$ with a class in $K_0(A)$ given by differences of classes of projections over $\tilde{A}$ is given by $\langle [U], [p]\rangle = \text{index}(pU_n p)$ on $p\mathcal{H}^n$.

As we shall see the brown douglas fillmore theorem is essentially the fact that the index pairing defines an isomorphism

$$K^1(A) \to \text{hom}(K_1(A), \mathbb{Z})$$

For $A = C(X)$. Assuming this is true we need two additional facts first.
The pointwise determinant gives an isomorphism of $K_1(C(X)) \cong H^1(X)$ where $H^1(X)$ is the group $[X, \mathbb{C} - 0]$ (under pointwise multiplication).
$H^1(X)$ is freely generated by the functions $z - \lambda$ where λ is some point in a bounded component of $\mathbb{C}\backslash X$ (we choose one point for each).
Proof of the brown douglas fillmore theorem:
As we have already seen $\mathrm{Ext}(X) \cong K^1(C(X))$ the correspondence is that for $T \in \mathrm{Ext}(X)$ one get's an injective extension $C(X) \to \mathcal{Q}(\mathcal{H})$ that is semisplit and is given by an abstract toeplitz extension which is in turn given by a projection P on $\mathcal{H}_1$ where $P\mathcal{H}_1 = \mathcal{H}$. The representation of $C(X)$ on $\mathcal{H}_1$ is given by the functional calculus on the normal operator T_1 (where the identity function z goes to T_1 ,note that by definition $PT_1P \sim T$). If $[f] \in H^1(X) \cong K_1(C(X))$ it is easy to see that the index pairing between $[f]$ and $[P]_{K^1(C(X))} \leftrightarrow [T]_{\mathrm{Ext}(X)}$ is given by $\mathrm{index}(Pf(T_1)P)$. In particular if $f = z - \lambda$ then this is just $\mathrm{index}(P(T_1 - \lambda I)P) = \mathrm{index}(T - \lambda I : \mathcal{H} \to \mathcal{H})$. The bdf theorem is a restatement of the fact that the index function gives an isomorphism $K^1(C(X)) \to \mathrm{hom}(K_1(C(X)), \mathbb{Z}) = \mathrm{hom}(H^1(X), \mathbb{Z})$.
The proof of the fact that $K^1(A) \to \mathrm{hom}(K_1(A), \mathbb{Z})$ is an isomorphism for some C^* algebra (or a rationalized version of this) is algebraic topological in nature and is done in a way similar with poincare duality.
Similarly to the bott periodicity and six term exact sequence in K theory and there is another six term exact sequence in K homology (which is a special case of that in KK theory)

$$\begin{array}{ccccc} K^1(A/J) & \longrightarrow & K^1(A) & \longrightarrow & K^1(J) \\ \uparrow & & & & \downarrow \\ K^0(J) & \longleftarrow & K^0(A) & \longleftarrow & K^0(A/J) \end{array}$$

. One proves that the index pairing is compatible with the boundary maps (and functorial homomorphisms which is trivial) and therefore obtains a morphism of the above exact sequence to the $\mathrm{hom}(-, Z)$ version of the K theory exact sequence . (the latter is not always exact of course but in cases of interests to us one encounters only free abelian groups,and anyway one can take the rationalized version)
Therefore one can prove for increasingly complicated C^* algebras that the index pairing defines an isomorphism.For $A = \mathbb{C}$ it's a simple excercise.The reader is referred to [23].

1.3.3 Fredholm modules

Now we are going to give an alternative definition of K homology that's naturally encountered in geometry.
Definition:**Fredholm module**
A fredholm module $(\rho, \mathcal{H}, F)$ over a separable C^* algebra A is given by a separable hilbert space $\mathcal{H}$ together a representation ρ of A and an operator F such that.

$$(F^2 - 1)\rho(a),\ (F - F^*)\rho(a),\ F\rho(a) - \rho(a)F$$

are compact . As in the case of spectral triples the fredholm module can be ungraded or graded i.e. $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ is graded together with a grading operator γ and the representation of A is by even operators whereas F is odd. In fact spectral triples (if we take the C^* algebra to be unital) are almost the same thing as fredholm modules given a spectral triple $(A, \mathcal{H}, D)$ we can take $F = D(I + D^2)^{-1/2}$ and obtain a fredholm module.
We will also need to define p-multigraded fredholm modules,see [23]
We will get a group out of fredholm modules (or unitary equivalence classes thereof with the obvious meaning) under direct sum. A fredholm module $[x]$ is degenerate if the above three operators are identically 0.This should represent the zero element a way to see this is that if we take the countable direct sum $[\Sigma x]$ of $[x]$ then this still represents a fredholm module and $[\Sigma x] + [x] \cong [\Sigma x]$ (unitary equivalence).
Therefore we factor out degenarate fredholm modules.
Next we impose that fredholm modules that are operator homotopic (meaning there is a continuous homotopy of fredholm modules $(\rho, \mathcal{H}, F_t$,$t \to F_t$ should be continous in operator norm. In this way we obtain the K-homology group $KK^{-p}(A)$ generated by p-multigraded fredholm modules.
We have that this definition of K-homology (for $p = -1, 0$) is equivalent to the one given in last section.

- We have a map $K^0(A) \to KK^0(A)$.
  Take a unitary $U \in D_\rho(A)$ and form the graded fredholm module $(\rho \oplus 0, \mathcal{H} \oplus \mathcal{H}, \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix})$
- We also have a map $K^1(A) \to KK^1(A)$.
  Take a projection $P \in D_\rho(A)$ and form the ungraded fredholm modules $(\rho, \mathcal{H}, 2P - 1)$

The above maps are isomorphisms ,towards this : Every fredholm module $[x] \in KK^{-p}(A)$ has an additive inverse fredholm module $[x^{op}]$ (the one with opposite multigrading and opposite operator F^{op}) such that $[x \oplus x^{op}]$ is homotopic to a degenerate element.Therefore any class in $KK^{-p}(A)$ is represented by a single fredholm module.
(two fredholm modules differ by compact perturbation if $(F - F')\rho(a)$ is compact these are obviously homotopic)

Lemma :Every fredholm module $(\rho, \mathcal{H}, F)$ is homotopic to one $(\rho, \mathcal{H}, F')$ such that F' is self adjoint and $||F'|| \leq 1$:
First take $\frac{F+F'}{2}$ that is a compact perturbation therefore assume that F is already self adjoint. Next take $\phi(F)$ (for the function $\phi(\lambda) = \lambda, \lambda \in [-1, 1]$, $\phi(\lambda) = -1$ if $\lambda \leq -1$ and $\phi(\lambda) = 1$ if $\lambda \geq 1$) $\phi(F)$ is a compact perturbation of F because they project to the same element in the quotient C^* algebra $D(A)/I$ where $I = \{T \in D(A) | T\rho(a) \in \mathcal{K}(\mathcal{H})$
Normalization of fredholm modules : A Fredholm modules is normalized if $F - F^* = F^2 - 1 = 0$ (For example the above maps produce normalized fredholm modules) Any class in $KK^{-p}(A)$ is represented by a normalized fredholm module.
Continue from the last lemma and take the equivalent fredholm module $(\rho \oplus 0, \mathcal{H}, \begin{pmatrix} F & (1-F^2)^{1/2} \\ (1-F^2)^{1/2} & -F \end{pmatrix})$
Without too much additional effort one can prove that the above maps are isomorphic. Inspecting the above one sees that we can actually define K homology with normalized fredholm modules.

The index pairing

As we did before we can directly define an index pairing between $KK(A)$ and K theory (assume A is unital the nonunital case is similar)

- Pairing of $KK^0(A)$ and $K_0(A)$
  Take a class in $KK^0(A)$ represented by a graded fredholm module $(\rho, \mathcal{H}, F)$ where $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ and $F = \begin{pmatrix} 0 & F^- \\ F^+ & 0 \end{pmatrix}$ and a class $[p] \in K_0(A)$ where p is a projection in $M_n(A)$ which can be made to act evenly on $\mathcal{H}^n = (\mathcal{H}^+)^n \oplus (\mathcal{H}^-)^n$ as $P = \rho(p)$. The index is given by

  $$\text{index}((P(F^+)^n P) : (\mathcal{H}^+)^n \to (\mathcal{H}^-)^n)$$

  Note that $P(F^-)^n P$ is an inverse modulo compacts ,this pairing has a simple geometrical meaning as we shall see (like that described in the twisting of dirac operators in the section about spectral triples)

- Pairing of $KK^1(A)$ and $K_1(A)$
  Take a class in $KK^1(A)$ represented by an ungraded fredholm module $(\rho, \mathcal{H}, F)$ and a class $[u] \in K_1(A)$ for a unitary $u \in M_n(A)$ which acts on $\mathcal{H}^n$ as $\rho(u) = U$.Denote $P = \frac{1+F}{2}$ it is a projection up to compacts.The index is given by:

  $$\text{index}(P^n U P^n - (1 - P^n)) : \mathcal{H}^n \to \mathcal{H}^n$$

  Note that $P^n U P^n - (1 - P^n)$ is essentially unitary.

The above are well defined pairings between abelian groups and they are exactly the same as the ones defined before (in the second one we need to take a normalized fredholm module so that P is actually a projection). In a subsequent section we are going to see how this pairing can be computed using chern charachters and prove index theorems in the context of cyclic cohomology and homology.

The product

We are going to construct a product $KK(A_1) \times KK(A_2) \to KK(A_1 \otimes A_2)$ which we are going to interpret geometrically later.This construction is at the heart of KK theory.
We need some things first: We know that a self-adjoint operator is fredholm if and only if 0 is not in the essential spectrum ,is not an accumulation point of the (essential) spectrum ,the following is an immediate consequence of this.
An odd ,self adjoint operator F on a graded hilbert space $\mathcal{H}$ is fredholm iff for some $\epsilon > 0$ $F^2 - \epsilon I$ is positive in $\mathcal{Q}(\mathcal{H})$
We usually care about index(F^+) which is just $\dim(\ker F)^+ - \dim(\ker F)^-$ (the graded dimension), we will denote this just by index(F).

If $FF' + F'F$ is positive in $\mathcal{Q}(\mathcal{H})$ then index$(F) = $ index(F'):
$t \to \cos(\frac{\pi}{2}t)F + \sin(\frac{\pi}{2}t)F'$ is a homotopy of graded fredholm operators. The product of two classes given by

fredholm modules $(\rho_1,\mathcal{H}_1,F_1)$ and $(\rho_2,\mathcal{H}_2,F_2)$ should be given by a fredholm module whose hilbert space and representation are given by tensor products $\mathcal{H}_1\otimes\mathcal{H}_2$ and $\rho_1\otimes\rho_2$ .It remains to determine the operator F, intiutively the following relation between the indices should hold index$(F)=$ index(F_1)index(F_2).
Definition: An operator F on $\mathcal{H}_1\otimes\mathcal{H}_2$ is alligned with F_1 and F_2 if

$$F(F_1\otimes_g I)+(F_1\otimes_g I)F\geq 0 \text{ modulo compacts}$$

$$F(I\otimes_g F_2)+(I\otimes_g F_2)F\geq 0 \text{ modulo compacts}$$

($\otimes_g$ is the graded tensor product of operators :$T_1\otimes_g T_2(v_1\otimes v_2)=(-1)^{\deg(T_2)\deg(v_1)}T_1v_1\otimes T_2v_2$)
There exist such fredholm operators and any of them has index index(F_1)index(F_2).
$F=F_1\otimes_g I+I\otimes_g F_2$ is aligned with F_1,F_2 (for example $F(F_1\otimes_g I)+(F_1\otimes_g I)F=2(F_1^2\otimes_g I)\geq 0$. To show F is fredholm note that for small ϵ:

$$F^2-\epsilon I=(F_1^2-\epsilon I)\otimes_g I+I\otimes_g F_2^2=F_1^2\otimes_g I+I\otimes_g(F_2^2-\epsilon I)\geq 0 \,\text{modulo}\, \mathcal{K}(\mathcal{H}_1)\otimes\mathcal{B}(\mathcal{H}_2),\mathcal{B}(\mathcal{H}_1)\otimes\mathcal{K}(\mathcal{H}_2)$$

Lemmae A1,A2 show that it is positive modulo $\mathcal{K}(\mathcal{H}_1)\otimes\mathcal{K}(\mathcal{H}_2)=\mathcal{K}(\mathcal{H}_1\otimes\mathcal{H}_2)$.Therefore is fredholm by the above.
Lemma A1: If I,J are ideals in a C^* algebra and a is positive modulo both then it is positive modulo $I\cap J$:
Positivity of a modulo an ideal is equivalent to $a^-=\frac{a-|a|}{2}\in I$
Lemma A2: If I,J are closed ideals then $I\cap J=IJ$
write positive self adjoint elements in $I\cap J$ as squares of elements. If F' is another graded fredholm operator aligned with F_1,F_2 then $FF'+F'F\geq 0$ by construction and they have the same index.To determine the index of F note that $\ker F=\ker F^2=\ker(F_1^2\otimes_g I+I\otimes_g F_2^2)=\ker F_1\otimes\ker F_2$
$F=F_1\otimes_g I+I\otimes_g F_2$ is not yet the product we are looking for cause it doesn't satisfy F^2-I but modifying it by adding "weights" is the actual construction of the product.
The weights will be positive operators N_1,N_2 on $\mathcal{H}_1\otimes\mathcal{H}_2$ with the following properties:
$N_1^2+N_2^2=I$
N_1,N_2 commute modulo compact operators with $F_1\otimes_g I, I\otimes_g F_2$
$N_1(F_1\otimes_g I)^2\sim N_1$ and $N_2(I\otimes_g F_2)\sim N_2$
These are obtained with kasparov technical theorem.
Then

$$F=N_1^{\frac{1}{2}}(F_1\otimes_g I)N_1^{\frac{1}{2}}+N_2^{\frac{1}{2}}(I\otimes_g F_2)N_2^{\frac{1}{2}}$$

defines an odd fredholm operator that is aligned with F_1,F_2 and $F^2\sim I$. Now we do these in the context of fredholm modules we .
In analogy with a previous statement we have the following important lemma :
Important lemma: If $(\rho,\mathcal{H},F_0)$,$(\rho,\mathcal{H},F_1)$ are fredholm modules such that $\rho(a)(F_0F_1+F_1F_0)\rho(a^*)\geq 0$ modulo compacts for every $a\in A$ then F_0,F_1 are operator homotopic.

Proof:Denote by $I\subset D(A)$ the ideal of T such that $T\rho(a),\rho(a)T\in\mathcal{K}(\mathcal{H})$.
$F_0F_1+F_1F_0$ commutes modulo I with F_0,F_1
$\rho(a)(F_0F_1+F_1F_0)\rho(a^*)\geq 0$ modulo compacts implies that there is a positive operator S such that $S-(F_0F_1+F_1F_0)\in I$,this follows from the lemma:
If $T\in D(A)$ is selfadjoint modulo I (which is the case for $F_0F_1+F_1F_0$) and $\rho(a)T\rho(a^*)\geq 0$ modulo compacts then T is positive modulo I.
Take $(T+T^*)/2$ therefore it suffices to prove it for T self-adjoint.Write $T=T^+-T^-$ then $T^+T^-=T^-T^+=0$ and for any a, $R=\rho(a)T\rho(a^*)=\rho(a)T^+\rho(a^*)-\rho(a)T^-\rho(a^*)=R^+-R^-$ and R^+R^-,R^-R^+ are compact.We then get R^- is compact therefore $T^-\in I$.
The path

$$F_t=(\cos tF_0+\sin tF_1)(1+\cos t\sin tS)^{-\frac{1}{2}}$$

is an operator homotopy between F_0,F_1 We say that the fredholm module $(\rho_1\otimes\rho_2=\rho,\mathcal{H}_1\otimes\mathcal{H}_2=\mathcal{H},F)$ is aligned with $(\rho_{1,2},\mathcal{H}_{1,2},F_{1,2})$ if

$$\rho(a)(F(F_1\otimes_g I)+(F_1\otimes_g I)F)\rho(a^*)\geq 0 \text{ modulo compacts}$$

$$\rho(a)(F(I\otimes_g F_2)+(I\otimes_g F_2)F)\rho(a^*)\geq 0 \text{ modulo compacts}$$

and if $\rho(a)F$ derives $\mathcal{K}(\mathcal{H}_1)\otimes\mathcal{B}(\mathcal{H}_2)$ Then $(\rho,\mathcal{H},F)$ represents the product. An F is constructed using the sort of construction described above and then the above lemma is used to show that any other F satisfying the above is operator homotopic to the one we constructed.The details are omitted.
This product is associative and the unit element is given by the generator in $KK^0(\mathbb{C})$

Fredholm modules arising from geometry

K homology originally was developed from abstracting the properties of elliptic operators.
Every p-multigraded elliptic pseudodifferential operator P on sections of a p-multigraded vector bundle S over M gives rise to a p-multigraded fredholm module $(\rho, L^2(S), \phi(P))$ in $KK^{-p}(C_0(M))$ where $C_0(M)$ acts as multiplication by functions, and ϕ is a normalizing odd function such that $\phi(t) \to \pm 1$ when $t \to \pm\infty$ such as tanh. Any two normalizing functions give the same class.
The kasparov product of the fredholm modules given by elliptic operators D_1, D_2 is given by the fredholm module of the operator $D_1 \times D_2 = D_1 \otimes_g I + I \otimes_g D_2$ on the bundle $S_1 \otimes S_2$. One has two check the conditions of the kasparov product ,delicate analysis is involved.
In geometric terms the index pairing between a K homology class given by the dirac operator D (graded or ungraded) and a K theory class given by a vector bundle E is the index of the operator D_E .For example about the pairing of $KK^0(C(M))$ and $K_0(C(M)) \cong K^0(M)$if we represent a vector bundle $[E]$ by a projection in $M_N(C(M))$ then we get the index of the operator $P\phi(D)^n P : PS \otimes \mathbb{C}^n \to PS \otimes \mathbb{C}^n$ which is the same thing as the index of the operator $\phi(PD^nP)$ since these are compact perturbations of each other which is the same thing as the index of $\phi(D_E)$ since D_E and PD^nP have the same symbol and are therefore compact perturbations of each other and the same thing as the index of D_E by the spectral decomposition.

1.4 KK theory

KK theory is a generalization of both K homology and K theory The index pairing as well as the product we saw in the last section are special cases of the product in KK theory. It uses hilbert modules to be defined so let's review this material first. See also [3],[11],[13],[26],[25]

1.4.1 Hilbert modules

Let A, B be a C^* algebras Defintion :A Hilbert B module is a complex vector space E which is also a right B module equiped with a B valued inner product $\langle , \rangle$ which is linear in the second variable (and conjugate linear in the first such that the following hold.:

- $\langle x, yb \rangle = \langle x, y \rangle b$ (and $\langle xb, y \rangle = b^* \langle x, y \rangle$)
- $\langle x, y \rangle^* = \langle y, x \rangle$
- $\langle x, x \rangle \geq 0$ and 0 iff $x = 0$

The norm $||x|| = ||\langle x, x \rangle||^{\frac{1}{2}}$ turns E into a normed vector space which should be complete. If not we refer to it as a pre-hilbert module,
We have classical inequalities :

$$||eb|| \leq ||e|| ||b| \text{ and } ||\langle e, f \rangle|| \leq ||e|| ||f||$$

These show that a pre hilbert module can be completed into a hilbert module.
Examples include :

- B itself is a hilbert B-module with inner product $\langle a, b \rangle = a^* b$
- $\mathcal{H}_B$ the module of sequences $(b_1, b_2, b_3, ...)$ such that the inner product $\langle (a_1, a_2, ...), (b_1, b_2, ...) \rangle = \sum a_i^* b_i$ converges in B. When $B = \mathbb{C}$ this is just l^2

For E_1, E_2 hilbert B modules we denote $\mathcal{L}_B(E_1, E_2)$ the space of linear maps that are also B-module maps such that $T : E_1 \to E_2$ has an adjoint :$\langle Te_1, e_2 \rangle = \langle e_1, T^* e_2 \rangle$. (this is not always the case). One shows with the principle of uniform boundedness that T is actually bounded. The norm satisfies $||T|| = ||T^*||$ and $\mathcal{L}_B(E_1, E_2)$ is closed in this norm. Furthermore $\mathcal{L}_B(E, E) = \mathcal{L}_B(E)$ is a C^* algebra.
The generalization of compact operators is the subspace $\mathcal{K}_B(E_1, E_2)$,the closed linear span of maps of the form $\Theta_{x,y} : E_1 \to E_2$ (where $x \in E_2$ and $y \in E_1$) where $\Theta_{x,y}(e_1) = x\langle y, e_1 \rangle$. $\mathcal{K}_B(E)$ is an ideal in $\mathcal{L}_B(E)$ and we have that the multiplier algebra $\mathcal{M}(\mathcal{K}_B(\mathcal{H}))$ is just $\mathcal{L}_B(\mathcal{H})$
Kasparov stabilization theorem :If E is a countably generated Hilbert B-module then $E \oplus \mathcal{H}_B \cong \mathcal{H}_B$
We have the following constructions with hilbert B modules: We denote $E_1 \oplus E_2$ the direct sum hilbert B module.
Pushout
If $f : B \to A$ is a surjective homomorphism between C^* algebras then we have the pushout $f_*(E)$ hilbert A module defined as follows :

Denote by N the submodule of x such that $f(\langle x, x\rangle) = 0$ and then consider the prehilbert A-module E/N with inner product $\langle x+N, y+N\rangle = f(\langle x, y\rangle)$. (You will need two facts to see why this works first f sends positive elements to positive elements and the cauchy schwarz inequality for the inner product $f(\langle , \rangle)$.)Then complete E/N to get $f_*(E)$ .The reader is encouraged about what this does in simple situations.
Internal tensor product .
If E is a hilbert B module and F is a hilbert A module and $\phi : B \to \mathcal{L}_A(F)$ is a *-homomorphism (which makes F an A, B bimodule) We form the algebraic tensor product $E \otimes_B F$ which is a right A module in an obvious way and we define an A valued inner product $\langle e_1 \otimes f_1, e_2 \otimes f_2\rangle = \langle f_1, \phi(\langle e_1, e_2\rangle)f_2\rangle$ which is well defined and A-linear. As before consider the A-submodule N of $E \otimes_B F$ of z such that $\langle z, z\rangle = 0$ and complete the quotient pre-hilbert A module $E \otimes_B F/N$ to get $E \otimes_\phi F$.
Every $F \in \mathcal{L}_B(E)$ corresponds via a *homomorphism to a $F \otimes_\phi I \in \mathcal{L}_A(E \otimes_f F)$ (It suffices to check that $F \otimes I$ maps N to itself but this holds because it is adjoinable)
We have that the pushout $f_*(E)$ is just the internal tensor product with A via $f : B \to \mathcal{L}_A(A)$: $E \otimes_f A$ .The above homomorphism takes $\mathcal{K}_B(E)$ to $\mathcal{K}_A(E \otimes_f A)$ in this case.
External tensor product.
There is also an external tensor product of a hilbert B module E and a hilbert C module F yielding a hilbert $B \otimes C$ module $E \otimes F$. Assume that B or C nuclear so that the tensor product is unique. .

stabilization

A Hilbert B-module E is called countably generated when there is a countable set $\{x_n\}$ in E such that span of the set $\{x_n b : n \in \mathbb{N}, b \in B\}$ is dense in E. A set $\{x_n\}$ in E with this property is called a set of generators for E.

[Kasparov's stabilization theorem]. If E is a countably generated Hilbert B-module, then $E \oplus H_B \approx H_B$. Let $\{\eta_n\} \subseteq E$ be a countable set of generators for E, chosen such that for each $n \in \mathbb{N}, \eta_n = \eta_m$ for infinitely many other $m \in \mathbb{N}$. After normalizing each η_n we can assume that $\|\eta_n\| \leq 1$ for all n. Let ϵ_i be the element of H_B whose coordinates are all zero, except at the i-th place where there is 1 , the unit in B. Define $T : H_B \to E \oplus H_B$ to be the element of $\mathcal{L}_B(H_B, E \oplus H_B)$ given by $T(\epsilon_i) = \left(2^{-i}\eta_i, 4^{-i}\epsilon_i\right), i \in \mathbb{N}$. Then $T = \sum_i 2^{-i}\Theta_{(\eta_i, 2^{-i}\epsilon_i), \epsilon_i} \in \mathcal{K}_B(H_B, E \oplus H_B)$. Fix $n \in \mathbb{N}$. For every other $m \in \mathbb{N}$ such that $\eta_n = \eta_m$ we have that $(\eta_n, 2^{-m}\epsilon_m) = T(2^m\epsilon_m) \in \operatorname{Ran} T$. Since there are infinitely many such m, we see that $(\eta_n, 0)$ is contained in the closure of Ran T. But then $(0, \epsilon_n) = 4^n(T(\epsilon_n) - 2^{-n}(\eta_n, 0))$ is also in this closure. Since T is a B-module map and $\{(\eta_n, 0), (0, \epsilon_n) : n \in \mathbb{N}\}$ generates a dense B-submodule of $E \oplus H_B$, we conclude that T has dense range. Note that

$$\begin{aligned} T^*T &= \sum_{i,j} 2^{-(i+j)}\Theta_{\epsilon_i(<\eta_i,\eta_j>+<2-i\epsilon_i,2-j\epsilon_j>,\epsilon_j} \\ &= \sum_i 4^{-2i}\Theta_{\epsilon_i,\epsilon_i} + \left(\sum_i 2^{-i}\Theta_{(\eta_i,0),\epsilon_i}\right)^* \left(\sum_i 2^{-i}\Theta_{(\eta_i,0),\epsilon_i}\right) \\ &\geq \sum_i 4^{-2i}\Theta_{\epsilon_i,\epsilon_i}. \end{aligned}$$

The latter operator is obviously positive and has dense range in H_B so it is strictly positive by . Since T^*T dominates this element in $\mathcal{K}_B(H_B)$ it must also be strictly positive, i.e. have dense range by . Therefore $|T| = (T^*T)^{\frac{1}{2}}$ has also dense range. Define $V : H_B \to E \oplus H_B$ to be the element in $\mathcal{L}_B(H_B, E \oplus H_B)$ given by $V(|T|x) = Tx, x \in H_B$. Since

$$\begin{aligned} \langle V(|T|x), V(|T|y)\rangle &= \langle Tx, Ty\rangle \\ &= \langle x, T^*Ty\rangle = \langle |T|x, |T|y\rangle \end{aligned}$$

for all $x, y \in H_B$ and $\operatorname{Ran} V$ contains $\operatorname{Ran} T$, we conclude that V defines the desired isomorphism.

1.4.2 The KK-groups

The definition of KK theory is similar to the definition of K-homology via fredholm modules ,except that now instead of hilbert spaces we are using hilbert modules.
Definition :An (A, B) hilbert bimodule is a hilbert B module E toghether with a representation $A \to \mathcal{L}_B(E)$. We also have the notion of graded hilbert bimodule if E is graded (say with grading operator γ and the representation of A is even.
A **Kasparov A,B-module** is a triple (ϕ, E, F) where E is a (graded or ungraded) hilbert (A, B) bimodule and

$F \in \mathcal{L}_B(E)$ is an element of degree 1. such that :

$$(F^2-1)a, (F-F^*)a, [F,a] \in \mathcal{K}_B(E)$$

Denote $\mathbb{E}(A,B)$ the set of kasparov A,B-modules .
These come with an obvious direct sum.
A degenerate kasparov module is one for which the above 3 operators are 0. Denote these by $\mathbb{D}(A,B)$.

The pullback of $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A,B)$ by $\psi : C \to A$ is : $\psi^*(\mathcal{E}) = (E, \phi \circ \psi, F) \in \mathbb{E}(C,B)$
The pushforward by $f : B \to C$ is defined by the kasparov module $f_*(\mathcal{E}) = (\phi \hat{\otimes} I, E \hat{\otimes}_f C, F \hat{\otimes}_f I)$,by the remarks on the internal tensor product this is still a kasparov module in $\mathbb{E}(A,C)$. We have that $g_*(f_*(\mathcal{E})) \cong (g \circ f)_*(\mathcal{E})$.
Two Kasparov A,B-modules $\mathcal{E}_1, \mathcal{E}_2$ are isomorphic if there is an isometry of (graded) hilbert B modules intertwining the the representations of A and the operators F_1, F_2 .

Homotopy Let IB denote $B \otimes C[0,1]$ we have obvious evaluation homomorphisms $\pi_t : IB \to B$
Two Kasparov A,B modules are homotopic if there is a kasparov A, IB-module such that they are isomorphic to the pushforwards of it under the evaluation homomorphisms π_0, π_1.
Slightly weaker relation than this is operator homotopy two kasparov modules (E, ϕ, F_0) and (E, ϕ, F_1) are operator homotopic if there is a homotopy of kasparov modules (E, ϕ, F_t) .This relation is contained in the former , just take The A, IB bimodule $(IE, I\phi, IF)$.(The A, IB bimodule structure is $(I\phi(a)e)(t) = \phi(a)e(t)$ and $eb(t) = e(t)b(t)$ and IF acts as $IF(e)(t) = F_t e_t$.)

The KK-group as in the case of K homology we turn the set of isomorphism classes of kasparov modules into a group under direct sum of kasparov modules and the additional requirements that degenerate modules to represent the zero element and homotopic elements represent the same element. It is a nontrivial result that if one uses the equivalence relation of operator homotopy instead of homotopy then one gets the same groups which we shall denote by $KK(A,B)$. We say groups because as in K homology for every kasparov module there is a similarly defined module which is shown to be an additive inverse.Pushforwards and pullbacks descend to well defined maps on KK groups.
It holds that degenerate modules are homotopic to 0. For this take a degenerate module (E, ϕ, F) and then take the A-IB module $(E \otimes C_0[0,1), I\phi, IF)$ giving a homotopy to the 0 module.
We have the analogous lemma as in K homology that if :
$(E, \phi, F), (E, \phi, F') \in \mathbb{E}(A,B)$ and $\phi(a)(FF' + F'F)\phi(a)^* \geq 0$ modulo $\mathcal{K}_B(E)$ then F, F' are operator homotopic. The proof are exactly the same.
For example $KK(\mathbb{C}, \mathbb{C})$ is easily seen to be $\mathbb{Z}$ which follows from our result that the index is a bijective map from operator homotopy classes of fredholm operators to $\mathbb{Z}$.

1.4.3 The Product

The most general form of the product is

$$Kk(A_1, B_1 \otimes D) \times KK(A_2 \otimes D, B_2) \to KK(A_1 \otimes A_2, B_1 \otimes B_2)$$

We shall see how to construct the product $KK(A,B) \times KK(B,C) \to KK(A,C)$. Take classes in $KK(A,B)$ and $KK(B,C)$ given by kasparov modules (E_1, ϕ, F_1) and (E_2, f, F_2) ,we form the internal tensor product $E_{12} = E_1 \otimes_f E_2$ (which has an obvious grading) and is a hilbert A, C bimodule.
What should a product operator $F_1 \times F_2$ be ,If we try $F_1 \otimes I + I \otimes F_2$ as we did for K-homology then we run into trouble because $I \otimes F_2$ is not a well defined operator. For that we have the notion of an F_2 connection:
Let $T_x \in \mathcal{L}_B(E_2, E_{12})$ be given by $T_x(e_2) = x \otimes_f e_2$ the adjoint is given by $T_x^*(e_1 \otimes_f e_2) = f(\langle x, e_1 \rangle)e_2$.
Consider the operator $\tilde{T}_x = \begin{pmatrix} 0 & T_x^* \\ T_x & 0 \end{pmatrix} \in \mathcal{L}_B(E_2 \oplus E_{12})$ Note that the (grading) degrees of $T_x, T_x^*, \tilde{T}_x$
An element $F \in \mathcal{L}_B(E_{12})$ (of degree 1) is called an F_2 connection when $[\tilde{T}_x, F_2 \oplus F] \in \mathcal{K}_B(E_2 \oplus E_{12})$ (graded commutator) for all $x \in E_1$ This means in particular that

$$T_x F_2 - F T_{\gamma_1(x)} \in \mathcal{K}_B(E_2, E_{12})$$

$$F_2 T_x^* - T_{\gamma_1(x)}^* F \in \mathcal{K}_B(E_{12}, E_2)$$

F_2 connections always exist :write later

Definition of the kasparov product : The Kasparov product of $\mathcal{E}_1 = (E_1, \phi_1, F_1) \in \mathbb{E}(A,B)$ and $\mathcal{E}_2 = (E_2, \phi_2, F_2) \in \mathbb{E}(B,C)$ is $\mathcal{E}_{12} = (E_{12}, \phi \otimes_{\phi_2} \mathrm{id} = \tilde{\phi}, F) \in \mathbb{E}(A,C)$ for any F satisfying :

$$F \text{ is an } F_2-\text{connection}$$

$$\tilde{\phi}(a)((F_1 \otimes_{\phi_2} I)F + F(F_1 \otimes_{\phi_2} I))\tilde{\phi}(a)^* \geq 0 \bmod \mathcal{K}_C(E_{12})$$

Kasparov products do always exist ,first one finds an F_2 connection G and sets $F = M(F_1 \otimes_{\phi_2} I) + NG$ where M, N are positive weights that are obtained through kasparov technical theorem. Then one shows that any F' satisfying the above is homotopic to F by using the important lemma .
This then descends to a product $KK(A,B) \times KK(B,C) \to KK(A,C)$,we are not going to reproduce the entire procedure here as it wouldn't have been much enlightening to the reader who is refered to [skandalis,elements of kk theory] for the details.
we just review the basic ingredients.

Construction of the Kasparov product

Connection always exist

See [11]

a) For any $G \in \mathcal{L}(\mathcal{E})$ satisfying $[d, G] \in \mathcal{R}(\mathcal{E}')$ $(\forall d \in D)$, there exist G connections on $\mathcal{E}$. b) The space of G-connections is affine; the associated vector space is the space of 0 connections:

$$\left\{\Omega \in \mathcal{L}(\mathcal{E}''), \Omega(k\hat{\otimes}1) \in \mathcal{R}(\mathcal{E}'') \quad \text{and} \quad (k\hat{\otimes}1)\Omega \in \mathcal{R}(\mathcal{E}'') \quad \text{for all} \quad k \in \mathcal{R}(\mathcal{E})\right\}.$$

c) If $\tilde{G}$ is a G-connection and $k \in \mathcal{R}(\mathcal{E})$ then $[\tilde{G}, k\hat{\otimes}1] \in \mathcal{R}(\mathcal{E}'')$.

Proof. If $P \in \mathcal{L}_A(E_1)$ is a projection of degree zero and $F \in \mathcal{L}_B(E_{12})$ is an F_2-connection for E_1 then the operator $(P \otimes_f id) F (P \otimes_f id)$ is an F_2-connection for PE_1. To see this it suffices to observe that

$$\begin{aligned} (P \otimes_f id) T_x &= T_x, \\ (P \otimes_f id) T_{S_{E_1}(x)} &= T_{S_{E_1}(x)}, x \in PE_1, \end{aligned}$$

and that

$$\begin{aligned} (P \otimes_f id) \mathcal{K}_B(E_2, E_{12}) &\subseteq \mathcal{K}_B(E_2, PE_1 \otimes_f E_2), \\ \mathcal{K}_B(E_{12}, E_2)(P \otimes_f id) &\subseteq \mathcal{K}_B(PE_1 \otimes_f E_2, E_2). \end{aligned}$$

For example, for $x \in PE_1$, we find

$$\begin{aligned} & T_x F_2 - (P \otimes_f id) F (P \otimes_f id) T_{S_{S_1}(x)} \\ & = \\ & = T_x F_2 - (P \otimes_f id) F T_{S_{E_1}(x)} \\ & = (P \otimes_f id)\left(T_x F_2 - F T_{S_{E_1}(x)}\right) \in (P \otimes_f id) \mathcal{K}_B(E_2, E_{12}) \\ & \subseteq \mathcal{K}_B(E_2, PE_1 \otimes_f E_2). \end{aligned}$$

Therefore by the stabilization theorem it is enough to construct a G-connection on $\mathcal{E} = \mathcal{H}\hat{\otimes}_{\mathbf{C}}\tilde{D}$ ($\tilde{D}$ is obtained from D by adjoining a unit which acts as the identity in $\mathcal{E}'$. But then $\tilde{G} = 1_{\mathcal{H}}\hat{\otimes}_{\mathbf{C}}G \in \mathcal{L}\left(\mathcal{E}\hat{\otimes}_{\tilde{D}}\mathcal{E}'\right) = \mathcal{L}\left(\mathcal{H}\hat{\otimes}_{\mathbf{C}}\mathcal{E}'\right)$ is a G-connection (the set of $\xi \in \mathcal{E}$ such that conditions of Definition A. 1 are satisfied is a closed $\tilde{D}$ submodule of $\mathcal{E}$ as $[\tilde{D}, G] \in \mathcal{R}(\mathcal{E}')$, and contains $\mathcal{H}\hat{\otimes}_{\mathbf{C}}\mathbf{C}$). Call a connection of the form $\left(P\hat{\otimes}_D 1_{\mathcal{E}'}\right)\left(1_{\mathcal{H}}\hat{\otimes}_{\mathbf{C}}G\right)\left(P\hat{\otimes}_D 1_{\mathcal{E}'}\right)$ a Grassmann connection.
(b) is trivial
(c) It is enough to prove it for $k = \theta_{\xi,\eta}$. But $\theta_{\xi,\eta}\hat{\otimes}1 = T_\xi T_\eta^*$.

Kasparov techincal theorem

quasicentral approximate units

The following is going to be very useful. Let J be an ideal in a C*-algebra B. An approximate unit $\{u_\alpha\}$ for J is quasicentral for B if $\|bu_\alpha - u_\alpha b\| \to 0$, for every $b \in B$. As is the case with ordinary approximate units, if B is separable we may require our quasicentral approximate unit to be a sequence, while if B is non-separable we must settle for a net.

Existence of quasicentral units:

If J is an ideal in a separable C*-algebra B then there is an approximate unit for J which is quasicentral for B. In fact, if $\{u_n\}$ is any approximate unit for J then there is a quasicentral approximate unit $\{v_n\}$ for which each v_n is a finite convex combination of the elements $\{u_n, u_{n+1}, \ldots\}$.

Proof: [HR]
The following is proved by the use of the above device
Kasparov technical theorem: Let B be a graded σ-unital C^*-algebra, let E_1, E_2 be σ-unital subalgebras of $\mathcal{M}(B)$ and let $\mathcal{F}$ be a separable closed linear subspace of $\mathcal{M}(B)$. Assume that

1. $\underline{\beta}_B(E_i) = E_i, i = 1, 2$, and $\underline{\beta}_B(\mathcal{F}) = \mathcal{F}$,
2. $E_1 E_2 \subseteq B$
3. $[\mathcal{F}, E_1] \subseteq E_1$.

Then there exist elements $M, N \in \mathcal{M}(B)$ of degree 0 such that $N + M = 1, N, M \geq 0, ME_1 \subseteq B, NE_2 \subseteq B$ and $[N, \mathcal{F}] \subseteq B$
Proof:[elements of KK theory]

existence

Let $(\mathcal{E}, F) \in KK(A, D)$ and $(\mathcal{E}', F') \in KK(D, B)$. Put $\mathcal{E}'' = \mathcal{E}\hat{\otimes}_D\mathcal{E}'$. It is an A, B bimodule.
Theorem: There exists an F' connection F'' (of degree one) on $\mathcal{E}$ such that: (a) $(\mathcal{E}'', F'')$ is a Kasparov bimodule. (b) $\left[F'', F\hat{\otimes}1\right] = P + h$ where $P \geq 0$ and

$$h \in \Im = \{k \in \mathcal{L}(\mathcal{E}''), kA \subset \mathcal{R}(\mathcal{E}''), Ak \subseteq \mathcal{R}(\mathcal{E}'')\}.$$

Such a connection is unique up to operatorial homotopy; the class of $(\mathcal{E}'', F'')$ in $KK(A, B)$ is the Kasparov product$(\mathcal{E}, F) \otimes_D (\mathcal{E}', F')$.

Proof. Existence. Let G be an F' connection on $\mathcal{E}$. Let E_1 be the C^*-subalgebra of $\mathcal{L}(\mathcal{E}'')$ generated by $\mathcal{R}(\mathcal{E})\hat{\otimes}1$ and $\mathcal{R}(\mathcal{E}'')$. Let E_2 be the (separable) C^*-subalgebra of $\mathcal{L}(\mathcal{E}'')$ generated by $\left\{G^2 - 1, G - G^*, [G, F\hat{\otimes}1], [G, a](a \in A)\right\}$. Let $\mathcal{F}$ be the (separable) vector space spanned by $F\hat{\otimes}1, G$ and A. Finally put $E = \mathcal{R}(\mathcal{E}'')$. As all elements of E_2 are 0 -connections $E_1 \cdot E_2 \subseteq E$. Using A. 2 (c) we get $[\mathcal{F}, E_1] \subseteq E_1$. Apply kasparov technical theorem and get $M, N \in \mathcal{L}(\mathcal{E}''), M \geq 0, N \geq 0, M + N = 1$ such that $ME_1 \subseteq E, NE_2 \subseteq E, [M, \mathcal{F}] \subseteq E$. Put then $F'' = M^{1/2}F\hat{\otimes}1 + N^{1/2}G$. One gets easily that $(\mathcal{E}'', F'')$ is a Kasparov bimodule. As $M \cdot E_1 \subseteq E, M^{1/2}$ is a 0 -connection; as $[M, F\hat{\otimes}1] \in E, M^{1/2}(F\hat{\otimes}1)$ is also a 0 -connection. By A. 2 (b) $N^{1/2}$ is a 1-connection. Hence F'' is an F' connection. Finally $\left[F'', F\hat{\otimes}1\right] = M^{1/2}[F\hat{\otimes}1, F\hat{\otimes}1]$ modulo $\mathcal{R}(\mathcal{E}'')$. But as $2M^{1/2}\left(F^2\hat{\otimes}1\right) = 2M^{1/2}$ modulo $\Im$ we get the positivity condition.

uniqueness

Uniqueness. Let first G_0 and G_1 be two F' connections. Let E_2 be the C^*-subalgebra of $\mathcal{L}(\mathcal{E}'')$ generated by $\left\{G_0 - G_1, G_0^2 - 1, G_0 - G_0^*, \left[G_0, F\hat{\otimes}1\right], [G_0, a]\, a \in A\right\}$ and $\mathcal{F}$ the subspace spanned by $F\hat{\otimes}1, G_0, G_1, A$. Apply then kasparov technical theorem (with E_1 and E as defined above). Put $F_t'' = M^{1/2}(F\hat{\otimes}1) + N^{1/2}\left((1-t)G_0 + tG_1\right)$. It now remains to prove that if G is an F' connection satisfying (a) and (b), and if M, N are constructed as above, we can join G and $F'' = M^{1/2}(F\hat{\otimes}1) + N^{1/2}G$ by a norm continuous path of G-connections satisfying (a) and (b). Let $Q_t = (tM)^{1/2}(F\hat{\otimes}1) + (1 - tM)^{1/2}G$. Write $[F\hat{\otimes}1, G] = P + h$ with $P \geq 0$ and $h \in \Im$. Put $Z_t = 1 + t^{1/2}(1-t)^{1/2}P$, and $F_t'' = Q_t Z_t^{-1/2}$. One checks easily that $|Q_t|^2 - Z_t \in \Im$. Moreover $[A, Q_t] \subset \mathcal{R}(\mathcal{E}'')$; hence $[A, Z_t] \subset \mathcal{R}(\mathcal{E}'')$. Thus $[A, F_t''] \subset \mathcal{R}(\mathcal{E}'')$, and $|F_t''|^2 - 1 = Z_t^{-1/2}\left(|Q_t|^2 - Z_t\right)Z_t^{-1/2} \in \Im$. Also $[Q_t, Z_t] \in \Im$, so that $F_t'' - F_t''^* \in \Im$. We thus get that $(\mathcal{E}'', F_t'')$ is a Kasparov A, B bimodule. As Q_t is a G-connection and P is a 0 -connection, F_t'' is a G-connection. Finally $[F\hat{\otimes}1, P] \in \Im$ so that $\left[F\hat{\otimes}1, Z_t^{-1/2}\right] \in \Im$ and hence $\left[F\hat{\otimes}1, F_t''\right] = Z_t^{-1/4}\left(2(tM)^{1/2} + (1-t)^{1/2}P\right)Z_t^{-1/4} + h_t$ where $h_t \in \Im$. Thus F_t'' is the desired homotopy between $F_0'' = G$ and $F_1'' = F''$.

1.4.4 Properties

The basic properties of the product can be summarized in the following: 3.2. Abstract properties of $KK(A, B)$. Let A an B be two C^*-algebras. In order to simplify our presentation, we assume that A and B are separable. Here is the list of the most important properties of the KK functor. - $KK(A, B)$ is an abelian group. - Functorial properties The functor KK is covariant in B and contravariant in A : if $f : B \to C$ and $g : A \to D$ are two homomorphisms of C^*-algebras, there exist two homomorphisms of groups:

$$f_* : KK(A, B) \to KK(A, C) \text{ and } g^* : KK(D, B) \to KK(A, B).$$

In particular $id_* = id$ and $id^* = id$. - Each *-morphism $f : A \to B$ defines an element, denoted by $[f]$, in $KK(A, B)$. We set $1_A := [id_A] \in KK(A, A)$. - Homotopy invariance $KK(A, B)$ is homotopy invariant.

Recall that the C^*-algebras A and B are homotopic, if there exist two *-morphisms $f : A \to B$ and $g : B \to A$ such that $f \circ g$ is homotopic to id_B and $g \circ f$ is homotopic to id_A. Two homomorphisms $F, G : A \to B$ are homotopic when there exists a $*$-morphism $H : A \to C([0,1], B)$ such that $H(a)(0) = F(a)$ and $H(a)(1) = G(a)$ for any $a \in A$. - Stability If $\mathcal{K}$ is the algebra of compact operators on a Hilbert space, there are isomorphisms:

$$KK(A, B \otimes \mathcal{K}) \simeq KK(A \otimes \mathcal{K}, B) \simeq KK(A, B).$$

More generally, the bifunctor KK is invariant under Morita equivalence. - Suspension If E is a C^*-algebra there exists an homomorphism

$$\tau_E : KK(A, B) \to KK(A \otimes E, B \otimes E)$$

which satisfies $\tau_E \circ \tau_D = \tau_{E \otimes D}$ for any C^*-algebra D. - Kasparov product There is a well defined bilinear coupling:

$$\begin{array}{ccc} KK(A, D) \times KK(D, B) & \to & KK(A, B) \\ (x, y) & \mapsto & x \otimes y \end{array}$$

called the Kasparov product. It is associative, covariant in B and contravariant in A: if $f : C \to A$ and $g : B \to E$ are two homomorphisms of C^*-algebras then

$$f^*(x \otimes y) = f^*(x) \otimes y \text{ and } g_*(x \otimes y) = x \otimes g_*(y).$$

If $g : D \to C$ is another *-morphism, $x \in KK(A, D)$ and $z \in KK(C, B)$ then

$$h_*(x) \otimes z = x \otimes h^*(z).$$

Moreover, the following equalities hold:

$$f^*(x) = [f] \otimes x, g_*(z) = z \otimes [g] \text{ and } [f \circ h] = [h] \otimes [f].$$

In particular

$$x \otimes 1_D = 1_A \otimes x = x.$$

The usual K-theory groups appears as special cases of KK-groups:

$$KK(\mathbb{C}, B) \simeq K_0(B),$$

while the K-homology of a C^*-algebra A is defined by

$$K^0(A) = KK(A, \mathbb{C}).$$

Any $x \in KK(A, B)$ induces a homomorphism of groups:

$$\begin{array}{ccc} KK(\mathbb{C}, A) \simeq K_0(A) & \to & K_0(B) \simeq KK(\mathbb{C}, B) \\ \alpha & \mapsto & \alpha \otimes x \end{array}$$

In most situations, the induced homomorphism $KK(A, B) \to \mathrm{Mor}\,(K_0(A), K_0(B))$ is surjective. Thus one can think of KK-elements as homomorphisms between K groups.

Morita equivalence recall ,the notion of morita equivalence of C^* algebras we introduced earlier.A morita equivalence gives a KK-equivalence as follows .
If C^* algebras A, B are morita equivalent and ${}^AE_B$ is an imprimitivity bimodule then the kasparov module given by ${}_AE_B$ concentrated in degree 0. Together with the 0 operator gives the KK equivalence of A,B. Note that A acts as $\mathcal{K}_B(E)$ because $\overline{\langle E, E\rangle}_A = A$ and any $\langle x, y\rangle_A$ acts as $\langle x, y\rangle_A e = x\langle y, e\rangle_B$.
The product of ${}_AE_B$ and ${}_BE_A^{op}$ is easily seen to be the identity in $KK(A, A)$.

The KK element of a family of order 0 psueodifferential operators

Let X be a smooth manifold and Y a locally compact parameter space. In this section we shall first interpret the construction of continuous families, indexed by Y, of pseudo-differential operators on X, as yielding a map $\Psi^* : K(T^*X \times Y) \to KK(X, Y)$. The Kasparov product of two such families can be computed from a formula at the symbol level see [skandalis]

1.5 Groupoids ,algebroids, C^* algebras of groupoids

1.6 Lie groupoids

A Lie groupoid $G \rightrightarrows M$ is a smooth category (the morphisms ,as well as the objects manifolds).
Namely it consists of two manifolds G and M, called respectively the groupoid and the base, together with:

- two submersions $s, r : G \to M$ called respectively the source projection and target projection,
- a smooth immersion map 1: $x \mapsto 1_x, M \to G$ called the object (unit) inclusion map
- and a partial multiplication $(h, g) \mapsto hg$ in G defined on the set $G * G = \{(h, g) \in G \times G \mid s(h) = r(g)\}$
- A smooth (involution) inversion map $\gamma \to \gamma^{-1}$ where γ^{-1} is a two sided inverse :$\gamma\gamma^{-1} = 1_{r(\gamma)}$ and $\gamma^{-1}\gamma = 1_{s(\gamma)}$.

all subject to the following conditions:

1. $s(hg) = s(g)$ and $r(hg) = r(h)$ for all $(h, g) \in G * G$
2. $s(1_x) = r(1_x) = x$
3. $s(\gamma^{-1}) = r(\gamma)$ and $r(\gamma^{-1}) = s(\gamma)$

Also Denote:
The left-translation corresponding to g is $L_g : G^{s(g)} \to G^{r(g)}, h \mapsto gh$;
the right-translation corresponding to g is $R_g : G_{r(g)} \to G_{s(g)}, h \mapsto hg$.
Examples
Any manifold M may be regarded as a Lie groupoid on itself with every element a unity

1. $M \times M \rightrightarrows M$ with the obvious groupoid structure is called the pair groupoid.
2. Let $q : M \to Q$ be a surjective submersion. Then
$$R(q) = M \times \times_q M = \{(x, y) \in M \times M \mid q(x) = q(y)\}$$
is a Lie groupoid on M with respect to the restriction of the pair groupoid structure.
3. The groupoid of homotopy classes of paths between two points $\Pi(M) \rightrightarrows M$ is the fundamental groupoid of M.
4. Let (E, q, M) be a vector bundle. Let $\Phi(E)$ denote the set of all vector space isomorphisms $\xi : E_x \to E_y$ for $x, y \in M$. Then $\Phi(E)$ is a Lie groupoid on M .
5. Let $G \times M \to M$ be a smooth action of a Lie group G on a manifold M with $r(g \times m) = gm$ and $s(g \times m) = m$,multiplication $(g_1, g_2 m) \circ (g_2, m) = (g_1 g_2, m)$ and object inclusion $m \to (1, m)$.This is the action groupoid

Definition: A **bisection** of G is a smooth map $\sigma : M \to G$ which is right-inverse to $\alpha : G \to M$ and is such that $\beta \circ \sigma : M \to M$ is a diffeomorphism. The set of bisections of G is denoted $\mathscr{B}(G)$. A Local bisection is defined on an open set $U \subset M$ the set of which is denoted $\mathscr{B}_U(G)$.There is an obvious way of composing (local) bisections $\sigma' \star \sigma(x) = \sigma'(r(\sigma(x)))\sigma(x)$. Denote the left and right translations by bisections as L_σ, R_σ
Local bisections exist: For $\gamma \in G$ with $s(\gamma) = x$ there is $x \in U$ a bisection $\sigma : U \to G$ for which $\sigma(x) = \gamma$:

Proof:We just have to pick a subspace of dimension $\dim(M)$ of $T_\gamma G$ which is transverse to both $\ker ds$ and $\ker dt$.This subspace will be tangent to a submanifold of G for which s, r restrict to local diffeomorphisms around γ.
Corollary:Let $G \rightrightarrows M$ be a Lie groupoid. For each $x \in M$, $r_x : G_x \to M$ is of constant rank.

Proof Take $g, h \in G_x$. Then $j = gh^{-1}$ is defined and so there is a local bisection $\sigma \in \mathscr{B}_U G$ with $r(h) \in U$ and $\sigma((r(h)) = j$.

Now $L_\sigma : G_x^U \to G_x^V$, where $V = (r \circ \sigma)(U)$, maps h to g and $r_x \circ L_\sigma = (\beta \circ \sigma) \circ r_x$. Hence the ranks of r_x at g and h are equal.
From the constant rank theorem we get: For all $x, y \in M, G_x^y$ is a closed embedded submanifold of G_x, of G^y and of G. In particular, each vertex group G_x^x is a Lie group.
Also for each $x \in M$, the orbit $\mathscr{O}_x = r_x(G_x)$ is an immersed submanifold of M.Itis isomorphic to the quotient space of the (right) group action of G_x^x on G_x via $r_x : G_x / G_x^x \to \mathscr{O}_x$.
Composition of local bisections is well defined and has a groupoid structure.This also holds for 1-jets of bisections ,this groupoid $J^1 G$ will be useful later on.

Pullback groupoids

Consider (lie) groupoid $G \rightrightarrows M$ and any map $f : N \to M$. The pullback groupoid is defined as

$$f^{\Downarrow}(G) = N \times_{f,r} G \times_{s,f} N \rightrightarrows N$$

with the obvious groupoid structure:

- $r(x, g, y) = x$ and $s(x, g, y) = y$
- $(x, g, y) \circ (y, h, z) = (x, gh, z)$
- $1_x = (x, 1_f(x), x)$ and $(x, g, y)^{-1} = (y, g^{-1}, x)$

If f is transversal to G (see the next section) (which is the case when f is a surjective submersion) then $f^{\Downarrow}G$ is a lie groupoid.

1.6.1 Lie algebroids

Definition: A Lie algebroid on a manifold M is a vector bundle (A, q, M) together with a vector bundle map $a : A \to TM$ over M, called the anchor of A, and a bracket $[,] :, \Gamma A \times \Gamma A \to \Gamma A$ which is $\mathbb{R}$-bilinear and alternating, satisfies the Jacobi identity, and is such that

$$[X, uY] = u[X, Y] + a(X)(u)Y, \quad \text{(Leibniz rule)}$$
$$a([X, Y]) = [a(X), a(Y)],$$

A vertical vector field on G is one that is tangent to the s-fibers, namely it is s related to the 0 vector field on M.
Right invariant vector fields on G denoted by $\mathfrak{X}^R(G)$ are vector fields on G tangent to the s-fibers and R_g related ,this is meaningful since R_g is a diffeomorphism between s-fibers.
We explain ,a section X of A correspnds to a right invariant vector field on G via
The simplest examples of Lie algebroids are Lie algebras, Lie algebra bundles, and the tangent bundle to a manifold.
Another example :The trivial Lie algebroid with structure algebra $\mathfrak{g}$
On $TM \oplus (M \times \mathfrak{g})$ define an anchor $a = \pi_1 : TM \oplus (M \times \mathfrak{g}) \to TM$ and a bracket

$$[X \oplus V, Y \oplus W] = [X, Y] \oplus \{X(W) - Y(V) + [V, W]\}.$$

Then $TM \oplus (M \times \mathfrak{g})$ is a Lie algebroid on M.

The Lie algebroid of a Lie groupoid

The Lie algebroid of G is defined to be the pullback of the subbundle $\ker(ds)$ by the object inclusion map. $A = \cup_{x \in M} T_{1_x}(G_x)$ with the natural vector bundle structure over M which it inherits from TG. The Lie bracket is placed on the module of sections of A via the correspondence between sections of A and right-invariant vector fields on G.

$$\Gamma(A) \leftrightarrow \mathfrak{X}^R(G)$$

The anchor map $a : A \to TM$ is given by Tr restricted on the unit space M.

We explain:a vertical vector field on G is one that is tangent to the s-fibers, namely it is s related to the 0 vector field on M.
Right invariant vector fields on G denoted by $\mathfrak{X}^R(G)$ are vector fields on G tangent to the s-fibers and R_g related ,this is meaningful since R_g is a diffeomorphism between s-fibers.
From the definition we immediately conclude that $\mathfrak{X}^R(G)$ is a Lie subalgebra of all vector fields on G. A section X of A correspnds to a right invariant vector field X^R on G via $X^R(\gamma) = TR_\gamma(X(r(\gamma))$ and conversely a right invariant vector field restricted to M is a section of A.
We immediately get that X^R is r-related to $a(X)$ hence we get the second defining identity $a([X, Y]) = [a(X), a(Y)]$.Now it remains to prove the Leibniz identity:
For $u \in C^\infty(M) : uY \in \Gamma(A)$ corresponds to $(u \circ r)Y^R$.Therfore

$$[X, uY] \leftrightarrow [X^R, (u \circ r)Y^R] = X^R(u \circ r)Y^R + (u \circ r)[X^R, Y^R] =$$
$$= Tr(X^R)(u)Y^R + (u \circ r)[X^R, Y^R] \leftrightarrow a(X)(u)Y + [X, Y]$$

example: The lie algebroid of the pair groupoid is the tangent bundle lie algebroid.

The exponential map

For a $X\Gamma A$ consider the flow ϕ_t^X of the right invariant vector field X^R.
From the "relatedness" properties of X^R we immediately get that

- $s \circ \phi_t^X = s$ namely ϕ_t is a flow along the s fibers G_x.
- $R_g \circ \phi_t^X = \phi_t^X \circ R_g$ the flow is right invariant
- If ψ_t^X denotes the flow of $a(X)$ then $r \circ \phi_t^X = \psi_t^X$

Starting the flow from the unit space we can get local bisections as suggested by the following:
Let $W \subseteq M$ be an open subset, and take $X \in \Gamma_W A$. Then for each $x_0 \in W$ there is an open neighbourhood U of x_0 in W, called a flow neighbourhood for X, an $\varepsilon > 0$, and a unique smooth family of local bisections $\operatorname{Expt} X \in \mathscr{B}_U G, |t| < \varepsilon$, such that;

1. $\frac{d}{dt} \operatorname{Exp} tX\big|_0 = X$
2. $\operatorname{Exp} 0X = \operatorname{id} \in \Gamma_U G$
3. $\operatorname{Exp}(t+s)X = (\operatorname{Exp} tX) \star (\operatorname{Exp} sX)$, whenever $|t|, |s|, |t+s| < \varepsilon$
4. $\operatorname{Exp} -tX = (\operatorname{Exp} tX)^{-1}$
5. $\{r \circ \operatorname{Exp} tX : U \to U_t\}$ is a local 1-parameter group of transformations for $a(X) \in \Gamma_W TM$ in U

The exponential map $X \to \operatorname{Exp} X$ sends local sections of ΓA to local bisections.

pullback lie algebroids

Definition A smooth map $f : N \to M$ is transverse to a Lie groupoid $G \rightrightarrows M$ when for all $x \in N$: $Tf\,(T_x N) + a(A)_{f(x)} = T_{f(x)} M$.
Let A be a lie algebroid over M.The pullback Lie algebroid $f^! A$ of A over a smooth map $f : N \to M$ is described by the vector bundle $TN \times_{Tf, f^*(a)} f^*(A)$ over N. Namely the fiber over $y \in N$ is given by $Z \oplus X$ where $Z \in T_y N$ and $X \in A_{f(y)}$ such that $Tf(Z) = a(X)$.The anchor is given by $a^!(Z \oplus X) = Z$
This should be viewed as the pullback in the category of vector bundles over N of $Tf : TN \to f^*TM$ and $f^*(a) : f^*A \to f^*TM$.It fits in a pullback square:

$$\begin{array}{ccc} f^! A & \longrightarrow & A \\ \downarrow & & \downarrow a \\ TN & \xrightarrow{Tf} & TM \end{array}$$

$f^! A$ is of course a well defined vector bundle in case f is transverse to A (which is always the case if f is a surjective submersion). General sections of $f^! A$ are expressions of the form :

$$Z \oplus \left(\sum u_i \otimes X_i\right) \quad u_i \in C^\infty(N), \quad \text{such that} \quad T(f)\,(Z(y)) = \sum u_i(y) a(X_i(f(y))) \quad \forall y \in N$$

The bracket is defined by

$$\left[Z_1 \oplus \left(\sum u_i \otimes X_i\right), Z_2 \oplus \left(\sum v_j \otimes Y_j\right)\right] =$$
$$[Z_1, Z_2] \oplus \left(\sum u_i v_j \otimes [X_i, Y_j] + \sum Z_1\,(v_j) \otimes Y_j - \sum Z_2\,(u_i) \otimes X_i\right)$$

Example The pullback of the lie algebra $\mathfrak{g}$ over the map $M \to$ point is the trivial lie algebroid $M \times \mathfrak{g}$
Fact: The algebroid of a pullback groupoid (over a transverse map) is the pullback lie algebroid (of the groupoid).
The s-fiber of $f^{\Downarrow} G$ is identified with $N \times_{f,r} G_{f(y)}$ the above fact is then clear.We will be mainly interested in the pullback Lie algebroid as vector bundle or as the infinitesimal object of a Lie groupoid.

representations

Let A be a Lie algebroid on M and let E be a vector bundle, also on M. A representation of A on E is a morphism of Lie algebroids over M,

$$\rho : A \to \mathfrak{D}(E).$$

Example:Let E be a vector bundle on M, and let ∇ be a flat connection in E . Then $X \mapsto \nabla_X$ is a representation of TM on E.

lie algebroid cohomology

Lie algebroid cohomology $H^\bullet(A)$ is defined in direct analogy to de Rham cohomology is defined by the complex $(\Gamma(\wedge A^*), \mathrm{d}_A)$.where d_A is the chevalley eilenberg operator:
Explicitly, for $\omega \in \Gamma\left(\wedge^k A^*\right)$

$$\begin{aligned}\left(\mathrm{d}_A \omega\right)\left(X_1, \ldots, X_{k+1}\right) = & \sum_{i<j}(-1)^{i+j} \omega\left(\left[X_i, X_j\right], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right) \\ & + \sum_{i=1}^{k+1}(-1)^{i+1} a\left(X_i\right) \omega\left(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}\right)\end{aligned}$$

Of course $d_A^2 = 0$ in the case of de Rham cohomology for $X \in \Gamma(A)$, on $\Gamma(A^*)$ there are the insertion operator i_X (of degree -1) the Lie derivative $\mathcal{L}_X$ (of degree 0) defined on functions as $\mathcal{L}_X(f) = a(X)f$ and the exterior derivative d_A is of degree 1. These satisfy the usual identities.
$d_A \circ i_X + i_X \circ d_A = \mathcal{L}_X$, $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,X]}$, $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$

1.6.2 Lie groupoid cohomology

Lie groupoid cohomology is a generalization of alexander spanier cohomology.(See the section about connes moscovici index theorem) First we have to define $B_p\mathcal{G}$: the space of p-arrows:

$$B_p \mathcal{G} = \left\{\left(g_1, \ldots, g_p\right) \mid g_i \in \mathcal{G}, \mathrm{s}\left(g_i\right) = \mathrm{r}\left(g_{i+1}\right)\right\}.$$

This is a manifold.

There are $p+1$ face maps, where

$$\partial_i : B_p \mathcal{G} \rightarrow B_{p-1} \mathcal{G}, i = 0, \ldots, p$$

amounts to 'omitting the i-th base point'. Explicitly,

$$\partial_i\left(g_1, \ldots, g_p\right) = \begin{cases}\left(g_2, \ldots, g_p\right) & i = 0 \\ \left(g_1, \ldots, g_i g_{i+1}, \ldots, g_p\right) & 0 < i < p \\ \left(g_1, \ldots, g_{p-1}\right) & i = p.\end{cases}$$

For $p = 1$ we have $\partial_0(g) = \mathrm{s}(g), \quad \partial_1(g) = \mathrm{t}(g)$. There are also degeneracy maps $\epsilon_i : B_p\mathcal{G} \rightarrow B_{p+1}\mathcal{G}$, 'repeating the i-th base point' by inserting a trivial arrow. That is,

$$\epsilon_i\left(g_1, \ldots, g_p\right) = \left(g_1, \ldots, g_i, m_i, g_{i+1}, \ldots, g_p\right).$$

Definition : The complex $(\mathrm{C} \bullet (\mathcal{G}), \delta)$ of differentiable groupoid cochains is given by

$$C^p(\mathcal{G}) = C^{\infty}\left(B_p \mathcal{G}\right), \quad \delta = \sum_{i=0}^{p+1}(-1)^i \partial_i^*$$

it is not hard to check to verify that δ does indeed square to zero. Of interest to us is the localized version of these complexes, denoted $C_M^\bullet(\mathcal{G})$ whose cochains are the germs of functions on $B_p\mathcal{G}$ along the p-diagonal $M \subseteq B_p\mathcal{G}$.It's cohomology is the Lie groupoid cohomology.

The cohomology of $C_M^\bullet(M \times M)$ is by definition the alexander spanier cohomology of M.
As in the case of Alexander spanier cohomology there is an isomorphism from Lie groupoid cohomology to Lie algebroid cohomology:
There is a chain map

$$\Phi : C_M^\bullet(\mathcal{G}) \rightarrow \Gamma\left(\wedge^\bullet A^*\right)$$

Defined by

$$\Phi(\phi)(X_1, X_2, ..., X_k)_m = \sum_{\sigma \in S_k} (-1)^{\operatorname{sgn}(\sigma)} R_{X_{\sigma(1)}} R_{X_{\sigma(2)}} ... R_{X_{\sigma(k)}} \phi_(m)$$

Where for $X \in \Gamma(A)$ and $\phi \in C_M^k(\mathcal{G})$: $R_X\phi \in C_M^{k-1}(\mathcal{G})$

$$R_X(\phi)(g_2, ..., g_k) = \frac{d}{d\tau}\phi(\mathrm{Exp}(\tau\mathrm{X})(r(g_2)), g_2, g_3, ..., g_k)|_{\tau=0}$$

Fact :Φ is a quasi isomorphism ,see [Weinstein,Xu].

1.7 Groupoid C* algebras

The groupoid $C*$ algebra consists of functions on G together with multiplication (compostition) in an appropriate sense so let:

$\Omega^{1/2}$ be the complex line bundle of half densities $\Omega^{1/2}(\ker ds \oplus \ker dr)$ on the vector bundle $\ker ds) \oplus \ker dr$ over G.Notice that for fixed $\gamma \in G$:

$$\bigwedge^{top}((\ker ds \oplus \ker dr)_\gamma) = \bigwedge^{top} \ker ds_\gamma \otimes \bigwedge^{top} \ker dr_\gamma$$

Using this we get

$$\Omega^{1/2}(\ker ds \oplus \ker dr) \cong \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt)$$

The composition rule for sections a, b of this vector bundle is something of the form $a \circ b = \int_{G^{r\gamma}} a(\gamma_1)b(\gamma_1^{-1}\gamma)$ .One way to see this is Is if we consider the manifold $G^{(2)} = G \times_{s,r} G$ of composable pairs with projections $G^{(2)} \underset{\mathrm{pr}_1}{\overset{\mathrm{pr}_2}{\longrightarrow}} G$ and multiplication $G^{(2)} \xrightarrow{m} G$. The convolution product of $a, b \in \Gamma_c(G; \Omega^{1/2})$ is defined by integrating $\mathrm{pr}_1^* a \cdot \mathrm{pr}_2^* b$ over the fibers of m.

The involution $i : \gamma \to \gamma^{-1}$ interchanges the subbundles $\ker ds, \ker dt$ therefore acts naturally on $\Omega^{1/2}$. The involution $*$ on $\Gamma_c(G; \Omega^{1/2})$ acts as $f^*(\gamma) = \overline{i^*(f(\gamma^{-1}))}$

For $\gamma \in G$ the fiber $m^{-1}(\gamma)$ is parametrized by $\eta \in r^{-1}(r(\gamma)) \to (\eta, \eta^{-1}\gamma)$ (or by $\beta \in s^{-1}(s(\gamma)) \to (\gamma\beta^{-1}, \beta)$) this roughly gives the standard expression $a \circ b = \int_{r^{-1}(r(\gamma))} a(\eta)b(\eta^{-1}\gamma)$.

Note that the parametrizations above imply isomorphisms

$$\ker dm \cong \mathrm{pr}_1^*(\ker dr) \cong \mathrm{pr}_2^*(\ker ds)$$

Also it's easy to see that we have isomorphisms (through right and left compositions)

$$\mathrm{pr}_1^*(\ker ds) \cong m^*(\ker ds) \quad \text{and} \quad \mathrm{pr}_2^*(\ker dr) \cong m^*(\ker dr)$$

Therefore $\mathrm{pr}_1^* a, \mathrm{pr}_2^* b$ can be identified as sections of

$$\Omega^{1/2}(m^*(\ker ds)) \otimes \Omega^{1/2}(\ker dm) \quad \text{and} \quad \Omega^{1/2}(\ker dm) \otimes \Omega^{1/2}(m^*(\ker dt))$$

respectively and moreover $\mathrm{pr}_1^* a \cdot \mathrm{pr}_2^* b$ is a section of

$$\Omega^{1/2}(m^*(\ker ds)) \otimes \Omega^{1/2}(\ker dm) \otimes \Omega^{1/2}(\ker dm) \otimes \Omega^{1/2}(m^*(\ker dt)) \cong m^*(\Omega^{1/2}) \otimes \Omega^1(\ker dm)$$

Integrating over the fiber $m^{-1}(\gamma)$ the density part we get a well defined section of $\Omega_\gamma^{1/2}$. Explicit expression of these are as follows:

For $\tau_s \otimes \tau_r \in \bigwedge^{top} \ker ds_\gamma \otimes \bigwedge^{top} \ker dt_\gamma$,$\gamma \in G$ and denote $\gamma_2 = \gamma_1^{-1}\gamma$ for $\gamma_1 \in G^{r(\gamma)}$

$$a \circ b(\gamma)[\tau_s \otimes \tau_r] = \int_{G^{r(\gamma)}} a(\gamma_1)[R_{\gamma_2^{-1}*}(\tau_s) \otimes [\bullet]]b(\gamma_2)[(R_\gamma \circ i)_*[\bullet] \otimes L_{\gamma_1^{-1}*}(\tau_r)]$$

Note that the above is a density on the manifold $G^{r\gamma}$.Integration of the pullback density by the diffeomorphism $L_\gamma \circ i : G_{s(\gamma)} \to G^{r(\gamma)}$, $\gamma_2 \to \gamma\gamma_2^{-1} = \gamma_1$ yields that the above is equal to :

$$a \circ b(\gamma)[\tau_s \otimes \tau_r] = \int_{G_{s(\gamma)}} a(\gamma_1)[R_{\gamma_2^{-1}*}(\tau_s) \otimes (L_\gamma \circ i)_*[\bullet]]b(\gamma_2)[[\bullet] \otimes L_{\gamma_1^{-1}*}(\tau_r)]$$

The involution is given by :

$$a^*(\gamma)[\tau_s \otimes \tau_r] = a(\gamma^{-1})[i_*\tau_r \otimes i_*\tau_s]^*$$

(Transpose operator)

Here is a quick verification that this is actually a star algebra.

(associativity): Denote $\gamma_2 = \gamma_1^{-1}\gamma\gamma_3^{-1}$ for $\gamma_1 \in G^{r(\gamma)}$, $\gamma_3 \in G_{s(\gamma)}$ and $\gamma_{12} = \gamma_1\gamma_2$.

$$(a \circ b) \circ c(\gamma)[\tau_s \otimes \tau_r] = \int_{G_{s(\gamma)}} a \circ b(\gamma_{12})[R_{\gamma_3^{-1}*}(\tau_s) \otimes (L_\gamma \circ i)_*[\bullet]]c(\gamma_3)[[\bullet] \otimes L_{\gamma_{12}^{-1}*}(\tau_r)] =$$

$$\int_{G_{s(\gamma)}} \left(\int_{G^{r(\gamma)}} a(\gamma_1)[(R_{\gamma_2^{-1}}R_{\gamma_3^{-1}})_*(\tau_s) \otimes [\diamond]]b(\gamma_2)[(R_{\gamma_{12}} \circ i)_*[\diamond] \otimes (L_{\gamma_1^{-1}}L_\gamma \circ i)_*[\bullet]] \right) c(\gamma_3)[[\bullet] \otimes L_{\gamma_{12}^{-1}*}(\tau_r)]$$

$$a\circ(b\circ c)(\gamma)[\tau_s\otimes\tau_r]=\int_{G^{r(\gamma)}}a(\gamma_1)[R_{\gamma_{23}^{-1}*}(\tau_s)\otimes[\diamond]]b\circ c(\gamma_{23})[(R_\gamma\circ i)_*[\diamond]\otimes L_{\gamma_1^{-1}*}(\tau_r)]=$$

$$\int_{G^{r(\gamma)}}a(\gamma_1)[R_{\gamma_{23}^{-1}*}(\tau_s)\otimes[\diamond]]\left(\int_{G_{s(\gamma)}}b(\gamma_2)[(R_{\gamma_3^{-1}}R_\gamma\circ i)_*[\diamond]\otimes(L_{\gamma_{23}}\circ i)_*[\bullet]]c(\gamma_3)[[\bullet]\otimes(L_{\gamma_2^{-1}}L_{\gamma_1^{-1}})_*(\tau_r)]\right)$$

These are identical double integrals , now the star identity $(a\circ b)^*=b^*\circ a^*$ can be proved in the same manner and the pullback by the diffeomorphism $i:G^{r(\gamma}\to G_{s(\gamma^{-1})}$ shall be used. This algebra has a star representation π_x on each hilbert space $L_2(G_x)$ which has an obvious form. More precisely for each $x\in G_0$,$L_2(G_x)$ is by definition half densities on the manifold G_x namely sections of the vector bundle $\Omega^{1/2}(\ker ds)$ restricted there. For $\xi\in L_2(G_x),\gamma\in G_x$,$\tau_s\in\bigwedge^{top}\ker ds_\gamma$ it is given by:

$$\pi_x(a)\xi(\gamma)[\tau_s]=\int_{G^{r(\gamma)}}a(\gamma_1)[R_{\gamma_2^{-1}*}(\tau_s)\otimes[\bullet]]\xi(\gamma_2)[(R_\gamma\circ i)_*[\bullet]]$$

The fact that this is a star representation can be proved along the same lines as before.

Remark:haar systems

Usually one is given a G-invariant family of measures λ_x on G_x (this means that the pushforward of $\lambda_{r(\gamma)}$ on $G_{r(\gamma)}$ under the right multiplication by γ map $R_\gamma:G_{r(\gamma)}\to G_{s(\gamma)}$ is $\lambda_{s(\gamma)}$,one can do the same with left translation) The family of λ_x is then called a right (or left) haar system.
In case a haar system is given the C^* algebra $C^*(G)$ can be defined with functions. The product is going to be given by $a\circ b(\gamma)=\int_{G_{s(\gamma)}}a(\gamma\gamma_1^{-1})b(\gamma_1)d\lambda_{s(\gamma)}(\gamma_1)$ and involution by $f^*(\gamma)=\overline{f(\gamma^{-1})}$.
$C_c(G,\Omega^{1/2})$ completed with the norm $\|a\|=\max_{x\in G_0}\|\pi_x(a)\|$ is the reduced C^* algebra $C^*_{\text{red}}(G)$. This is the one we will mainly use ,we shall refer to it by just the groupoid C^* algebra $C^*(G)$.
There is also the full C^* algebra to define it first we introduce the L_1 norm

$$\|f\|_1=\sup(\max(\int_{G_x}|f(\gamma)|,\int_{G_x}|f(\gamma^{-1})|))$$

The inequalities $\|f\circ g\|_1\le\|f\|_1\|g\|_1$ and $\|f^*\|=\|f\|_1$ hold.
The full C^* algebra is the completion of $C_c(G,\Omega^{1/2})$ with

$$\|f\|_{\text{full}}=\sup_\pi\|\pi(f)\|$$

where π is a representation of G such that $\|\pi(f)\|\le\|f\|_1$.
Obviously for the representations π_x (by a minkowski inequality) $\|\pi_x(f)\|_{L_2}\le\|f\|_1$

Alternate version of the convolution algebra

$$\Gamma(G,r^*\bigwedge^{\text{top}}A^*)$$

Carries also a convolution product that is associative. This can be seen as follows :sections of this are basically top forms on the fibers G_x so we can integrate along fibers,explicitly the convolution can be given by:

$$a\circ b(\gamma)[\alpha_{r(\gamma)}]=\int_{G_{s(\gamma)}}a(\gamma h^{-1})[\alpha_{r(\gamma)}]b(h)[R_{\gamma^{-1}*}[-]]=\int_{G^{r(\gamma)}}a(h)[\alpha_{r(\gamma)}]b(h^{-1}\gamma)[R_{h*}\circ i_*[-]]$$

1.8 Morita equivalence

Morita equivalence of rings typical refers to the equivalence of categories of left modules $\text{mod}_A\cong\text{mod}_B$ over two rings. This equivalence can be shown to be implemented by two bimodules ${}_BP_A,{}_AQ_B$ such that $P\otimes_AQ\cong B$ and $Q\otimes_AP\cong A$ such that the equivalence is implemented by the functors $P\otimes_A(-)$ and $Q\otimes_B(-)$. Let's see how this applies to C^* algebras and it's geometric counterpart for groupoids , we are going to further see that a morita equivalence for Lie groupoids gives a morita equivalence of their C^* algebras.

1.8.1 Strong morita equivalence of C^* algebras

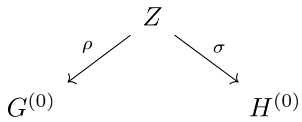
Definition: An (A, B) imprimitivity bimodule is a hilbert bimodule ${}_AE_B$ with A and B valued inner products $\langle , \rangle_A$ (antilinear in it's second argument) and $\langle , \rangle_B$ such that the following hold :

1. $\overline{\langle E, E\rangle_A} = A$ and $\overline{\langle E, E\rangle_B} = B$
2. $\langle a \cdot x, y\rangle_B = \langle x, a^* \cdot y\rangle_B$ and $\langle x \cdot b, y\rangle_A = \langle x, y \cdot b^*\rangle_A$
3. $\langle x, y\rangle_A \cdot z = x \cdot \langle y, z\rangle_B$

The typical example is the $\mathcal{K}(\mathcal{H}) - \mathbb{C}$ bimodule $\mathcal{H}$ where $\mathcal{H}$ is a hilbert space.

1.8.2 Morita equivalence of Lie groupoids

In a similar fashion to the algebraic morita equivalence it works as follows . A left (resp. right) representation space for a groupoid G is a space X over G^0 $X \xrightarrow{\sigma} G^0$ such that there is an action of G on X defined as :
$G \times_{r,\sigma} X \to X$ where $G \times_{s,\rho} X = \{(g, x)|s(g) = \rho(x)\}$ and it satisfies $\rho(g \cdot x) = r(g)$ so that it is an actions in the sense of groupoids.A right action is defined analogously.
Definition A (G, H) equivalence for lie groupoids is a space Z (a manifold) that is simultaneously a left G-space and a right H-space , there are surjective submersions $\rho : Z \to G^0$,$\sigma : Z \to H^0$



The actions are $G \times_{s,\rho} Z \to Z$ and $Z \times_{\sigma,r} H \to Z$ the actions commute and they induce bijections $Z/H \xrightarrow{\rho} G^0$ and $G\backslash Z \xrightarrow{\sigma} H^0$. Another definition is given by a Lie groupoid X and $G^0 \xrightarrow{i_1} X^0$,$H^0 \xrightarrow{i_2} X^0$ are transverse to the groupoids G, H and meet all orbits then G, H are isomorphic to the pullback groupoids by $i_1^*(X), i_2^*(X)$. This definition implies the first if we let Z be the following space $G^0 \times_{i_1,\sigma} X \times_{\rho,i_2} H^0$ with the obvious actions by G, H . The trivial example is the $(G, G_x^x)-$equivalence G_x , where G is a transitive lie groupoid such as the fundamental groupoid.

1.8.3 $G \sim_{\textbf{morita}} H \Rightarrow C^*(G) \sim_{\textbf{morita}} C^*(H)$

The morita equivalence of lie groupoids gives an equivalence of their C^* algebras as introduced later .
The imprimitivity bimodule is going to be extracted from the space Z :consider $C_c(Z)$ which is going to be completed later into a hilbert bimodule after we introduce to it the structure of a $C_c(G), C_c(H)$ (pre)-imprimitivity bimodule . Details why this works can be found in [31]
If $\phi \in C_c(Z), f \in C_c(G), g \in C_c(H)$ then:
a left $C_c(G)$ action is given by

$$f \cdot \phi(z) = \int_{G^{\rho(z)}} f(\gamma)\phi(\gamma^{-1}z)d\gamma$$

and a right $C_c(H)$-action is given by

$$\phi \cdot g(z) = \int_{H^{\sigma(z)}} \phi(z \cdot \eta)g(\eta^{-1})d\eta$$

The $C_c(H)$ valued inner product is (which is independent of z with $\sigma(z) = r(\eta)$ due to the translational invariance of the haar measure)

$$\langle \phi, \psi\rangle_{C_c(H)}(\eta) = \int_{G^{\rho(z)}} \overline{\phi(\gamma^{-1} \cdot z)}\psi(\gamma^{-1} \cdot z \cdot \eta)d\gamma$$

The $C_c(H)$ valued inner product is (which is independent of z with $\rho(z) = r(\gamma)$)

$$\langle \phi, \psi\rangle_{C_c(G)}(\eta) = \int_{G^{\sigma(z)}} \phi(\gamma^{-1} \cdot z \cdot \eta)\overline{\psi(z \cdot \eta)}d\eta$$

1.9 Representations

Hilbert bundles arise in a standard way when a commutative C^*-algebra $C_0(X)$ is represented non-degenerately on a Hilbert space. In that case there is a probability measure μ (called a basic measure) on X and a representation of the Hilbert space as a Hilbert bundle $L^2(X, \{H_u\}, \mu)$ with respect to which the elements of $C_0(X)$ act as diagonalizable operators. So the operator associated with $g \in C_0(X)$ on the original Hilbert space is now identified as an operator on the L^2-space of sections of the bundle, and given explicitly by $u \to g(u)I_u$ where I_u is the identity operator on H_u. (Think of the spectral theorem!)
A representation of the locally compact groupoid G is defined by a Hilbert bundle $\left(G^0, \{H_u\}, \mu\right)$ where μ is a measure on G^0 (that also gives a measure on G by fiber integration) and, for each $x \in G$, a unitary element $L(x) \in B\left(H_{d(x)}, H_{r(x)}\right)$ such that: (i) $L(u)$ is the identity map on H_u for all $u \in G^0$; (ii) $L(x)L(y) = L(xy)$ for ν^2-a.e. $(x, y) \in G^2$; (iii) $L(x)^{-1} = L\left(x^{-1}\right)$ for ν-a.e. $x \in G$; (iv) for any $\xi, \eta \in L^2\left(G^0, \{H_u\}, \mu\right)$, the function

$$x \to \langle L(x)\xi(d(x)), \eta(r(x))\rangle$$

is measurable on G.
As commented earlier, we are interested in linking up representations of G with representations of $C_c(G)$. $C_c(G)$ is a normed *-algebra under the I-norm, and that all representations of $C_c(G)$ (on a Hilbert space) considered in the book are assumed to be I-norm continuous. Since $C_c(G)$ is separable (trivial), every representation of $C_c(G)$ generates a separable C^*-algebra, and such C^*-algebras can always be realized on a separable Hilbert space. We can assume then that every representation of $C_c(G)$ under consideration is on a separable Hilbert space.

A representation L of the locally compact groupoid G "integrates up" to give
a representation $\pi_L : C_c(G) \to B\left(L^2(\mathcal{H})\right)$, where $\mathcal{H} = L^2\left(G^0, \{H_u\}, \mu\right)$, and where π_L is given by:

$$\langle \pi_L(f)\xi, \eta\rangle = \int_G f(x)\langle L(x)(\xi(d(x))), \eta(r(x))\rangle d\nu_0(x).$$

Desintagration theorem

: The above defines a representation π_L of $C_c(G)$ of norm ≤ 1 on $\mathcal{H} = L^2\left(G^0, \{H_u\}, \mu\right)$.
Let G be a lie groupoid. Then every representation of $C_c(G)$ is of the form π_L for some representation L of G, and the correspondence $L \to \pi_L$ preserves the natural equivalence relations on the representations of G and the representations of $C_c(G)$. The proof is very technical, the first step is to show that from a representation of $C_c(G)$ one can obtain a representation of the functions of the base space ,thus by a version of the spectral theorem one get's a field of hilbert spaces on the base space. The reader is refered to [36].

1.10 Tangent groupoid

Connes defines the tangent groupoid $\mathbb{T}M$ as the adiabatic groupoid of the pair groupoid (see previous section) explicitly the underlying space is:

$$TM \times \{0\} \cup M \times M \times (0, 1]$$

the groupoid structure is given by:

$$(x, X, 0) \circ (x, Y, 0) = (x, X + Y, 0)$$

on the part $TM \times \{0\}$ (denote this groupoid $\mathbf{T}M$)

$$(x, y, t) \circ (y, z, t) = (x, z, t)$$

on the part $M \times M \times (0, 1]$ (denote B_M this groupoid). The unit space is $M \times [0, 1]$ with the obvious range(r) ,source(s) and inclusion maps. The smooth structure is given in some equivalent ways, one requires that in coordinates charts U the map $TU \times [0, \epsilon) \to \mathbb{T}M$ given by

$$(z, Z, t) \to (z + tZ, z, t) \qquad t > 0$$
$$(z, Z, 0) \to (z, Z, 0)$$

The smooth structure outside of $t = 0$ is trivial, one checks that this indeed defines a smooth structure (that can also be defined invariantly using a metric and exponential maps).One can also do this by specifying the smooth functions (see smooth structure of deformation to normal cone). The tangent groupoid is the union of a closed and open subgroupoid $\mathbb{T}M = \mathbf{T}M \cup M \times M \times (0,1]$ and due to this decomposition one has an exact sequence of C^* algebras:

$$0 \to C^*(B_M) \xrightarrow{\iota_*} C^*(\mathbb{T}M) \xrightarrow{ev_{0*}} C^*(\mathbf{T}M) \to 0$$

ev_t denotes evaluation at t
The exactness of this sequence can be deduced from the general result about the deformation to normal cone for groupoids,an elementary proof can be found in the appendix.
It turns out that $C^*(B_M)$ is isomorphic to $\mathcal{K} \otimes C_0((0,1]) = C_0((0,1],\mathcal{K})$ where $\mathcal{K}$ is the C^* algebra of compact operators on $L_2(M)$ (a separable hilbert space) and is contractible : to be made clear shortly.

Whereas $C^*(\mathbf{T}M) \cong C_0(T^*M)$ the isomorphism via the fourier transform on tangent spaces :straightforward, recall that $C^*(\mathbf{T}M)$ is the completion of the compactly supported smooth functions on TM (roughly ,a metric gives us a haar system) when viewed as convolutional operators on the L_2 of the tangent spaces and the norm of a convolution operator on L_2 is equal to the maximum norm of it's fourier transform.

Now the map

$$K_0(C^*(\mathbb{T}M)) \xrightarrow{[ev_{0*}]} K_0(C^*(\mathbf{T}M))$$

in K theory is invertible , this follows from the fact that $C^*(B_M)$ is contractible and six term exact sequence. Connes then defines:
(note that $K_0(C^*(\mathbb{T}M)) \cong K_0(C_0(T^*M)) \cong K^0(T^*M)$ and $C^*(\mathbb{T}M_{t=1}) \cong C^*(M \times M) \cong \mathcal{K}$)
Analytic index:

$$\mathrm{Ind}_a = [ev_{1*}] \circ [ev_{0*}]^{-1} : K^0(T^*M) \to K_0(C^*(\mathbb{T}M_{t=1})) \cong K_0(\mathcal{K}) \cong \mathbb{Z}$$

And it turns out that this is exactly the analytic index map of atiyah and singer.

1.10.1 $C^*(\mathbb{T}M)$

Before we go any further let's inspect the groupoid $C^*(\mathbb{T}M)$. Specifically we inspect what happens in the part given by subgroupoid B_M. The underlying topological spaces of B_M and $M \times M \times (0,1]$ are the same but as riemannian manifolds are completely different as B_M blows up at zero.This is reflected in the groupoid structure.
$C^*(\mathbb{T}M)$ is given through a completion of $C_c(\mathbb{T}M, \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$ In B_M such sections are given by $\Gamma(M \times M \times (0,1], \Omega^{1/2} \otimes \Omega^{1/2})$ and the fiber over (x,y,t) is $\Omega_x^{1/2} \otimes \Omega_y^{1/2}$ and it is easy to see that the L_2 spaces they act on are $\Omega^{1/2}(M)$ through integration of a 1-density: $\Omega_x^{1/2} \otimes \Omega_y^{1/2} \otimes \Omega_y^{1/2} = \Omega_x^{1/2} \otimes \Omega_y^1$. Therefore such sections on $M \times M \times \{t\}$ can be viewed as integral operators on $L_2(M)$ and under any realization of $L_2(M)$ because the norms would be essentially the same.(For example we could take a measure on M giving a nonzero section of $\Omega^{1/2}(M)$ and then realize $L_2(M)$ through functions ,the norms would be equivalent) . Conversely any bounded operator on $L_2(M)$ that has a continuous kernel can be given by a section in $\Gamma(M \times M, \Omega^{1/2} \otimes \Omega^{1/2})$.

Furthermore we have a well defined $*-$representation of $\Gamma(M \times M \times \{t\}, \Omega^{1/2} \otimes \Omega^{1/2})$ on $L_2(M, \Omega^{1/2})$ (denote it $L_2(M)$ from now on) and the norm is defined as the operator norm. The norm on $\Gamma(M \times M \times (0,1], \Omega^{1/2} \otimes \Omega^{1/2})$ is defined to be the supremum over all t , in short we conclude that:

$$\Gamma(M \times M \times (0,1], \Omega^{1/2} \otimes \Omega^{1/2}) \quad \text{isometrically embedded in} \quad C((0,1],\mathcal{K})$$

Now $C^*(B_M)$ is given by the completion of sections with compact support $C_c(M \times M \times (0,1], \Omega^{1/2} \otimes \Omega^{1/2})$ and these are contained in the C^*- subalgebra $C_0((0,1],\mathcal{K})$ and it is easy to see that this is actually the completion therefore is equal to $C^*(B_M)$ as stated earlier. From now on denote $C((0,1],\mathcal{K}) = \mathcal{K}_\infty$ and $C_0((0,1],\mathcal{K}) = \mathcal{K}_0$

Now let's see what elements of $C^*(\mathbb{T}M)$ near $t = 0$ look like. One could take a metric on M so that the bundle $\Omega^{1/2} \otimes \Omega^{1/2}$ is trivialised (the bundle over $\mathbf{T}M$ is similar and is also trivialised with a metric) and then talk about elements of $C_c(\mathbb{T}M, \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$ as functions on $\mathbb{T}M$) but that is completely wrong.

Instead of using the coordinates (y,z,t) that apply only to $t > 0$ to see what sections look like locally we will use the coordinates (z,Z,t) around points $(z,Z,0) \in \mathcal{T}M$ given earlier. The transition between these systems of coordinates on $t > 0$ are given by:

$$(y,z,t) \to (z, \frac{y-z}{t}, t) \qquad (z,Z,t) \to (z + tZ, z, t)$$

A local section around a point in $t > 0$ in the first coordinate chart is given by $|dy|^{1/2} \otimes |dz|^{1/2}$.
Whereas a local section around $(z_0, Z_0, 0)$ in the second coordinate chart is given by $|dZ|^{1/2} \otimes |dZ|^{1/2} = |dZ|^1$
The explanation for the second $|dZ|^{1/2}$ is the bundle trivialization of $\ker dt : Z' \to (-tZ', Z', 0)$. Then it is easy to see that a general local section around $(z_0, Z_0, 0)$ is given by $k(z, Z, t)|dZ|^{1/2}$ and is represented in the first coordinate chart as:

$$k(z, Z, t) \leftrightarrow \frac{1}{t^n} k(z, \frac{y - z}{t}, t)|dy|^{1/2} \otimes |dz|^{1/2}$$

We conclude that a compactly supported section that is described in the two parts $U \times U \times (0, \epsilon)$ and $\mathbf{T}U$ for any local coordinate chart U as:

$$\frac{1}{t^n} K_U(y, z, t)|dy|^{1/2} \otimes |dz|^{1/2} \qquad \text{in} \qquad U \times U \times (0, \epsilon)$$

$$k_U(z, Z)|dZ|^1 \qquad \text{in} \qquad \mathbf{T}U$$

represents an element of $C_c(\mathbb{T}M, \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$ if :

$$(z, Z, t) \to K_U(z + tZ, z, t)$$

$$(z, Z, 0) \to k_U(z, Z)$$

is smooth. In other words that the composite function K_U, k_U is smooth on $\mathbb{T}U$ and then $C^*(\mathbb{T}M)$ is taken as the completion of these functions , note that $k(x, X)|dX|^1$ acts as a convolution operator on $L_2(T_xM)$ and it's norm is given by the max norm of the fourier transform.

1.10.2 Asymptotic pseudodifferential calculus

What we are about to do next is formulated correctly with the use of half densities instead of functions and sections of vector bundles tensored with half densities $\Gamma(\Omega^{1/2} \otimes E)$ instead of sections $\Gamma(E)$ of E itself. But for notational simplicity and to capture the ideas we are going to write down functions and pretend they are half densities. The plan here is to take a pseudodifferential operator $P = Op_p$ with symbol $p(x, \xi)$ on a manifold and get a family of operators P_t indexed by t whose symbols under some sense are $p_t(x, \xi) = p(x, t\xi)$. Write $P_t = Op^t_p = Op_{p_t}$ to indicate such a construction and p_t for the symbol of P_t .
Moreover for negative order psedodifferential operators we want to obtain an element of $C^*(\mathbb{T}M)$ that on $M \times M \times (0, 1]$ is given by P_t (in the sense of compact operators on $L_2(M)$ see earlier discussion) and on $\mathbf{T}M$ is given for each $x_0 \in M$ as the operator on $L_2(T_xM)$ whose symbol is the frozen symbol $p(x_0, \xi)$ on x_0.
The first step is to see how that works out on an open subset $U \subset \mathbb{R}^n$ when $p \in \mathcal{S}(T^*U)$ (schwarz class). So for $f \in L_2(U)$:

$$P_t f(x) = \int p(x, t\xi)e^{i\langle x,\xi\rangle} \widehat{f}(\xi)d\xi = \int p(x, t\xi)e^{i\langle x-y,\xi\rangle} f(y)d\xi dy \overset{\xi \to \xi/t}{=} \int \left(\frac{1}{t^n} p(x, \xi)e^{i\langle \frac{x-y}{t},\xi\rangle} d\xi \right) f(y)dy =$$

$$= \int \frac{1}{t^n} \widehat{p}(x, \frac{y - x}{t})f(y)dy = \int \frac{1}{t^n} K(x, y, t)f(y)dy$$

Whereas for g in $L_2(T_{x_0}U)$ we get :

$$P_{0x_0} g(X) = \int p(x_0, \xi)e^{i\langle X,\xi\rangle} \widehat{g}(\xi) = \int \left(p(x_0, \xi)e^{i\langle X-Y,\xi\rangle} d\xi\right) g(Y)dY =$$

$$\int \widehat{p}(x_0, Y - X)g(Y)dY = \int k(x_0, X - Y)g(Y)dY$$

From our previous discussion we see that the kernels

$$\frac{1}{t^n} K(x, y, t) = \frac{1}{t^n} \widehat{p}(x, \frac{y - x}{t})$$

$$k(x, X) = \widehat{p}(x, -X)$$

comprise an element of $C_0(\mathbb{T}U, \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$.We also need that the operators P_t, P_{0x} are uniformly bounded ,for the P_{0x} it's trivial whereas for the P_t we see that the $\frac{1}{t^n} K(x, y, t)$ have a bound of the form $\frac{(1/t)^n}{1+((1/t)|y-x|)^N}$ and this yields that they are uniformly L_1 with respect to x, y. Therefore we get a well defined map

$$\mathcal{S}(T^*U) \to C^*(\mathbb{T}U)$$

Still working on $\mathbb{R}^n$ we would like to consider the family of operators for $t > 0$ given by the symbol $p(x, t\xi)$ where $p \in \mathsf{Sym}^m_K$ compactly supported in K with respect to x acting on sobolev spaces as $Op_{p(x,t\xi)} : W_0^{s+m}(U) \to W_0^s(U)$ for some bounded open set $K \subset U$.(refer to $W_0^s(U)$ as just W^s) We prove the following very important properties:

Lemma 0: $Op_{p(x,t\xi)} : W^{s+m} \to W^s$ has uniformly bounded norm which is bounded by norms of p in Sym^m_K and is continous in the operator norm with respect to t.
Proof: same as the one without the deformation.

Lemma 1a: If $p \in \mathsf{Sym}^m_K$ and $q \in \mathsf{Sym}^k_K$ then

$$(p_t \circ q_t)(x, \xi) - p_t(x, \xi)q_t(x, \xi) = r^t(x, t\xi)$$

Where r^t is a uniformly bounded family of symbols in Sym^{m+k-1}_K tending to 0 so

$$Op^t_p \circ Op^t_q - Op^t_{pq} = K_t \quad : W^{s+m+k} \to W^s$$

Where K_t is a uniformly bounded compact operator tending to 0 in norm.
Proof: Same as the proof of the composition formula.

Lemma 1b: If $p \in \mathsf{Sym}^m_K$ a matrix valued symbol acting on trivial vector bundles $E = \mathbb{C}^{m_1} \to F = \mathbb{C}^{m_2}$ over U carrying some hermitian structure (not trivial) then

$$(p_t)^\dagger(x, \xi) - (p^*)_t(x, \xi) = r^t(x, t\xi)$$

Where r^t is a uniformly bounded family of symbols in Sym^{m-1}_K tending to 0 so

$$(Op^t_p)^* - Op^t_{p^*} = L_t \quad : W^{s+m}(F) \to W^s(E)$$

Where K_t is a uniformly bounded family of compact operators tending to 0.
Proof: Same as the proof of the adjoint formula.

Let's record these properties in case we are dealing with symbols in $C_c(T^*U)$ for a bounded open subset of $\mathbb{R}^n$ in that case we have

$Op_{p_t} : L_2(U) \to L_2(U)$ is uniformly bounded and compact for each t thus it defines an element of $\mathcal{K}_\infty$. ($\mathcal{K} = \mathcal{K}(L_2(U))$)

$Op^t_p \circ Op^t_q - Op^t_{pq}$ and $(Op^t_p)^* - Op^t_{\bar{p}}$ are compact operators tending to 0 as $t \to 0$ thus they are elements of the closed ideal $\mathcal{K}_0$.
So we get a well defined $*-$homomorphism

$$C_c(T^*U) \to \mathcal{K}_\infty/\mathcal{K}_0$$

The usual proof that $*-$homomorphisms have norm at most 1 works in this case so that we can also get a continous extension to $C_0(T^*U)$ that has also norm 1.
Lemma 2: $Op^t : C_c(T^*U) \to \mathcal{K}_\infty/\mathcal{K}_0$ has norm ≤ 1 .

Now suppose we are given a symbol (of order m) $p(x, \xi) : E_x \to F_x$ which is given exactly by a bundle morphism in $\pi^*E \to \pi^*F$ above T^*M .We want to define a family of uniformly bounded operators $(Op_p)^t : W^{s+m}(E) \to W^s(F)$ namely an element of $C((0,1), \mathcal{B}(W^{s+m}(E), W^s(F))$ and we want this to be uniquely defined modulo $C_0((0,1), \mathcal{K}(W^{s+m}(E), W^s(F))$,we call this equivalence ,for notational simplicity ignore the bundles.
Let's describe the standard procedure first we will be using for defining operators indexed by t. Fix a quadratic partition of unity (ψ_j) subordinate to coordinate patches cover (U_j, ϕ_j) . Use pushforward of symbols (If $\sigma \in C(T^*X)$ and $\phi : X \to Y$ is a diffeomorphism then $\phi_*\sigma \in C(T^*Y)$ and $\phi_*\sigma(\phi(x), \eta) = \sigma(x, \phi'(x)^T\eta)$) (also ϕ_j^* denotes pullback of half densities that we pretend are functions) Then the family of operators on $L_2(M)$ are given by

$$Op^t_p = \sum \phi_j^*(\psi_j \circ Op^t_{\phi_{j*}p} \circ \psi_j)(\phi_j^{-1})^* = \sum \psi_j \circ Op^t_p \circ \psi_j$$

then we already know that any two constructions will have the same principal symbol for every t and thus they differ by an element in $C((0,1),\mathcal{K}(W^{s+m},W^s))$ Moreover require that if we apply this to a symbol in $C_c(T^*M)$ (and by extent to $C^0(T^*M)$) the resulting family Op_p^t together with the element in $C^*(\mathbf{T}M)$ given by $p(x,\xi)$ comprise a well defined element of $C^*(\mathbb{T}M)$
observation : The distributional kernel of $p(x,t\xi)$ decays faster than any power of t outside the diagonal:
For $\phi,\psi \in C_c(U)$ with disjoint supports from lemma 1a we see $\phi\circ p_t - p_t\circ\phi = t(r^t)_t$ repeated application will give $(\mathrm{ad}\phi)^N p_t = t^N(r^t)_t$ for some $r^t \in \mathrm{Sym}^{\mathrm{m-N}}$ but also $(\mathrm{ad}\phi)^N p_t\psi = \phi^N p_t\psi = t^n K^t\psi$ for the kernel K^t of r^t.

Using the above observation and Lemma 1a it's not hard to see that the second requirement is going to be automatically satisfied cause $\psi_j\circ Op_p^t\circ\psi_j = \psi_j^2\circ Op_p^t + \psi_j(Op_p^t\circ\psi_j - \psi_j\circ Op_p^t)$ defines an element in $C^*(\mathbb{T}M)$ that is given by $\psi_j^2\circ Op_p^t$ on $C^*(\mathbf{T}M)$. Having verified that it is quite clear that we have a bounded linear map $C_c(T^*M)\to C^*(\mathbb{T}M)$ that can be extended to $C_0(T^*M)$ and is the identity when composed with ev_{0*}. Now it remains to verify that any choice of coordinate charts give equivalent operator families.We state the relevant result.
Lemma 1c: Let $\kappa : X\to Y$ be a diffeomorphism of open sets and $p(x,\xi)\in \mathsf{Sym}^m$ compactly supported in X then

$$Op^t_{\kappa * p} - (\kappa^{-1})^* Op_p^t \kappa^* \quad : W^{s+m}(Y)\to W^s(Y)$$

is a compact operator that tends to 0 as $t\to 0$.
Proof: The same procedure as in the case of coordinate invariance applies.
So if we have different partitions of unity and charts we can state that (using lemma 1a) :
$Op_p^t = \sum\psi_j\circ Op_p^t\circ\psi_j = \sum\psi_j\psi_i'\circ Op_p^t\circ\psi_j\psi_i' \cong \sum\psi_j^2\psi_i'\circ Op_p^t\circ\psi_i' = \sum\psi_i'\circ Op_p^t\circ\psi_i' = Op_p'^t$
It is also easy to see why lemmae 1a,1b carry over to manifolds from now on we are going to take these for granted as well as the invariance of the definition in $C((0,1),\mathcal{B}(W^{s+m}(E),W^s(F))/C_0((0,1),\mathcal{K}(W^{s+m}(E),W^s(F))$,note that the composition and adjoint of such classes are also well defined. Also note that when defining $p^\dagger$ we want it to define $Op_{p^\dagger} = (Op_p)^*$ through a fixed standard procedure. Note that if we are dealing with differential operators then we do not need any of these ,their deformation is invariantly defined and lemmae 1a, 1b are trivial for differential operators (in lemma 1a even if just p is a differential operator and that's all we need. So we could do what's next just for differential operator in which case things are much simpler.

1.10.3 Comparison of analytic indices

Now we are going to prove that the analytic index defined by connes is the same as the one defined by atiyah singer (fredholm index).The plan is simple we are going to use a standard device which gives projections out of arbitrary operators to obtain an element in $K_0(C^*(\mathbb{T}M))$ whose images under $[ev_0^*]$ and $[ev_1^*]$ are respectively the symbol class and the fredholm index of the operator.Thus showing the equivalence of the two index maps.
Consider an elliptic pseudodifferential operator of positive order m

$$Op_p = P : C^\infty(E)\to C^\infty(F)$$

acting between hermitian vector bundles with elliptic symbol $p(x,\xi)$ ($|p(x,\xi)e| > \delta|\xi|^m|e|$ for $|\xi|\geq 1$)
Consider the two families of operators acting on $L_2(E\oplus F)$ for $t>0$.

$$Q_t = \begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i\end{pmatrix}^{-1} \qquad \text{and} \qquad Op_q^t \quad \text{where the symbol} \quad q = \begin{pmatrix} -i & p^* \\ p & -i\end{pmatrix}^{-1}$$

The first is a family of resolvents of elliptic self adjoint operators whereas the second is an asymptotic pseudodifferential operator of order $-m$.
We are now going to show that $Q_t - Op_{q_t}\in\mathcal{K}_0$.

$$Op_q^t \in C((0,1),\mathcal{B}(L_2(E\oplus F),W^m(E\oplus F))/C_0((0,1),\mathcal{K}(L_2(E\oplus F),W^m(E\oplus F))$$

$$\begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i\end{pmatrix} \in C((0,1),\mathcal{B}(W^m(E\oplus F),L_2(E\oplus F))/C_0((0,1),\mathcal{K}(W^m(E\oplus F),L_2(E\oplus F))$$

Also note that due to lemma 1b

$$\begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i\end{pmatrix} \cong \begin{pmatrix} -i & (Op_{p^*}^t) \\ Op_p^t & -i\end{pmatrix} = Op_{q^{-1}}^t$$

Therefore

$$\begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i\end{pmatrix}\circ Op_q^t \cong Op_{q^{-1}}^t\circ Op_q^t \cong 1$$

So we have that:

$$\begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i \end{pmatrix} \circ Op_q^t = I + K_t$$

For a family K_t of compact operators tending to 0.But then $I + K_t$ has an inverse $I + K'_t$ for $K'_t \in \mathcal{K}_0$ and $t \to 0$ Therefore we have managed to recover the resolvent for $t \to 0$: $Q_t = Op_q^t \circ (I + K'_t)$ the result follows immediately.
Now since $Q_t - Op_{q_t} \in \mathcal{K}_0$ it defines an element of $C^*(\mathbb{T}M, E \oplus F)$ therefore we get a well defined element $Q_t = Op_q^t + (Q_t - Op_q^t)$ of $C^*(\mathbb{T}M, E \oplus F)$ that is given by Q_t on $t > 0$ and given by the symbol q in $C^*(\mathbf{T}M)$, that's all we needed.

Consider the element $U_t = \begin{pmatrix} i & (Op_p^t)^* \\ Op_p^t & i \end{pmatrix} \circ \begin{pmatrix} -i & (Op_p^t)^* \\ Op_p^t & -i \end{pmatrix}^{-1} = I + 2iQ_t$ of $\tilde{C}^*(\mathbb{T}M, E \oplus F)$ take also the bundle automorphism $\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and view it also as $\varepsilon : C^\infty(E \oplus F) \to C^\infty(E \oplus F)$.Because ε anticommutes with $D_t = \begin{pmatrix} 0 & (Op_p^t)^* \\ Op_p^t & 0 \end{pmatrix}$. We get that εU_t is self adjoint and also $(\varepsilon U_t)^2 = 1$.
Now add orthogonally vector bundles to E and F to make them trivial: $E \oplus E' = \mathbb{C}^{N_1}$ and $F \oplus F' = \mathbb{C}^{N_2}$. Extend U_t to act on $C^\infty((E \oplus E') \oplus (F \oplus F'))$ by acting as the identity on $E' \oplus F'$. Also extend ε to be the automorphism $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ on $\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}$. We still have that εU_t is self adjoint and also $(\varepsilon U_t)^2 = 1$.
Now we see that

$$\frac{1}{2}(\varepsilon U_t + I) \quad \text{defines a projection in} \quad \tilde{C}^*(\mathbb{T}M, E \oplus E' \oplus F \oplus F') = \tilde{C}^*(\mathbb{T}M, \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}) = \mathcal{M}_N(\tilde{C}^*(\mathbb{T}M))$$

Which is given by the symbol $\frac{1}{2}(\varepsilon u + I)$ on $\tilde{C}^*(\mathbf{T}M, \mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2}) = \mathcal{M}_N(\tilde{C}^*(\mathbf{T}M)) = \mathcal{M}_N(\tilde{C}_0(T^*M))$
Where u is the (matrix) symbol that acts as $\begin{pmatrix} i & p^* \\ p & i \end{pmatrix} \circ \begin{pmatrix} -i & p^* \\ p & -i \end{pmatrix}^{-1} = I + 2iq$ on $\pi^*E \oplus \pi^*F$ and as the identity on $\pi^*E' \oplus \pi^*F'$
So we get a well defined element $[\frac{1}{2}(\varepsilon U_t + I)] \in K_0(\tilde{C}^*(\mathbb{T}M))$ and finally a well defined element :

$$b = [\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon U_t + I)] \in K_0(C^*(\mathbb{T}M))$$

Fredholm index

The image of b under ev_{1*} is $[\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon U + I)] \in K_0(\mathcal{K})$. Consider the action of D on $C^\infty(E \oplus F)$, we know from the spectral theory of selfadjoint elliptic operators that D has a discrete spectrum in $\mathbb{R}$ and the eigenfunctions form an orthonormal basis of $L_2(E \oplus F)$
Consider the homotopy of operators : $T \in [1, +\infty)$

$$U^T = (TD + i) \circ (TD - i)^{-1} \ : L_2(E \oplus F) \to L_2(E \oplus F)$$

U^T is $-I$ on $\ker D$ and converges in norm to the identity on $(\ker D)^\perp$ thus U^T converges to $(I - P_{\ker D}) - P_{\ker D} = I - 2P_{\ker D}$. The extension (by the identity on $E' \oplus F'$) of U^T on $C^\infty(\mathbb{C}^{N_1} \oplus \mathbb{C}^{N_2})$ converges in norm to the operator $I - 2P_{\ker D}$ where $P_{\ker D}$ is extended by zero to $E' \oplus F'$. Thus we get

$$[\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon U + I)] = [\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon(I - 2P_{\ker D}) + I)] = [\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon + I) - \varepsilon P_{\ker D}]$$

and this is mapped under the trace isomorphism $\mathrm{Tr} : K_0(\mathcal{K}) \to \mathbb{Z}$ to $\mathrm{Tr}(\varepsilon P_{\ker D}) = \dim(\ker P) - \dim(\ker P^*)$

Symbol class

The image of b under ev_{0*} is $[\frac{1}{2}(\varepsilon + I)] - [\frac{1}{2}(\varepsilon u + I)] \in K_0(C_0(T^*M))$ as a difference of elements in $K_0(\tilde{C}_0(T^*M))$. Of course $\tilde{C}_0(T^*M) = C(T^*M^+)$ and the element in $K^0(T^*M)$ that we are going to get is by definition the difference of the image bundles of the projection valued functions $\frac{1}{2}(\varepsilon + I), \frac{1}{2}(\varepsilon u + I)$ on T^*M^+ which are of course equal at ∞. For simplicity of notation denote by E the pullback bundle π^*E over T^*M. E cannot be seen as a vector bundle on T^*M^+ because it cannot be trivialised near infinity whereas $E \oplus E' \cong \mathbb{C}^{N_1}$ can and also $[E \oplus E']$ is the image bundle of $\frac{1}{2}(\varepsilon + I)$.
It remains to describe the image bundle of $\frac{1}{2}(\varepsilon u + I)$.
Consider the action of $\frac{1}{2}(\varepsilon u + I)$ on $E \oplus F$ .On T^*M right compose it with $\begin{pmatrix} -i & p^* \\ p & -i \end{pmatrix}$ it gives

$$\frac{1}{2}(\varepsilon u + I) \begin{pmatrix} -i & p^* \\ p & -i \end{pmatrix} = \frac{1}{2} \left(\begin{pmatrix} i & p^* \\ -p & -i \end{pmatrix} + \begin{pmatrix} -i & p^* \\ p & -i \end{pmatrix} \right) = \begin{pmatrix} 0 & p^* \\ 0 & -i \end{pmatrix}$$

For $|\xi| \le 1$ we can see that the image bundle is isomorphic to F however the above is not good for describing the image bundle near ∞. For $|\xi| \ge 1$ further right compose with $(p^*)^{-1}$ to get

$$\begin{pmatrix} 1 & 0 \\ -i(p^*)^{-1} & 0 \end{pmatrix}$$

So there the image bundle is isomorphic to E. In short we conclude (after considering the action on $E' \oplus F'$) the following :The image bundle of $\frac{1}{2}(\varepsilon u + I)$ over T^*M^+ is isomorphic to $[L_E \oplus E']$ where :
$L_E \oplus E'$ is $F \oplus E'$ for $|\xi| \le 1$ and $E \oplus E'$ for $|\xi| \ge 1$ and thus (can be trivialized and be a well defined bundle near ∞) with clutching function on the unit sphere bundle $S(T^*M)$ given by: $cp^*; F \oplus E' \to E \oplus E'$. (the value of c doesn't matter).
So we get that the symbol class in $K^0(T^*M)$ is exactly $[E \oplus E'] - [L_E \oplus E']$.
One last thing :similarly consider the vector bundle $L_F \oplus F'$ over T^*M^+ which is $E \oplus F'$ for $|\xi| \le 1$ and $F \oplus F'$ for $|\xi| \ge 1$ with clutching function on $S(T^*M)$ given by : $cp : E \oplus F' \to F \oplus F'$.
We then have that $[E \oplus E'] - [L_E \oplus E'] = [L_F \oplus F'] - [F \oplus F']$:
$[L_E \oplus E' \oplus L_F \oplus F']$ is the bundle that is given by $E \oplus E' \oplus F \oplus F'$ both for $|\xi| \le 1$ and $|\xi| \ge 1$ but has clutching function on $S(T^*M)$ given by

$$\begin{pmatrix} 0 & p^* \\ -p & 0 \end{pmatrix} : E \oplus E' \oplus F \oplus F' \to E \oplus E' \oplus F \oplus F'$$

(It is skew symmetric) which is homotopic to the identity clutching function through the homotopy of clutching functions :

$$\begin{pmatrix} 1-t & tp^* \\ -tp & 1-t \end{pmatrix} \qquad \text{(invertible)}$$

Thus $[L_E \oplus E' \oplus L_F \oplus F'] = [E \oplus E' \oplus F \oplus F']$.
$[L_F \oplus F'] - [F \oplus F']$ is exactly the symbol class of p as defined by Atiyah and Singer
(see Lawson and Michelson Spin Geometry).

1.11 DNC

1.11.1 Deformation to normal cone

Recall the deformation to the normal cone construction for a submanifold $V \subset M$ with normal bundle N_V and a diffeomorphism $\theta : N_V \to U$ where U is a tubular neighborhood of V in M

$$DNC(M,V) = M \times R^* \cup N_V \times 0$$

it's smooth structure is given by the requirement that $\Theta : (x, \xi, t) \to DNC(M,V)$
$(\theta(x, t\xi), t)$ for $t \neq 0$
$(x, \xi, 0)$ for $t = 0$
is a diffeomorphism of $N_V \times R$ on it's image. The manifold with boundary $DNC_+$is taken to be the restriction to positive t $M \times R_+^* \cup N_V \times 0$. In the case of groupoids observe that $\ker ds$ restricted on the unit submanifold can be identified with it's normal bundle . The adiabatic groupoid is $DNC_+(G, G_0)$ and is explicitly given by

$$\mathbb{T}G = (A \times \{0\}) \cup G \times \mathbb{R}^* \rightrightarrows M \times \mathbb{R}$$

It is a groupoid with composition rule:
$(\gamma_1, t) \circ (\gamma_2, t) = (\gamma_1\gamma_2, t)$
$(x, \xi, 0) \circ (x, \eta, 0) = (x, \xi + \eta, 0)$
We get a short exact sequence of C^* algebras of groupoids

$$0 \to C_0(\mathbb{R}^*) \otimes C^*(G) \to C^*(\mathbb{T}G) \to C^*(A) \to 0$$

where the above maps are roughly inclusion and evaluation .For the details and proof of exactness of this sequence refer the reader to [androulidakis,skandalis]. If we choose a metric on A the C^* algebra of this groupoid becomes functions with convolution on fibers which through fourier transform (recall the norm) becomes functions on the dual lie algebroid (with max norm). The $K0$ theory of this C^* algebra is where symbols live. Due to the fact that the first term of the above exart sequence is a contractible C^* algebra and six term exact sequence the K theory map $[ev0] : K_0(C^*(G_{ad})) \to K_0(C^*(A)) = K_0(C_0(A^*))$ is invertible. The index map is defined as

$$[ev_1] \circ [ev_0]^{-1} : K_0(C_0(A^*)) \to K_0(C^*(G))$$

1.12 Proof of atiyah singer

This section follows [13]

1.12.1 Groupoids used in the proof

In this section we are going to give the proof of the atiyah singer index theory using just the deformation to normal cone construction. This proof uses just geometrical constructions and gives the description of the topological index using these and then puts both the analytical and topological index in a commmutative diagram.One encounters such proofs in algebraic topology, the hard part is to prove that the classical topological index coincides with the one given here. This fact uses KK theory which is a more powerful device to deal with K theory. Let $N \to M$ be a vector bundle (later to be a the normal bundle) .Let

$$^*p^*(TM) = N \times_M TM \times_M N \rightrightarrows N$$

be the pullback groupoid of TM along the submersion $N \to M$. (See morita equivalence) It is morita equivalent to TM and the space giving this equivalence (following the observation) is $p^*(N)$ (pullback vector bundle) with the obvious actions.It's Lie algebroid is trivially isomorphic to TN as the pullback lie algebroid of $N \to M$
The thom groupoid TG_N is by definition the deformation groupoid of $^*p^*(TM)$ therefore as a space

$$TG_N = TN \times \{0\} \cup {}^*p^*(TM) \times (0,1]$$

and has the given smooth structure
We also need the tangent thom groupoid

$$\mathbb{T}G_N = \mathbb{T}N \times \{0\} \cup {}^*(p \times \mathrm{id}_{[0,1]})^*(\mathbb{T}M) \times (0,1]$$

This is not a deformation groupoid though a functorial description of it's smooth structure can be given.(see later) As a space it's comprised of the following parts

- (n_1, n_2, t, s) for t and $n_1, n_2 \in N$
- $(n_1, n_2, X_m, 0, s)$ for $s > 0$ and n_1, n_2 belonging to the fiber over m
- $(Y_n, 0, 0)$ at the corner $t, s = 0$ and Y_n a tangent vector at $n \in N$

Note that on $t > 0$ this space doesn't blow up when $s \to 0$. Locally in coordinates it's smooth structure near a point in $t = s = 0$ is given by the requirement that the map $(\mathbb{R}^n, \mathbb{R}^k) \times \mathbb{R}^k \times \mathbb{R}^n \times [0,1) \times [0,1) \to \mathbb{T}G_N$

- $(n, n^v, X, t, s) \to (n + tX + (s+t)n^v, n, t, s)$ when $t > 0$
- $(n, n^v, X, 0, s) \to (n + sn^v, n, 0, s)$ when $t = 0$
- $(n, n^v, X, 0, 0) \to (Y_n = X + n^v, 0, 0)$ when $t = s = 0$

1.12.2 Topological index

The topological index can be derived in this setting in the same manner that the analytic was using deformation groupoids. Recall that TM is morita equivalent to $^*p^*(TM)$ so are the corresponding C^* algebras and so we get an isomorphism in K theory $K_0(C^*(TM)) \cong K_0(C^*(^*p^*(TM))$. Composing with the map given by the deformation groupoid TG_N we get an element in $K_0(C*(TN))$ and the analytical index map applied to this gives a number which turns out to be the topological index. The proof of this fact uses KK-theory. So we get the topological index map through the following sequence:

$$K_0(C^*(TM)) \longleftarrow K_0(C^*(^*p^*(TM))) \longleftarrow K_0(C^*(TG_N)) \longrightarrow K_0(C^*(TN)) \xrightarrow{ind_a} \mathbb{Z}$$

The topological index map and the analytic index map fit into a diagram that is commutative and are therefore equal:

$$\begin{array}{ccccccc}
K_0(C^*(M \times M)) & \longleftarrow & K_0(C^*(N \times N)) & \xleftarrow{s=1} & K_0(C([0,1], C^*(N \times N))) & \xrightarrow{s=0} & K_0(C^*(N \times N)) \\
\uparrow & & \uparrow{\scriptstyle t=1} & & \uparrow & & \uparrow \\
K_0(C^*(\mathbb{T}M)) & \longleftarrow & K_0(C^*(^*(p \times \mathrm{id}_{[0,1]})^*(\mathbb{T}M))) & \longleftarrow & K_0(C^*(\mathbb{T}G_N)) & \longrightarrow & K_0(C^*(\mathbb{T}N)) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle ev_{t=0}} & & \downarrow \\
K_0(C^*(TM)) & \longleftarrow & K_0(C^*(^*p^*(TM))) & \longleftarrow & K_0(C^*(TG_N)) & \longrightarrow & K_0(C^*(TN))
\end{array}$$

The leftmost part of this diagram is by explicit morita equivalences described earlier and the commutativity follows immediately (K_0 can be viewed as projective modules in this case.)The rest of the diagram commutes trivially.

1.13 Equality with atiyah singer topological index

Here we are going to use the following groupoid which is closely related to a thom groupoid : Let $N \to X$ be a vector bundle. Think about applying the deformation to normal cone construction to a fiber N_x. N_x is a vector space so it's tangent bundle can be naturally identified with $N_x \oplus N_x$. Think of it as the fiber of the complexified bundle $E = N \times \mathbb{C} \cong N \oplus N$ .Therefore the deformation to normal cone has total space $E_x \times [0,1] = N_x \oplus N_x \times [0,1] \rightrightarrows N_x \times [0,1]$,the second summand N_x represents tangent vectors and the smooth structure is defined by the map $(v, V, t) \to (v + tV, v, t)$ for $t > 0$ and $(v, V, 0) \to (v, V, 0)$ being smooth. Now think of applying the deformation to normal cone construction simultaneously on every fiber one gets the groupoid I_N which can be thought of as the deformation of the groupoid $N \times_X N$ and it's total space is $E \times [0,1] \rightrightarrows N \times [0,1]$
Recall the section on the atiyah singer topological index , one has an embedding of the manifold M on an euclidean space and N is the normal bundle. Denote $q : TM \to M$ then $TN \to TM$ is isomorphic to $q^*N \oplus q^*N \to TM$ which is naturally a complex vector bundle. Recall that the atiyah singer topological index arises from the thom isomorphism $K^0(TM) \to K^0(TN)$ we will return to this later.

1.13.1 The thom element

For a complex hermitian vector bundle $p : E \to X$ which is a complexification of a real metric vector bundle $E = N \oplus N$ (so we can write local sections as $N + iN$ the second factor representing the imaginary part) we have the thom isomorphism $K_0(C(X), C(E))$ this can be implemented by the following KK theory element referred to as the thom element

$$\mathcal{T} = (C_0(E, p^*(\bigwedge E)), \rho, C) \in KK(C_0(X), C_0(E))$$

Where C is represented by the endomorphism field given by clifford multiplication multiplied by a normalizing constant:

$$C\omega(e_x) = \frac{1}{1 + ||e_x||^2}(e_x \wedge \omega(e_x) - \iota_{e_x}\omega(e_x))$$

Note that C is self adjoint ,and $C^2 \to I$ in the limit to infinity to the fibers.Moreover it is odd with respect to the grading $C_0(E, p^*(\bigwedge E)) = C_0(E, p^*(\bigwedge^{\text{even}} E)) \oplus C_0(E, p^*(\bigwedge^{\text{odd}} E))$
Let's see why multiplication with this element implements the classical thom isomorphism in K theory.
If we use K theory with compact supports then an element $[\xi] \in K^0(X)$ is represented by $[\xi_0, \xi_1; a]$ where a is an isomorphism outside a compact set and approximately unitary (give ξ_0, ξ_1 hermitian structures).
For a vector bundle $E \to X$ The thom isomorphism $i_! : K^0(X) \to K^0(E)$ sends $[\xi]$ to the element represented by the complex $\mathrm{Tot}(p^*\xi \otimes \Lambda E)$ where ΛE denotes the complex ... $\bigwedge^i E \xrightarrow{e\wedge} \bigwedge^{i+1} E \to ....$ This is in turn can be represented by the two term complex

$$\left[\xi_0 \otimes \Lambda_0 \oplus \xi_1 \otimes \Lambda_1; \xi_0 \otimes \Lambda_1 \oplus \xi_1 \otimes \Lambda_0; \theta = \begin{pmatrix} N(1 \otimes C) & M(\alpha^* \otimes 1) \\ M(\alpha \otimes 1) & -N(1 \otimes C) \end{pmatrix}\right]$$

where M and N are the multiplication operators by the functions $M(v) = \frac{1}{\|v\|^2+1}$ and $N = 1 - M$, respectively.
Proposition: Under the isomorphism $K^0(X) \simeq KK(\mathbb{C}, C_0(X))$ which is such that to the triple $[\xi_0; \xi_1; \alpha]$ there corresponds to the Kasparov module:

$$x = (C_0(X, \xi), 1, \widetilde{\alpha}), \quad \xi = \xi_0 \oplus \xi_1 \quad \text{and } \widetilde{\alpha} = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix}.$$

We have that the thom isomorphism is the same as the kasparov product with $\mathcal{T}$
Proof. $i_!(x)$ corresponds to $(\mathcal{E}, \widetilde{\theta})$ where:

$$\mathcal{E} = C_0(X, \xi) \underset{\rho}{\otimes} C_0(E, p^*(\Lambda E)) \simeq C_0(E, p^*(\xi \otimes \Lambda E)) \text{ and } \tilde{\theta} = \begin{pmatrix} 0 & \theta^* \\ \theta & 0 \end{pmatrix} \in \mathcal{L}(\mathcal{E}).$$

It will be useful to rewrite $\tilde{\theta}$ as

$$\widetilde{\theta} = M(\widetilde{\alpha}\hat{\otimes}1) + N(1\hat{\otimes}C)$$

Where we are using $\hat{\otimes}$ is the graded tensor product (see previous section).
Now we have to check that this is indeed the kasparov product $[\xi] \otimes \mathcal{T}$.
We will be using the following (the proof is trivial) .
Let X be a locally compact ,paracompact space and $V \to X$ a hermitian vector bundle , On the hilbert $C_0(X)$ module $C_0(X,V)$, $C_0(X)$ acts (on the left) as compact operators.
All of the following are justified by this.

- $M(\widetilde{\alpha}\hat{\otimes}1)$ is a 0 -connection on $\mathcal{E}$
  Let s be a section in $C_0(X,\xi)$ then:

$$M(\widetilde{\alpha}\hat{\otimes}1) \circ T_s = MT_{\tilde{a}(s)} \in \mathcal{K}(C_0(E,p^*(\bigwedge E)), C_0(E,p^*(\xi \otimes \bigwedge E)))$$
$$T_s \circ M(\widetilde{\alpha}\hat{\otimes}1) = MT^*_{\tilde{a}(s)} \in \mathcal{K}(C_0(E,p^*(\xi \otimes \bigwedge E)), C_0(E,p^*(\bigwedge E)))$$

  Note that T^*_s acts in an obvious way.

- $N(1\hat{\otimes}C)$ is a C-connection on $\mathcal{E}$

$$T_sC - N(1\hat{\otimes}C)T_{\gamma(s)} = T_sC - NT_sC = MT_sC \in \mathcal{K}(C_0(E,p^*(\bigwedge E)), C_0(E,p^*(\xi \otimes \bigwedge E)))$$
$$CT^*_s - T^*_{\gamma(s)}N(1\hat{\otimes}C) = MCT^*_s \in \mathcal{K}(C_0(E,p^*(\xi \otimes \bigwedge E)), C_0(E,p^*(\bigwedge E)))$$

This yields that $\widetilde{\theta} = M(\widetilde{\alpha}\hat{\otimes}1) + N(1\hat{\otimes}C)$ is a C-connection on $\mathcal{E}$. Secondly we have that

$$[\widetilde{\alpha}\hat{\otimes}1, \widetilde{\theta}] = 2M\widetilde{\alpha}^2\hat{\otimes}1 \geq 0 \quad \text{in} \quad \mathcal{L}(\mathcal{E})$$

(the above is a graded commutator) which proves that $(\mathcal{E}, \widetilde{\theta})$ represents the Kasparov product of x and T.

1.13.2 The inverse thom element

The inverse thom element is constructed using the standard procedure for the deformation groupoid I_N . One gets for the evaluation homomorphisms $e_0 : C^*(I_N) \to C^*(I_N|_{t=0}) = C^*(E_N)$ where E_N is the vector bundle groupoid $E \to N$ (projection to the first factor of $E = N \oplus N$)
and $e_1 : C^*(I_N) \to C_*(I_N|_{t=1}) = C^*(N \times_X N)$ the element $[e_0]^{-1} \otimes [e_1] \in KK(C^*(E_N), C^*(N \times_X N))$ $N \times_X N$ is morita equivalent to the trivial groupoid X (every point is a unit) as the corresponding pullback groupoid over $N \to X$. The morita equivalence KK element is given by $\mathcal{M} = (C(L_2(N)), k, 0)$ where $C(L_2(N))$ is the $C_0(X)$-hilbert module of sections of the continuous field of hilbert spaces $L_2(N_x)$ and $C^*(N \times_X N)$ acts as a continuous field of compact operators . (Note that we have the family of lebesgue measures on N_x that gives a haar system for $N \times_X N$.) Composing with this morita equivalence element we get

$$\mathcal{T}_{inv} = [e_0]^{-1} \otimes [e_1] \otimes \mathcal{M} \in KK(C^*(E_N), C_0(X))$$

Also note that the fourier transform on the second factor of E (using the metric structure) gives an isomorphism of $C^*(E_N)$ and $C_0(E)$.Use this to reinterpret $\mathcal{T}$ as an element in $KK(C_0(X), C^*(E_N))$.

1.13.3 $\mathcal{T}$ and $\mathcal{T}_{inv}$ are inverse to each other

In viewing $\mathcal{T}$ as a $C^*(E_N)$ hilbert module through an isomorphism with $C_0(E)$ we might as well apply the fourier transform with respect to the second factor of E to $C_0(E, p^*(\bigwedge E))$ and let $C^*(E_N)$ act by convolution on the second factor. ρ will still represent the obvious multiplication by functions and C is going to become :

$$C\omega(v+iw) = \int_{N_x \times N_x} e^{i\langle w-w',\xi\rangle} C(v+i\xi)\omega(v+iw')dw'd\xi$$

It suffices to show that $\mathcal{T} \otimes \mathcal{T}_{\text{inv}} = 1 \in KK(C_0(X), C_0(X))$ since we already know that thom isomorphism is an isomorphism.
The first step is to compute $\tilde{T} = \mathcal{T} \otimes [e_0]^{-1} \in KK(C_0(X), C^*(I_N))$ which is equivalent to $\tilde{T} \otimes [e_0] = \mathcal{T}$. Since the underlying hilbert module of $\mathcal{T}$ is given by the sections of E_N in the vector bundle $p^*(\bigwedge E)$ it makes perfect sense to consider $\tilde{T}$ as $(C^*(I_N, p^*(\bigwedge E)), ..., ...)$ where $p : I_N \to X$ is the obvious projection and $C^*(I_N, p^*(\bigwedge E))$ arises as a $C^*(I_N)-$hilbert module completion. The representation of $C_0(X)$ is going to be given by the obvious multiplication by functions. And to get a consistent endomorphism $\widetilde{C}$ with it's part at $t = 0$ one puts

$$\tilde{C}\omega(v+iw,t) = \int_{N_x \times N_x} e^{i\langle \frac{v-v'}{t},\xi\rangle} C(v+i\xi)\omega(v'+iw,t)\frac{dv'}{t^n}d\xi$$

for $t > 0$ and C for $t = 0$. Then it is easy to see that $\tilde{T} \otimes [e_0] = e_{0*}(\tilde{T}) = \mathcal{T}$. Then $\tilde{T} \otimes [e_1] = e_{1*}(\tilde{T})$ is represented by $(C^*(N \times_X N, p^*(\bigwedge E)), \rho_1, C_1)$ where ρ_1 is multiplication by functions and C_1 is given by the above formula for $t = 1$. Finally we compose with the morita equivalence.
Through

$$\omega(\cdot,\cdot) \otimes f \to \int_{N_x} \omega(v_x, \xi_x) f(\xi_x) d\xi$$

$$C^*(N \times_X N, p^*(\bigwedge E)) \otimes_{C^*(N \times_X N)} C(L_2(N))$$

is identified with $C(L_2(N, p^*(\bigwedge E)))$ (continous field of hilbert spaces) . So it turns out that $\tilde{T} \otimes [e_1] \otimes \mathcal{M} \in KK(C_0(X), C_0(X))$ is represented by the hilbert module $C(L_2(N, p^*(\bigwedge E)))$ (or a completed version of this) together with the endomorphism F_1 given by

$$F_1 \omega(v_x) = \int_{N_x \times N_x} e^{i\langle (v-v'),\xi\rangle} C(v + i\xi)\omega(v') dv' d\xi$$

Both $C_0(X)$ act as multiplication.From the section on KK theory such an element can be equivalently viewed as an element of $KK(\mathbb{C}, C(M)) = K(M)$ as the forgetful map $KK(C(M).C(M)) \to KK(\mathbb{C}, C(M)) \cong K(M)$ composed with the inclusion $K(M) \to KK(C(M).C(M))$ give the identity on elements represented by a family of endomorphisms on a hilbert bundle.This then is a family of pseudodifferential operators on the fibers $N_x \sim \mathbb{R}^n$ whose symbols are given by the bott elements λ_{N_x} . One expects that the family index is the identity.Which is indeed the case.It is proved through passing to $O(n)$ equivariant K theory (as is done in the section about thom isomorphism and) proving that the $O(n)$ equivariant index of this zero order operator over $\mathbb{R}^n$ is 1. In hormander this analytic index is represented by the complex $C^\infty(\mathbb{R}^n, \bigwedge^k \mathbb{C}^n) \xrightarrow{x\wedge + d} C^\infty(\mathbb{R}^n, \bigwedge^{k+1} \mathbb{C}^n)$ in [hormander] it is shown through a homotopy of hypoeliptic operator that the $O(n)$ equivariant index is 1 in $K_{O(n)}(T\mathbb{R}^n)$

1.13.4 Finishing the proof

We apply the above to the complex vector bundle $q^*N \oplus q^*N \to TM$. The groupoid I_{q*N} used before is not exactly isomorphic to the thom groupoid TG_N used in the before.But their C^* algebras are if one applies fourier transform on the fibers T_xM(with respect to a metric) .Note that the conolution product in $C^*(TG_N)$ is simultaneous convolution in the tangent space T_xM and convolution in the pair groupoid $N_x \times N_x$ (or convolution with respect with the vertical tangent space of TN at $t = 0$) so the above isomorphism should be clear. This isomorphism is apparently compatible with evaluations at $t = 0, 1$.
So we proved that the bottom row of the above diagram defines the thom isomorphism (if we identify $C^*(TM), C^*(TN)$ with $C_0(T^*M), C_0(T^*N)$).
Now it remains to show that the rightmost column defines the composition of the canonical inclusion (N is identified with an open subset of $\mathbb{R}^n$) $K_0(C^*(TN)) \to K_0(C^*(T\mathbb{R}^n))$ followed by bott periodicity $K_0(C^*(T\mathbb{R}^n)) = K_0(C_0(\mathbb{R}^{2n})) \cong \mathbb{Z}$.
This follows from the obvious commutativity of the diagram :

$$\begin{array}{ccc}
\mathbb{Z} \cong K_0(C^*(N \times N)) & \longrightarrow & K_0(C^*(\mathbb{R}^n \times \mathbb{R}^n)) \cong \mathbb{Z} \\
\uparrow & & \uparrow \\
K_0(C^*(\mathbb{T}N)) & \longrightarrow & K_0(C^*(\mathbb{T}\mathbb{R}^n)) \\
\downarrow & & \downarrow \\
K_0(C^*(TN)) & \longrightarrow & K_0(C^*(T\mathbb{R}^n))
\end{array}$$

and the fact that the analytic index map for $T^*\mathbb{R}^n$ is the same as bott periodicity . Or alternatively one can see that this is just the story developed in this section for a point imbedded in $\mathbb{R}^n$.

Chapter 2

2.1 Hochschild homology

Let A be a k algebra and M an A bimodule.
Consider the module $C_n(A,M) := M \otimes A^{\otimes n}$ (where $\otimes = \otimes_k$ and $A^{\otimes n} = A \otimes \ldots \otimes A, n$ factors). The Hochschild boundary is the k-linear map $b : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by the formula

$$b\,(m, a_1, \ldots, a_n) := (ma_1, a_2, \ldots, a_n) + \sum_{i=1} (-1)^i\,(m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) + (-1)^n\,(a_n m, a_1, \ldots, a_{n-1})$$

$d_i : M \otimes A^{\otimes n} \to M \otimes A^{\otimes n-1}$ given by

$$\begin{aligned} d_0\,(m, a_1, \ldots, a_n) &:= (ma_1, a_2, \ldots, a_n) \\ d_i\,(m, a_1, \ldots, a_n) &:= (m, a_1, \ldots, a_i a_{i+1}, \ldots, a_n) \quad \text{for} \quad 1 \le i < n \\ d_n\,(m, a_1, \ldots, a_n) &:= (a_n m, a_1, \ldots, a_{n-1}) \end{aligned}$$

With this notation one has

$$b = \sum_{i=0}^{n} (-1)^i d_i$$

1.1.2 Lemma. $b \circ b = 0$.
Proof. It is immediate to check that

$$d_i d_j = d_{j-1} d_i \quad \text{for} \quad 0 \le i < j \le n$$

Hochschild homology is by definition the the homology of the complex: $C_*(A,M)$.

$$\ldots \to M \otimes A^{\otimes n} \xrightarrow{b} M \otimes A^{\otimes n-1} \xrightarrow{b} \ldots \xrightarrow{b} M \otimes A \xrightarrow{b} M$$

For example $H_0(A,M) = M_A = M/\{am - ma \mid a \in A, m \in M\}$.
In the case where $M = A$ we denote $H_*(A,A)$ by $HH_*(A)$ the Hochschild complex $C_*(A)$ which is sometimes called cyclic bar complex.

Hochschild homology is functorial both in the module M and in the algebra A.
For example the group

$$H_0(A,M) = M_A = M/\{am - ma \mid a \in A, m \in M\}$$

Also one finds that $HH_0(A) = HH_0\left(\mathcal{M}_r(A)\right) = A/[A,A]$.

Bar resolution

Let $A^{\mathrm{e}} = A \otimes A^{\mathrm{op}}$ be the enveloping algebra of the associative and unital algebra A. The left A^{e}-module structure of A is given by $(a \otimes a')\,c = aca'$.
Consider the following complex, called the bar complex

$$C_*^{\mathrm{bar}} : \quad \ldots \to A^{\otimes n+1} \xrightarrow{b'} A^{\otimes n} \xrightarrow{b'} \ldots \xrightarrow{b'} A^{\otimes 2}$$

where $A^{\otimes 2}$ is in degree 0 and where $b' = \sum_{i=0}^{n-1}(-1)^i d_i$ (note that the sum is only up to $n-1$) The map $b' = \mu$: $A \otimes A \to A$ is an augmentation for the bar complex.
If A is a unital k-algebra. The complex C_*^{bar} is a projective (free in most cases) resolution of the A^{e}-module A with $s : A^{\otimes n} \to A^{\otimes n+1}, s(a_1, \ldots, a_n) = (1, a_1, \ldots, a_n)$, ,a contracting homotopy.
It is called the "bar resolution" of A.

Tor interpretation

It is easy to see that the bar resolution tensored with M over A^e is the hochschild complex,therefore

$$H_n(A, M) = \mathrm{Tor}_n^{A^e}(M, A).$$

normalized hochschild complex

If A is unital denote $\overline{A} = A/k \cdot 1$. Hochschild homology can be equivalently given by the complex $\overline{C}_*(A, M)$ where $\overline{C}_n(A, M) = M \otimes \overline{A}^{\otimes n}$. This is the quotient of hochschild complex by the subcomplex D_* of degenerate elements.Elements for which $a_i = 1$ for at least one i. It is easy to see that b vanishes on these elements therefore we get a well defined boundary map on the quotient subcomplex.
To prove that using the normalized complex we find hochschild homology, it suffices to see that:
The complex D_* is acyclic which in turn shows that the projection map $C_*(A, M) \to \bar{C}_*(A, M)$ is a quasi-isomorphism of complexes.
Consider $s_p : M \otimes A^{\otimes(n-1)} \to M \otimes A^{\otimes n}$, $s_p : m \otimes a_1 \otimes ... \otimes a_{n-1} \to m \otimes a_1 \otimes ... \otimes a_{p-1} \otimes 1 \otimes ... \otimes a_n$. Take the filtration $F_p D_*$ of the complex D_* where $F_p D_n$ is the submodule generated by the images of $s_1, s_2, ..., s_p$.
For $p \geq n$:$F_p D_n = D_n$ and for $p \leq 0$: $F_p D_n = 0$ therefore the filtration is bounded and the spectral sequence associated to the filtration converges to the homology of D_*.
It suffices to show that the homology of $G_{p,q} = F_p D_{p+q}/F_{p-1} D_{p+q}$ vanishes (the spectral sequence collapses at the E^1 page it turns out that s_p defines a contracting homotopy of this complex.

2.1.1 Morita invariance

Let M be a bimodule over the k-algebra A and let $\mathcal{M}_r(M)$ be the module of $r \times r$ matrices with coefficients in M. Bordering by zeroes

$$\alpha \mapsto \begin{bmatrix} & & & 0 \\ & \alpha & & \cdot \\ & & & \cdot \\ 0 & \cdot & 0 & 0 \end{bmatrix}$$

defines an inclusion inc: $\mathcal{M}_r(M) \to \mathcal{M}_{r+1}(M)$.
The (ordinary) trace map $\mathrm{tr} : \mathcal{M}_r(M) \to M$ is given by

$$\mathrm{tr}(\alpha) = \sum_{i=1}^{r} \alpha_{ii}$$

It is clear that tr is compatible with inc and defines $\mathrm{tr} : \mathcal{M}(M) \to M$. 1.2.1 Definition. The generalized trace map (or simply trace map)

$$\mathrm{tr} : \mathcal{M}_r(M) \otimes \mathcal{M}_r(A)^{\otimes n} \to M \otimes A^{\otimes n}$$

is given by

$$\mathrm{tr}(\alpha \otimes \beta \otimes \ldots \otimes \eta) = \sum \alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes \ldots \otimes \eta_{i_n i_0},$$

where the sum is extended over all possible sets of indices $(i_0, \ldots, i_n)$.
For $u_i \in \mathcal{M}_r(k), a_0 \in M$ and $a_i \in A$ for $i \geq 1$. The generalized trace map takes the form

$$\mathrm{tr}(u_0 a_0 \otimes \ldots \otimes u_n a_n) = \mathrm{tr}(u_0 \ldots u_n) a_0 \otimes \ldots \otimes a_n$$

The generalized trace map is a morphism of complexes from $C_*(\mathcal{M}_r(A), \mathcal{M}_r(M))$ to $C_*(A, M)$.
(Morita Invariance for Matrices): Let A be a unital k-algebra. Then for any $r \geq 1$ (including $r = \infty$) the maps

$$\mathrm{tr}_* : H_*(\mathcal{M}_r(A), \mathcal{M}_r(M)) \to H_*(A, M)$$

and

$$\mathrm{inc}_* : H_*(A, M) \to H_*\left(\mathcal{M}_r(A), \mathcal{M}_r(M)\right)$$

are isomorphisms and inverse to each other.

Proof. It is immediate that $\mathrm{tr} \circ \mathrm{inc} = id$, therefore it suffices to prove that $\mathrm{inc} \circ \mathrm{tr}$ is homotopic to id. In fact there is a homotopy $h = \sum(-1)^i h_i$ constructed as follows. For $i = (0, \ldots, n)$ let $h_i : \mathcal{M}_r(M) \otimes \mathcal{M}_r(A)^{\otimes n} \to \mathcal{M}_r(M) \otimes \mathcal{M}_r(A)^{\otimes n+1}$ be defined by the formula

$$\begin{aligned} h_i\left(\alpha^0, \ldots, \alpha^n\right) = & \sum E_{j1}\left(\alpha^0_{jk}\right) \otimes E_{11}\left(\alpha^1_{km}\right) \otimes \ldots \\ & \ldots \otimes E_{11}\left(\alpha^i_{pq}\right) \otimes E_{1q}(1) \otimes \alpha^{i+1} \otimes \alpha^{i+2} \otimes \ldots \otimes \alpha^n, \end{aligned}$$

where the sum is extended over all possible sets of indices $(j, k, m, \ldots, p, q)$. In this formula α^0 is in $\mathcal{M}_r(M)$ and the others α^s are in $\mathcal{M}_r(A)$;

More generally we have that: If R and S are Morita equivalent k-algebras and M is an R-bimodule, then there is a natural isomorphism

$$H_*(R, M) \cong H_*\left(S, Q \otimes_R M \otimes_R P\right)$$

inner derivation

Inner derivations act as zero on hochschild homology, inner derivations act on $C_n(A, M)$ as.

$$ad(u)\left(a_0, \ldots, a_n\right) = \sum_{0 \leq i \leq n}\left(a_0, \ldots, a_{i-1}, \left[u, a_i\right], a_{i+1}, \ldots, a_n\right).$$

It is easily checked that $ad(u)$ commutes with the Hochschild boundary. 1.3.3 Proposition. Let $h(u) : C_n(A, M) \to C_{n+1}(A, M)$ be the map of degree 1 defined by

$$h(u)\left(a_0, \ldots, a_n\right) := \sum_{0 \leq i \leq n}(-1)^i\left(a_0, \ldots, a_i, u, a_{i+1}, \ldots, a_n\right).$$

Then the following equality holds:

$$bh(u) + h(u)b = -ad(u).$$

Consequently $\mathrm{ad}\ (u)_* : H_n(A, M) \to H_n(A, M)$ is the zero map.

2.1.2 Hochschild cohomology

Hochschild cohomology of A with coefficients in M as

$$H^n(A, M) = H_n\left(\mathrm{Hom}_{A^e}\left(C^{\mathrm{bar}}_*(A), M\right)\right).$$

$$H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M)$$

The coboundary map β' in the Hom-complex is given by

$$\beta'(\phi) = -(-1)^n \phi \circ b'$$

for any cochain ϕ in $\mathrm{Hom}_{A^e}\left(C^{\mathrm{bar}}_n(A), M\right)$. Explicitly, such a cochain ϕ is completely determined by a k-linear map $f : A^{\otimes n} \to M$. The relationship is given by

$$\phi\left(a_0\left[a_1|\ldots|a_n\right] a_{n+1}\right) = a_0 f\left(a_1, \ldots, a_n\right) a_{n+1}.$$

Then the formula for the coboundary map is

$$\begin{aligned} \beta(f)\left(a_1, \ldots, a_{n+1}\right) = & a_1 f\left(a_2, \ldots, a_{n+1}\right) \\ & + \sum_{0 < i < n+1}(-1)^i f\left(a_1, \ldots, a_i a_{i+1}, \ldots, a_{n+1}\right) \\ & + (-1)^{n+1} f\left(a_1, \ldots, a_n\right) a_{n+1}. \end{aligned}$$

$H^0(A,M)$ is the subgroup of invariants of M

$$H^0(A,M)=M^A=\{m\in M\mid am=ma\quad \text{for any}\quad a \text{ in}\quad A\}.$$

For $n=1$ a 1-cocycle is a k-module homomorphism $D:A\to M$ satisfying the identity

$$D(aa')=aD(a')+D(a)a',\quad \text{for}\quad a\quad \text{and}\quad a'\in A.$$

Such a map is called a derivation (or sometimes a crossed homomorphism) from A to M and the k-module of derivations is denoted $\operatorname{Der}(A,M)$ (cf. 1.3.1). It is a coboundary if it has the form $ad_m(a)=[m,a]=ma-am$ for some fixed $m\in M; ad_m$ is called an inner derivation (or sometimes a principal crossed homomorphism). Therefore

$$H^1(A,M)=\operatorname{Der}(A,M)/\{\text{ inner derivations }\}.$$

It is sometimes called the group of outer derivations. In the particular case $M=A$ the module $H^1(A,A)$ is in fact a Lie algebra with Lie bracket given by $[D,D']=D\circ D'-D'\circ D$. Indeed it is immediate to check that $[D,D']$ is a derivation and that, if $D'=ad_u$ for some $u\in A$, then $[D,ad_u]=ad_{D(u)}$.

1.5.5 The Particular Case $M=A^*$. Notation.

There is defined an explicit map the other way round, called the cotrace map, as follows. Let $f\in C^n(A,M)$ and let $\alpha_1,\ldots,\alpha_n$ be in $\mathcal{M}_r(A)$. Then $F(\alpha_1,\ldots,\alpha_n)$ is a matrix in $\mathcal{M}_r(M)$ whose (i,j)-entry is

$$\sum f\left((\alpha_1)_{ii_2},(\alpha_2)_{i_2i_3},\ldots,(\alpha_n)_{i_nj}\right)$$

where the sum is extended over all possible sets of indices $(i_2,i_3,\ldots,i_n)$. The map of complexes $C^*(A,M)\to C^*(\mathcal{M}_r(A),\mathcal{M}_r(M)), f\mapsto F$ induces the cotrace map

$$\operatorname{cotr}:\ H^*(A,M)\to H^*(\mathcal{M}_r(A),\mathcal{M}_r(M)).$$

$$\begin{aligned}&\langle\operatorname{cotr}(f),x'\rangle=\langle f,\operatorname{tr}(x')\rangle\in M_A\\ &\text{for}\quad f\in H^n(A,M)\quad \text{and}\quad x'\in HH_n(\mathcal{M}_r(A))\end{aligned}$$

The cotrace map and inc* * are isomorphisms and inverse to each other.

1.5.7 Normalized Complex. Suppose that A is unital. Then the reduced complex $\bar{C}^*(A,M)$ is the subcomplex of $C^*(A,M)$ made up of the maps f which vanish on elements $(a_0,\ldots,a_n)$ such that one of the a_i 's $(i\neq 0)$ is 1 . The inclusion $\bar{C}^*\hookrightarrow C^*$ is a quasi-isomorphism.

2.2 Cyclic homology

2.1.0 Cyclic Group Action. The cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ action on the module $A^{\otimes n+1}$ is given by letting its generator $t=t_n$ act by

$$t_n(a_0,\ldots,a_n)=(-1)^n(a_n,a_0,\ldots,a_{n-1})$$

on the generators of $A^{\otimes n+1}$. It is then extended to $A^{\otimes n+1}$ by linearity; it is called the cyclic operator. Remark that $(-1)^n$ is the sign of the cyclic permutation on $(n+1)$ letters. Let $N=1+t+\ldots+t^n$ denote the corresponding norm operator on $A^{\otimes n+1}$.

2.1.2 The Cyclic Bicomplex. As an immediate consequence of Lemma 2.1.1, the following is a first quadrant bicomplex denoted $CC(A)$, and called the cyclic bicomplex:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{1-t} & A^{\otimes 3} & & \cdots\\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & \xleftarrow{N} & A^{\otimes 2} & \xleftarrow{1-t} & A^{\otimes 2} & & \cdots\\
\downarrow b & & \downarrow -b' & & \downarrow b & & \downarrow -b' & & \\
A & \xleftarrow{1-t} & A & \xleftarrow{N} & A & \xleftarrow{1-t} & A & & \cdots
\end{array}$$

2.1.4 Connes' Complex. The cokernel $A^{\otimes n+1}/(1-t)$ of the endomorphism $(1-t)$ of $A^{\otimes n+1}$ is the coinvariant space of $A^{\otimes n+1}$ for the action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$. Following A. Connes we denote it by $C_n^\lambda(A) := A^{\otimes n+1}/(1-t)$. By Lemma 2.1.1 the following is a well-defined complex

$$C_*^\lambda(A): \quad \ldots \xrightarrow{b} C_n^\lambda(A) \xrightarrow{b} C_{n-1}^\lambda(A) \xrightarrow{b} \ldots \xrightarrow{b} C_0^\lambda(A)$$

called Connes complex, and whose nth homology group is denoted $H_n^\lambda(A)$. The natural surjection $p : \operatorname{Tot} CC(A) \to C^\lambda(A)$ is the quotient map $A^{\otimes n+1} \to A^{\otimes n+1}/(1-t)$ on the first column and 0 on the others.

It is trivial to see that row number n is an acyclic complex except $H_0 = C_n^\lambda(A)$. As a consequence the homology of the bicomplex $CC(A)$ is canonically isomorphic to the homology of Connes' complex $C_*^\lambda(A)$.
We previously proved that the b'-complex is contractible when A is unital. So one can expect to simplify the double chain complex $CC(A)$ by getting rid of the contractible complexes (odd degree columns). To do this we use the following easy result.
(Killing Contractible Complexes). Let

$$\ldots \to A_n \oplus A'_n \xrightarrow{d=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}} A_{n-1} \oplus A'_{n-1} \to \ldots$$

be a complex of k-modules such that (A'_*, δ) is a complex and is contractible with contracting homotopy $h : A'_n \to A'_{n+1}$. Then the following inclusion of complexes is a quasi-isomorphism:

$$(id, -h\gamma) : (A_*, \alpha - \beta h\gamma) \hookrightarrow (A_* \oplus A'_*, d)\,.$$

Factoring out the odd degree columns we end up with:
Connes' Boundary Map B and the Bicomplex $\mathcal{B}(A)$:

$$\begin{array}{ccccc}
A^{\otimes 3} & \xleftarrow{B} & A^{\otimes 2} & \xleftarrow{B} & A \\
\downarrow b & & \downarrow b & & \\
A^{\otimes 2} & \xleftarrow{B} & A & & \\
\downarrow b & & & & \\
A & & & &
\end{array}$$

Where B is of course given by: $B = (1-t)sN$:
Explicitly $B : A^{\otimes n+1} \to A^{\otimes n+2}$ is given by

$$\begin{aligned} B(a_0, \ldots, a_n) = \sum_{i=0}^{n} & (-1)^{ni}(1, a_i, \ldots, a_n, a_0, \ldots, a_{i-1}) \\ & - (-1)^{ni}(a_i, 1, a_{i+1}, \ldots, a_n, a_0, \ldots, a_{i-1}) \end{aligned}$$

2.2.1 The bicomplex $\overline{\mathcal{B}}(A)$

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A \otimes \overline{A}^3 & \xleftarrow{\overline{B}} & A \otimes \overline{A}^2 & \xleftarrow{\overline{B}} & A \otimes \overline{A} & \xleftarrow{\overline{B}} & A \\
\downarrow b & & \downarrow b & & \downarrow b & & \\
A \otimes \overline{A}^2 & \xleftarrow{\overline{B}} & A \otimes \overline{A} & \xleftarrow{\overline{B}} & A & & \\
\downarrow b & & \downarrow b & & & & \\
A \otimes \overline{A} & \xleftarrow{\overline{B}} & A & & & & \\
\downarrow b & & & & & & \\
A & & & & & &
\end{array}$$

The Bicomplex $\overline{\mathcal{B}}(\boldsymbol{A})$. The (b,B)-bicomplex $\mathcal{B}(A)$ can be simplified further by replacing the Hochschild complexes by their normalizations (cf. 1.1.15). Let $\bar{A}=A/k$ and consider the new bicomplex $\overline{\mathcal{B}}(A)$: where $\bar{B}=sN: A\otimes\bar{A}^{\otimes n}\rightarrow A\otimes\bar{A}^{\otimes n+1}$ is given by the formula

$$\bar{B}\left(a_0,\ldots,a_n\right)=\sum_{i=0}^{n}(-1)^{ni}\left(1,a_i,\ldots,a_n,a_0,\ldots,a_{i-1}\right).$$

(Remark that the sign $(-1)^{ni}$ is exactly the sign of the involved cyclic permutation). In particular

$$\bar{B}(a)=(1,a),\quad \bar{B}\left(a,a'\right)=\left(1,a,a'\right)-\left(1,a',a\right).$$

If the context is clear we will often write simply B instead of $\bar{B}$. By Proposition 1.1.15 the normalization process does not change the homology of the columns. Therefore, by a standard spectral sequence argument (cf. 1.0.12) the surjective map of complexes $\mathcal{B}(A)\rightarrow\overline{\mathcal{B}}(A)$ is a quasiisomorphism. Thus we have proved the following:
2.1.10 Corollary. For any unital k-algebra A there is a canonical isomorphism

$$H_*(\operatorname{Tot}\overline{\mathcal{B}}(A))\cong HC_*(A).$$

The following are quasi-isomorphisms

$$\operatorname{Tot}\overline{\mathcal{B}}(A)\leftarrow\operatorname{Tot}\mathcal{B}(A)\hookrightarrow\operatorname{Tot}CC(A)\rightarrow C^{\lambda}(A),$$

(Connes' Periodicity Exact Sequence).From the short exact sequence $0\rightarrow CC(A)^{\{2\}}\rightarrow CC(A)\rightarrow CC(A)[2,0]\rightarrow 0$ (where the first term is the first two columns of the bicomplex $CC(A)$ and the last term is $CC(A)[2,0]$ shifted by 2 we obtain a natural long exact sequence

$$\ldots\rightarrow HH_n(A)\xrightarrow{I}HC_n(A)\xrightarrow{S}HC_{n-2}(A)\xrightarrow{B}HH_{n-1}(A)\xrightarrow{I}\ldots.$$

When A is unital this sequence can be deduced more simply from the bicomplex $\mathcal{B}(A)$ (or equivalently from $\overline{\mathcal{B}}(A)$), by considering the exact sequence of complexes

$$0\rightarrow C(A)\rightarrow\operatorname{Tot}(\mathcal{B}(A))\xrightarrow{S}\operatorname{Tot}(\mathcal{B}(A))[2]\rightarrow 0,$$

where the first map is the identification of $C(A)$ with the first column of $\mathcal{B}(A)$. Then the periodicity operator S is obtained by factoring out by this first column.Using this sequence ,examining the first terms and induction one proves that if a map between k algebras induces isomorphisms in hochschild homology then it does so in cyclic homology for example we get morita invariance:

The generalized trace map $\operatorname{tr}:\mathcal{M}_r(A)^{\otimes n+1}\rightarrow A^{\otimes n+1}$ (cf. 1.2.1) is compatible with the cyclic action.
(Morita Invariance for Cyclic Homology). For any $r\geq 1$ (including $r=\infty$) and any H-unital (e.g. unital) k-algebra A the map $\operatorname{tr}_*: HC_*\left(\mathcal{M}_r(A)\right)\rightarrow HC_*(A)$ is an isomorphism, with inverse induced by the inclusion inc : $A=\mathcal{M}_1(A)\hookrightarrow\mathcal{M}_r(A)$. More generally, if A and A' are Morita equivalent k-algebras, then there is a canonical isomorphism $HC_*(A)\cong HC_*\left(A'\right)$
2.2.10 Corollary. For any $r\geq 1$ (including $r=\infty$), the trace map induces an isomorphism $\operatorname{tr}_*: H_n^{\lambda}\left(\mathcal{M}_r(A)\right)\rightarrow H_n^{\lambda}(A)$ for any unital k-algebra $A(\mathbb{Q}\subset k)$.

2.2.2 Cyclic cohomology

Cyclic cohomology is the dual theory to cyclic homology:

$$HC^n(A):=H^n\left(\operatorname{Tot}CC^{**}(A)\right)$$

2.4.2 Connes' Definition. A cochain f in $C^n(A)$ is said to be cyclic if it satisfies the relation

$$f\left(a_0,\ldots,a_n\right)=(-1)^n f\left(a_n,a_0,\ldots,a_{n-1}\right),\quad a_i\in A.$$

These cyclic cochains form a sub- k-module of $C^n(A)$ denoted $C_{\lambda}^n(A)$.

Shuffle product

Let S_n be the symmetric group acting on the set $\{1,\ldots,n\}$. A (p,q)-shuffle is a permutation σ in S_{p+q} such that

$$\sigma(1)<\sigma(2)<\ldots<\sigma(p) \text{ and } \sigma(p+1)<\sigma(p+2)<\ldots<\sigma(p+q).$$

For any k-algebra A we let S_n act on the left on $C_n=C_n(A)=A\otimes A^{\otimes n}$ by:

$$\sigma.\left(a_0,a_1,\ldots,a_n\right)=\left(a_0,a_{\sigma^{-1}(1)},a_{\sigma^{-1}(2)},\ldots,a_{\sigma^{-1}(n)}\right).$$

In other words, if σ is a (p,q)-shuffle the elements $\{a_1,a_2,\ldots,a_p\}$ appear in the same order in the sequence $\sigma\cdot(a_0,\ldots,a_n)$ and so do the elements $\{a_{p+1},a_{p+2},\ldots,a_{p+q}\}$. Let A' be another k-algebra. The shuffle product

$$-\times-=sh_{pq}:C_p(A)\otimes C_q\left(A'\right)\to C_{p+q}\left(A\otimes A'\right)$$

is defined by the following formula:

$$\begin{aligned}&\left(a_0,a_1,\ldots,a_p\right)\times\left(a_0',a_1',\ldots,a_q'\right)\\=&\sum_\sigma \operatorname{sgn}(\sigma)\sigma.\left(a_0\otimes a_0',a_1\otimes 1,\ldots,a_p\otimes 1,1\otimes a_1',\ldots,1\otimes a_q'\right)\end{aligned}$$

2.3 Chern characters

In this section we are going to express the index pairing in terms of the pairing of cyclic Homology and cohomology.(The reader should refer to the section about these first.) The aim is to associate to a suitable representative of a K-theory class, respectively a K-homology class, a class in periodic cyclic homology, respectively a class in periodic cyclic cohomology, called a Chern character in both cases. The principal result is then that the pairing between the latter gives the index pairing of K-homology and K-theory.
In the context of spectral triples the result is:

$$\langle[x],[(\mathcal{A},\mathcal{H},\mathcal{D})]\rangle=-\frac{1}{\sqrt{2\pi i}}\left\langle\left[\operatorname{Ch}_*(x)\right],\left[\operatorname{Ch}^*(\mathcal{A},\mathcal{H},\mathcal{D})\right]\right\rangle$$

Note that we are using the (b,B) complexes:

$$\left(B\phi_m\right)\left(a_0,a_1,\ldots,a_{m-1}\right)=\textstyle\sum_{j=0}^{m-1}(-1)^{(m-1)j}\phi_m\left(1,a_j,a_{j+1},\ldots,a_{m-1},a_0,\ldots,a_{j-1}\right)$$
$$\left(b\phi_{m-2}\right)\left(a_0,a_1,\ldots,a_{m-1}\right)=\textstyle\sum_{j=0}^{m-2}(-1)^j\phi_{m-2}\left(a_0,a_1,\ldots,a_ja_{j+1},\ldots,a_{m-1}\right)+(-1)^{m-1}\phi_{m-2}\left(a_{m-1}a_0,a_1,\ldots,a_{m-2}\right)$$

The pairing between a (b,B)-cochain $\phi=\left(\phi_m\right)_{m=1}^M$ and a $\left(b^T,B^T\right)$-chain $c=\left(c_m\right)$ is given by ($M\in\mathbb{N}$ or $M=\infty$) is:

$$\langle\phi,c\rangle=\sum_{m=1}^M\phi_m\left(c_m\right).$$

2.3.1 Chern character on $K_0(A),K_1(A)$

- We recall that the Chern character $\operatorname{Ch}_*(u)$ of a unitary $u\in\mathcal{A}$ is the following (infinite) collection of odd chains $\operatorname{Ch}_{2j+1}(u)$ satisfying $b\operatorname{Ch}_{2j+3}(u)+B\operatorname{Ch}_{2j+1}(u)=0$,

$$\operatorname{Ch}_{2j+1}(u)=(-1)^j j!u^*\otimes u\otimes u^*\otimes\cdots\otimes u\quad(2j+2\text{ entries }).$$

- Similarly, the (b,B) Chern character of a projection p in an algebra $\mathcal{A}$ is an even (b,B) cycle with $2m$-th term, $m\geq 1$, given by

$$\operatorname{Ch}_{2m}(p)=(-1)^m\frac{(2m)!}{2(m!)}(2p-1)\otimes p^{\otimes 2m}.$$

  For $m=0$ the definition is $\operatorname{Ch}_0(p)=p$. It is a (b,B) cycle.
  With the trace inducing the (morita equivalence) isomorphism these can be extended to projections and unitaries over matrix rings of A.

2.3.2 The JLO cocycle

Connes then proved that the so called JLO cocycle is a representative for the Chern character of a spectral triple. We describe it explicitly . It is given on even spectral triples by an infinite sequence of cochains $(\mathrm{JLO}_{2k})_{k\geq 0}$ defined by

$$\mathrm{JLO}_{2k}\left(a_0, a_1, \ldots, a_{2k}\right) = \int_{\Delta} \operatorname{Trace}\left(\gamma a_0 e^{-t_0 \mathcal{D}^2}\left[\mathcal{D}, a_1\right] e^{-t_1 \mathcal{D}^2} \cdots e^{-t_{2k-1} \mathcal{D}^2}\left[\mathcal{D}, a_{2k}\right] e^{-t_{2k} \mathcal{D}^2}\right) dt_0 dt_1 \cdots dt_{2k}.$$

Here $\Delta = \left\{(t_0, t_1, \ldots, t_{2k}) \in \mathbb{R}^{2k+1} : t_j \geq 0, t_0 + t_1 + \cdots + t_{2k} = 1\right\}$ is the standard simplex.
In the odd case we have $(\mathrm{JLO}_{2k+1})_{k\geq 0}$ defined by

$$\mathrm{JLO}_{2k+1}\left(a_0, a_1, \ldots, a_{2k+1}\right) = \sqrt{2\pi i} \int_{\Delta} \operatorname{Trace}\left(a_0 e^{-t_0 \mathcal{D}^2}\left[\mathcal{D}, a_1\right] e^{-t_1 \mathcal{D}^2} \cdots e^{-t_{2k} \mathcal{D}^2}\left[\mathcal{D}, a_{2k}\right] e^{-t_{2k+1} \mathcal{D}^2}\right) dt_0 dt_1 \cdots dt_{2k+1}$$

As stated earlier the pairing between this cocycle and K theory (which exists under some assumptions) gives the index pairing.We mention this cocycle here because it is similar to other cocycles we are going to use later.

$$\langle [p], [(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = \langle [\mathrm{Ch}(p)], [\mathrm{JLO}(\mathcal{A}, \mathcal{H}, \mathcal{D})] \rangle = \sum_{k=0}^{\infty} \mathrm{JLO}_{2k}\left(\mathrm{Ch}_{2k}(p)\right)$$

2.3.3 The chern charachter of a fredholm module

The underlying theme of this first part is to " quantize " the usual calculus of differential forms. Letting $\mathscr{A}$ be an algebra over $\mathbf{C}$ we introduce the following operator theoretic definitions for

- the differential df of any $f \in \mathscr{A}$
- the graded algebra $\Omega = \oplus \Omega^q$ of differential forms
- the integration $\omega \to \int \omega \in \mathbf{C}$ of forms $\omega \in \Omega^n$,

In the context of a fredholm module (or spectral triple) these correspond to:

$$\begin{aligned} & df = i[\mathbf{F}, f] = i(\mathbf{F} f - f \mathbf{F}) \quad \forall f \in \mathscr{A}, \\ & \Omega^q = \left\{\Sigma f^0 df^1 \ldots df^q, f^j \in \mathscr{A}\right\}, \\ & \int \omega = \operatorname{Trace}(\varepsilon \omega) \quad \forall \omega \in \Omega^n. \end{aligned}$$

The data required for these definitions to have a meaning is an n-summable Fredholm module (H, F) over $\mathscr{A}$.

$$\mathrm{F} a - a \mathrm{F} \in \mathscr{L}^n(\mathrm{H})$$

where $\mathscr{L}^n(\mathrm{H})$ is the Schatten ideal .

Let then $\mathscr{A}$ be a not necessarily commutative algebra over $\mathbf{C}$ and $(\mathbf{H}, \mathbf{F})$ an n-summable (normalized) Fredholm module over $\mathscr{A}$. For any $a \in \mathscr{A}$, one has $da = i[\mathrm{F}, a] \in \mathscr{L}^n(\mathrm{H})$. For each $q \in \mathbf{N}$, let Ω^q be the linear span in $\mathscr{L}^{n/q}(\mathrm{H})$ of the operators

$$\left(a^0 + \lambda \mathrm{I}\right) da^1 da^2 \ldots da^q, \quad a^j \in \mathscr{A}, \lambda \in \mathbf{C}.$$

Since $\mathscr{L}^{n/q_1} \times \mathscr{L}^{n/q_2} \subset \mathscr{L}^{n/(q_1+q_2)}$ (follows from some Holder inequality)) one checks that the composition of operators, $\Omega^{q_1} \times \Omega^{q_2} \to \Omega^{q_1+q_2}$ endows $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ with a structure of a graded algebra. The differential $d, d\omega = i[\mathrm{F}, \omega]$ is such that

$$d^2 = 0, \quad d\left(\omega_1 \omega_2\right) = \left(d\omega_1\right) \omega_2 + (-1)^{\deg \omega_1} \omega_1 d\omega_2 \quad \forall \omega_1, \omega_2 \in \Omega.$$

Thus (Ω, d) is a graded differential algebra, with $d^2 = 0$. Moreover the linear functional $\int : \Omega^n \to \mathbf{G}$, defined by

$$\int \omega = \operatorname{Trace}(\varepsilon \omega) \quad \forall \omega \in \Omega^n$$

has the same properties as the integration of the trace of ordinary matrix valued differential forms on an oriented manifold, namely,

$$\int d\omega = 0 \quad \forall \omega \in \Omega^{n-1}, \qquad \int \omega_2\omega_1 = (-1)^{\deg\omega_1 \deg\omega_2} \int \omega_1\omega_2$$

for any $\omega_j \in \Omega^{q_j}, j = 1, 2, q_1 + q_2 = n$. Thus our construction associates to any n-summable Fredholm module (H, F) over $\mathscr{A}$ an n-dimensional cycle over $\mathscr{A}$ in the following sense.

Definition: A **cycle** of dimension n is a triple $\left(\Omega, d, \int\right)$ where $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ is a graded algebra over $\mathbf{C}, d$ is a graded derivation of degree I such that $d^2 = 0$, and $\int : \Omega^n \to \mathbf{C}$ is a closed graded trace on Ω.
Let $\mathscr{A}$ be an algebra over $\mathbb{C}$. Then a cycle over $\mathscr{A}$ is given by a cycle (Ω, d, f) and a homomorphism $\rho : \mathscr{A} \to \Omega^0$.

As we shall see a cycle of dimension n over $\mathscr{A}$ is essentially determined by its character, the $(n+1)$-linear function τ,

$$\tau\left(a^0, \ldots, a^n\right) = \int \rho\left(a^0\right) d\left(\rho\left(a^1\right)\right) d\left(\rho\left(a^2\right)\right) \ldots d\left(\rho\left(a^n\right)\right) \quad \forall a^j \in \mathscr{A}.$$

Moreover (cf. part II, proposition I), an $n+\mathrm{I}$ linear function τ on $\mathscr{A}$ is the character of a cycle of dimension n over $\mathscr{A}$ if and only if it is a cyclic cocycle.By the above remarks this will ultimately give the chern charachter of a fredholm module.

$$\mathrm{ch}^* : \{n \text{ summable Fredholm modules over } \mathscr{A}\} \to \mathrm{H}_\lambda^n(\mathscr{A}).$$

the character $\tau \in \mathbf{C}_\lambda^n(\mathscr{A})$ of any n summable Fredholm module over $\mathscr{A}$ is equal to o for n odd. Let us now restrict to even n 's. Also it turns out that the $(n+2k)$-dimensional character τ_{n+2k} of (H, F) is determined uniquely as an element of $\mathrm{H}_\lambda^{n+2k}(\mathscr{A})$ by the n-dimensional character τ_n of (H, F). More precisely, under the periodicity map $\mathrm{S} : \mathrm{H}_\lambda^n(\mathscr{A}) \to \mathrm{H}_\lambda^{n+2}(\mathscr{A})$ such that

$$\tau_{n+2k} = \mathrm{S}^k \tau_n \text{ in } \mathrm{H}_\lambda^{n+2k}(\mathscr{A}).$$

$$\mathbf{H}^*(\mathscr{A}) = \mathrm{H}_\lambda^*(\mathscr{A}) \otimes_{\mathbf{c}(\sigma)} \mathbf{C}$$

where $\mathbf{C}(\sigma)$ acts on $\mathbf{C}$ by $\mathrm{P}(\sigma)z = \mathrm{P}(\mathrm{I})z$ for $z \in \mathbf{C}$. The above results yield a map
ch * : { finitely summable Fredholm modules over $\mathscr{A}\} \to \mathrm{H}^*(\mathscr{A})$.
Moreover The following hold :
Two finitely summable Fredholm modules over $\mathscr{A}$ which are homotopic (among such modules) yield the same element of $\mathrm{H}^*(\mathscr{A})$.
One has a canonical pairing $\langle , \rangle$ between H^{ev} $(\mathscr{A})$ and $\mathrm{K}_0(\mathscr{A})$. see [7]

Now we are going to prove that for a summable fredholm module the pairing of the chern character with K theory gives the index pairing.

$$\langle ch_*[e], \mathrm{ch}^*(\mathrm{H}, \mathrm{F})\rangle = \langle [e], [(\mathrm{H}, \mathrm{F})]\rangle \quad \forall e \in \operatorname{Proj} \mathrm{M}_k(\mathscr{A}).$$

By the above remarks the left hand side can also be expressed as a pairing between K theory and H^{ev} $(\mathscr{A})$.
First we do it for $1-$summable fredholm modules:

The character of a 1-summable Fredholm module

Let $\mathscr{A}$ be an algebra over $\mathbf{C}$, with the trivial $\mathbf{Z}/2$ grading. Let $(\mathbf{H}, \mathbf{F})$ be a I-summable Fredholm module over $\mathscr{A}$. Lemma 1. - a) The equality $\tau(a) = \frac{1}{2}\operatorname{Trace}(\varepsilon\mathrm{F}[\mathrm{F}, a]), \forall a \in \mathscr{A}$, defines a trace on $\mathscr{A}$. b) The index map, $\mathrm{K}_0(\mathscr{A}) \to \mathbf{Z}$, is given by the trace τ :

$$\text{Index } \mathrm{F}_e^+ = (\tau \# \operatorname{Trace})(e) \quad \forall e \in \operatorname{Proj} \mathrm{M}_q(\mathscr{A}).$$

Proof. - a) Since $\mathscr{A}$ is trivially $\mathbf{Z}/2$ graded, one has $\varepsilon a = a\varepsilon$ for all $a \in \mathscr{A}$. As $\varepsilon\mathrm{F} = -\mathrm{F}\varepsilon$ one has $\varepsilon\mathrm{F}[\mathrm{F}, a] = \varepsilon\mathrm{F}^2 a - \varepsilon\mathrm{F}a\ \mathrm{F} = \varepsilon\mathrm{F}^2 a + \mathrm{F}a\varepsilon\mathrm{F} = \varepsilon a + \mathrm{F}a\varepsilon\mathrm{F}$ since $\mathbf{F}^2 = \mathrm{I}$. Thus,

$$\varepsilon\mathrm{F}[\mathrm{F}, a] = [\mathrm{F}, a]\varepsilon\mathrm{F}.$$

Then

$$\tau(ab)=\frac{1}{2}\operatorname{Trace}(\varepsilon\mathrm{F}[\mathrm{F},ab])=\frac{1}{2}\operatorname{Trace}(\varepsilon\mathrm{F}[\mathrm{F},a]b+\varepsilon\mathrm{F}a[\mathrm{F},b])$$
$$=\frac{1}{2}\operatorname{Trace}([\mathrm{F},a]\varepsilon\mathrm{F}b+[\mathrm{F},b]\varepsilon\mathrm{F}a)$$

which is symmetric in a and b. Thus $\tau(ab)=\tau(ba)$ for $a,b\in\mathscr{A}$.

b) Replacing $\mathscr{A}$ by $\mathrm{M}_q(\mathscr{A})$, and (H,F) by $(\mathrm{H}\otimes\mathbf{C}^q,\ \mathrm{F}\otimes\mathrm{I})$ we may assume that $q=\mathrm{I}$. Let $\mathrm{F}=\begin{bmatrix}\mathrm{o} & \mathrm{Q}\\ \mathrm{P} & \mathrm{o}\end{bmatrix}$ so that $\mathrm{PQ}=\mathrm{I}_{\mathrm{H}^-},\mathrm{QP}=\mathrm{I}_{\mathrm{H}^+}$. With $\mathrm{H}_1=e\mathrm{H}^+$, $\mathrm{H}_2=e\mathrm{H}^-$we let P' (resp. Q') be the operator from H_1 to H_2 (resp. H_2 to H_1) which is the restriction of $e\mathrm{P}e$ (resp. $e\mathrm{Q}e$) to H_1 (resp. H_2). Since $[\mathrm{F},e]\in\mathscr{L}^1(\mathrm{H})$ one has $\mathrm{P}'\mathrm{Q}'-\mathrm{I}_{\mathrm{H}_2}\in\mathscr{L}^1(\mathrm{H}_2),\mathrm{Q}'\mathrm{P}'-\mathrm{I}_{\mathrm{H}_1}\in\mathscr{L}^1(\mathrm{H}_1)$. Thus one has

$$\begin{aligned}
&\operatorname{Index}\mathrm{P}'=\operatorname{Trace}(\mathrm{I}_{\mathrm{H}_1}-\mathrm{Q}'\mathrm{P}')-\operatorname{Trace}(\mathrm{I}_{\mathrm{H}_2}-\mathrm{P}'\mathrm{Q}')\\
&=\operatorname{Trace}_{\mathrm{H}^+}(e-e\mathrm{Q}e\mathrm{P}e)-\operatorname{Trace}_{\mathrm{H}_-}(e-e\,\mathrm{P}\,e\mathrm{Q}e)\\
&=\operatorname{Trace}(\varepsilon(e-e\ \mathrm{F}e\ \mathrm{F}e)).\\
&\operatorname{Trace}(\varepsilon(e-e\mathrm{F}e\mathrm{F}e))=\operatorname{Trace}(\varepsilon(e-\mathrm{F}e\mathrm{F}e)e)=\operatorname{Trace}(\varepsilon\mathrm{F}(\mathrm{F}e-e\ \mathrm{F})e)\\
&=\frac{1}{2}\operatorname{Trace}([\mathrm{F},e]\varepsilon\mathrm{F}e+\varepsilon\mathrm{F}[\mathrm{F},e]e)=\frac{1}{2}\operatorname{Trace}(\varepsilon\mathrm{F}e[\mathrm{F},e]+\varepsilon\mathrm{F}[\mathrm{F},e]e)=\frac{1}{2}\operatorname{Trace}(\varepsilon\mathrm{F}[\mathrm{F},e])=\tau(e).
\end{aligned}$$

The Higher characters for a p-summable Fredholm module

Let (H,F) be a p-summable Fredholm module over $\mathscr{A}$. As explained before we shall associate to (H,F) an n-dimensional cycle over $\mathscr{A}$, where n is an arbitrary even integer such that $n\geqq p$.
For any $\mathrm{T}\in\mathscr{L}(\mathrm{H})$ such that $[\mathrm{F},\mathrm{T}]\in\mathscr{L}^1(\mathrm{H})$ let

$$\operatorname{Tr}_s(\mathrm{T})=\frac{\mathrm{I}}{2}\operatorname{Trace}(\varepsilon\mathrm{F}([\mathrm{F},\mathrm{T}])).$$

a) If T is homogeneous with odd degree, then $\operatorname{Tr}_s(\mathrm{T})=0$.
b) If $\mathrm{T}\in\mathscr{L}^1(\mathrm{H})$ then $\operatorname{Tr}_s(\mathrm{T})=\operatorname{Trace}(\varepsilon\mathrm{T})$.
c) One has $[\mathrm{F},\Omega^n]\subset\mathscr{L}^1(\mathrm{H})$ and the restriction of Tr_s to Ω^n defines a closed graded trace on the differential algebra Ω.
The above lemmas are easy and left to the reader.
Let $n=2m$ be an even integer, and (H,F) an $(n+1)$-summable Fredholm module over $\mathscr{A}$. Then the associated cycle over $\mathscr{A}$ is given by the graded differential algebra (Ω,d), the integral

$$\int\omega=(2i\pi)^m m!\operatorname{Tr}_s(\omega)\quad\forall\omega\in\Omega^n$$

and the homomorphism $\pi:\mathscr{A}\rightarrow\Omega^0\subset\mathscr{L}(\mathrm{H})$ of definition 1.

Proposition 5. - Let $n=2m,(\mathrm{H},\mathrm{F})$ be an $(n+\mathrm{I})$-summable Fredholm module over $\mathscr{A}$, and τ be the character of the cycle associated to (H,F),

$$\tau\left(a^0,\ldots,a^n\right)=(2i\pi)^m m!\operatorname{Tr}_s\left(a^0da^1\ldots da^n\right).$$

is a cyclic cocycle.

When the algebra $\mathscr{A}$ is not unital, one first extends $\varphi\in\mathbf{Z}_\lambda^n(\mathscr{A})$ to $\tilde{\varphi}\in\mathbf{Z}_\lambda^n(\mathscr{A})$, where $\tilde{A}$ is obtained from $\mathscr{A}$ by adjoining a unit, (note the cyclic property)

$$\widetilde{\varphi}\left(a^0+\lambda^0\mathbf{1},\ldots,a^n+\lambda^n\mathrm{I}\right)=\varphi\left(a^0,\ldots,a^n\right)\quad\forall a^j\in\mathscr{A},\lambda^j\in\mathbf{C}.$$

Then one applies the above formula, for $e\in\mathrm{M}_k(\tilde{\mathscr{A}})$.

Let $n = 2m$ and (H, F) an $(n+1)$ summable Fredholm module over $\mathscr{A}$. Then the index map $\mathrm{K}_0(\mathscr{A}) \to \mathbf{Z}$ is given by the pairing of $\mathrm{K}_0(\mathscr{A})$ with the class in $\mathrm{H}^n_\lambda(\mathscr{A})$ of the n-dimensional character τ_n of (H, F) :

$$\text{Index } \mathrm{F}^+_e = \langle [e], (\tau_n) \rangle \qquad \text{for } e \in \operatorname{Proj} \mathrm{M}_q(\mathscr{A}).$$

One has $\langle e, \tau_m \rangle = \frac{(-1)^m}{2} \operatorname{Trace}\left(\varepsilon \mathrm{F}[\mathrm{F}, e]^{2m+1}\right)$. As $[\mathrm{F}, e] = e[\mathrm{F}, e] + [\mathrm{F}, e]e$, one has

$$\operatorname{Trace}\left(\varepsilon \mathrm{F}([\mathrm{F}, e])^{2m+1}\right) = \operatorname{Trace}\left(\varepsilon \mathrm{F} e[\mathrm{F}, e][\mathrm{F}, e]^{2m}\right) + \operatorname{Trace}\left(\varepsilon \mathrm{F}[\mathrm{F}, e]e[\mathrm{F}, e]^{2m}\right).$$

Now $\varepsilon \mathrm{F} = -\mathrm{F}\varepsilon, \mathrm{F}[\mathrm{F}, e]^{2m+1} = -[\mathrm{F}, e]^{2m+1} \mathrm{F}$, so that

$$\begin{aligned} \operatorname{Trace}\left(\varepsilon \mathrm{F} e[\mathrm{F}, e]^{2m+1}\right) &= -\operatorname{Trace}\left(\mathrm{F}\varepsilon e[\mathrm{F}, e]^{2m+1}\right) \\ &= -\operatorname{Trace}\left(\varepsilon e[\mathrm{F}, e]^{2m+1} \mathrm{F}\right) = \operatorname{Trace}\left(\varepsilon e \mathrm{F}[\mathrm{F}, e]^{2m+1}\right). \end{aligned}$$

As $e[\mathrm{F}, e]^2 = [\mathrm{F}, e]^2 e$ we get

$$\begin{aligned} \operatorname{Trace}\left(\varepsilon \mathrm{F}[\mathrm{F}, e]^{2m+1}\right) &= 2 \operatorname{Trace}\left(\varepsilon e \mathrm{F}[\mathrm{F}, e]e[\mathrm{F}, e]^{2m}\right) = 2 \operatorname{Trace}\left(\varepsilon e \mathrm{F}[\mathrm{F}, e]e(e[\mathrm{F}, e]^2 e)^m\right) \\ &= 2(-1)^m \operatorname{Trace} \varepsilon(e - e\mathrm{F}e\mathrm{F}e)^{m+1} \end{aligned}$$

And then:

$$\begin{aligned} &\operatorname{Trace} \varepsilon(e - e\mathrm{F}e\mathrm{F}e)^{m+1} = \\ &\operatorname{Trace}_{\mathrm{H}^+} (e - e\mathrm{Q}e\mathrm{P}e)^{m+1} - \operatorname{Trace}_{\mathrm{H}^-} (e - e\mathrm{P}e\mathrm{Q}e)^{m+1} = \\ &\operatorname{Trace} (\mathrm{I}_{\mathrm{H}_1} - \mathrm{Q}'\mathrm{P}')^{m+1} - \operatorname{Trace} (\mathrm{I}_{\mathrm{H}_2} - \mathrm{P}'\mathrm{Q}')^{m+1} = \\ &\operatorname{Index}(P') \end{aligned}$$

2.3.4 Deformation and cyclic cocycles

The following is taken from [16].Revisit the section on comparison of analytic indices. We are going to use a very similar construction here (the two constructions are essentially the same) to obtain a projection over $C*(\mathbb{T}\mathbb{R}^n)$.Then we use cyclic cocycles to give an explicit formula for an index theorem on $\mathbb{R}^n$.
Consider a pseudodifferential operator on $\mathbb{R}^n$ that is given by a matrix valued symbol a.
On the part $t = 0$, this projection is given by the element

$$e_a = \begin{pmatrix} (1 + a^* a)^{-1} & (1 + a^* a)^{-1} a^* \\ a (1 + a^* a)^{-1} & a (1 + a^* a)^{-1} a^* \end{pmatrix}.$$

in $M_{2k}(\tilde{C}_0(T^*\mathbb{R}^n)) = M_{2k}(\tilde{C}^*(T\mathbb{R}^n))$ which is basically the projection on the graph (in $L_2(T^*\mathbb{R}^n)^k \oplus L_2(T^*(\mathbb{R}^n)^k)$ of a viewed as a multiplication operator on $L_2(T^*\mathbb{R}^n)^k$. It is easy to check that $\hat{e}_a = e_a - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2k}\left(C_0\left(T^*\mathbb{R}^n\right)\right)$
Similarly for a densely defined closed operator T on $L_2(\mathbb{R}^n)$ (for example for an elliptic pseudodifferential operator, T is then fredholm) consider the the projection on to it's graph given by

$$e = \begin{pmatrix} (1 + T^* T)^{-1} & (1 + T^* T)^{-1} T^* \\ T (1 + T^* T)^{-1} & T (1 + T^* T)^{-1} T^* \end{pmatrix}.$$

Where $e \in M_{2k}\left(\mathscr{K}\left(L^2\left(\mathbb{R}^n\right)\right)^\sim\right) = M_{2k}(\tilde{C}^*(M \times M))$ and it is easily checked that $e - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2k}\left(\mathscr{K}\left(L^2\left(\mathbb{R}^n\right)\right)\right) = M_{2k}(C^*(M \times M))$. and that the difference

$$[e] - \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right]$$

determines an element of $K_0\left(\mathscr{K}\left(L^2\left(\mathbb{R}^n\right)\right)\right)$. Moreover the same method used earlier shows that this element is exactly $[\ker T] - [\ker T^*]$ which corresponds to $\operatorname{index} T$ under the isomprphism $K_0(\mathcal{K}) \cong \mathbb{Z}$.
Now the element given by the grpah projection e_t on the graph of Op^t_a on $t > 0$ and by e_a on $t = 0$ is a well defined projection over $C^*(\mathbb{T}\mathbb{R}^n)$ as proved in section comparison of analytic indices.

the cyclic cocycle

Now consider the densely defined cyclic $2n-$cocycle on the algebra $\mathcal{K}^{\infty}$ of integral operators in $\mathcal{K}(L_2\mathbb{R}^n)$ with smooth kernels.

$$\omega\left(T_0, \ldots, T_{2n}\right)=\frac{(-1)^n}{n!} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) \operatorname{Tr}\left(T_0 \delta_{\sigma(1)}\left(T_1\right) \cdots \delta_{\sigma(2n)}\left(T_{2n}\right)\right)$$

Where $\delta_{2j-1}(T)=\left[D_j, T\right], \delta_{2j}(T)=\left[M_j, T\right]$ and $D_j=\frac{\partial}{\partial x_j}$ and M_j denotes pointwise multiplication by x_j. Because $\mathrm{Tr}(\delta_\mathrm{j}(\mathrm{T})=0$ and the derivations δ commute with one another it is easily checked that ω is indeed a cyclic cocycle.

The pairing between this cyclic cocycle and $K_0(\mathbb{K}(L_2(\mathbb{R}^n)))$ can be shown to be well defined and furthermore it gives the isomorphism $K_0(\mathcal{K}) \cong \mathbb{Z}$(by the above properties it can be defined on the unitization).

Also consider the cocycle ϵ_0 on $\mathcal{S}(T^*\mathbb{R}^n)$ defined by

$$\varepsilon_0\left(f^0, \ldots, f^{2n}\right)=\frac{1}{(2\pi i)^n n!} \int_{T*\mathbb{R}^n} f^0 df^1 \wedge \cdots \wedge df^n,$$

where $T^*\mathbb{R}^n$ is has the orientation given by the symplectic structure.

Under a certain sense the cocycle ω when calculated on deformed pseudodifferential operators converges to a cocycle on their symbols.That last cocycle is exactly ϵ_0.

$$\lim_{t\to 0} \omega(\mathrm{Op}^t_{a_0}, ..., \mathrm{Op}^t_{a_{2n}}) \to \epsilon_0(a_0, ..., a_{2n})$$

The reasons for that are the following two identities

$$\begin{aligned}
\delta_{2j-1}\left(\mathrm{Op}_a^t\right)=\left[D_j, \mathrm{Op}_a^t\right]=\mathrm{Op}_{\frac{\partial a}{\partial x_j}}^t \\
\delta_{2j}\left(\mathrm{Op}_a^t\right)=\left[M_j, \mathrm{Op}_a^t\right]=\mathrm{Op}_{it \frac{\partial a}{\partial \xi_j}}^t
\end{aligned}$$

the fact that the trace of a pseudodifferential operator with rapidly decaying symbol $p(x, \xi)$ is given by :

$$\mathrm{Tr}(\mathrm{Op}_p : L_2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)) = \int_{T^*\mathbb{R}^n} p(x,\xi) dx d\xi$$

and the symbolic calculus of asymptotic pseudodifferential operators discussed earlier.

If we apply this principle to the pairing of cyclic cohomology with K theory and use it on the deformed projection constructed earlier then we will get that formula for the index.

$$\text{index } P_a=\frac{1}{(2\pi i)^n n!} \int_{T*\mathbb{R}^n} \operatorname{tr}\left(\hat{e}_a\left(d\hat{e}_a\right)^{2n}\right),$$

2.4 Connes Moscovici localized index theorem

2.4.1 Alexander Spanier cohomology

Let us start recalling the definition of Alexander-Spanier cohomology with real coefficients, of a topological space M . With $q \geq 0$, let $C^q(M)$ be the vector space of all functions φ from M^{q+1} to $\mathbb{R}$; a coboundary homomorphism $\delta: C^q(M) \rightarrow C^{q+1}(M)$ is defined by the formula

$$(\delta\varphi)\left(x^0, \ldots, x^{q+1}\right)=\sum_{i=0}^{q+1}(-1)^i \varphi\left(x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{q+1}\right),$$

and $C^*(M)=\left\{C^q(M), \delta\right\}$ is a cochain complex over $\mathbb{R}$. Its cohomology is trivial, except in dimension 0 . The nontrivial cohomological information is concentrated in the subcomplex of "locally zero" cochains. An element $\varphi \in C^q(M)$ is said to be locally zero if there is an open covering $\mathscr{U}$ of M such that φ vanishes on the neighborhood $\mathscr{U}^{q+1}=\bigcup_{U \in \mathscr{U}} U^{q+1}$ of the q th diagonal of M. If φ is locally zero then, evidently, so is $\delta\varphi$. One obtains thus a subcomplex $C_0^*(M)=\left\{C_0^q(M), \delta\right\}$ of $C^*(M)$. The corresponding quotient complex $\bar{C}^*(M)=\left\{\bar{C}^q(M), \delta\right\}$ is called the Alexander-Spanier complex of M with coefficients in $\mathbb{R}$ and its graded cohomology space $\bar{H}^*(M)$ is called the Alexander-Spanier cohomology of M (with real coefficients). If $\varphi \in C^q(M)$, we shall denote by $\bar{\varphi}$ its image in $\bar{C}^q(M)$ and by $[\bar{\varphi}]$ the corresponding cohomology class. We are now going to identify alexander spanier cohomology with de rham cohomology.

First we need a lemma:(let $\mathscr{B}$) be a sufficiently fine covering)

If $\mathbf{x} = \left(x^0, \ldots, x^q\right) \in \mathscr{B}^{q+1}$ and $\mathbf{t} = (t_0, \ldots, t_q) \in \Sigma^q$, the function sending $y \in M$ to $\sum_{i=0}^q t_i d^2\left(x^i, y\right)$ has a minimum which is attained in a unique point $\sum_{i=1}^q t_i x^i \in M$. Moreover, this point depends differentiably on $(\mathbf{x}, \mathbf{t}) \in \mathscr{B}^{q+1} \times \Sigma^q$.

Given $\mathbf{x} \in \mathscr{B}^{q+1}$ we now define a C^∞ simplex $s_q[\mathbf{x}] : \Sigma^q \to M$ by setting

$$s_q[\mathbf{x}]\left(t_0, \ldots, t_q\right) = \sum_{i=0}^q t_i x^i.$$

Together with the covering $\mathscr{B}$, it will be convenient to fix a collection of functions $\chi = \{\chi_q\}_q \geq 0$ such that: (f) $\chi_q \in C^\infty\left(M^{q+1}\right)$, support $\chi_q \subset \mathscr{B}^{q+1}$ and $\chi_q \equiv 1$ on a neighborhood of the q th diagonal in M^{q+1} (g) $\chi_q\left(x^{\tau(0)}, \ldots, x^{\tau(q)}\right) = \chi_q\left(x^0, \ldots, x^q\right), \forall \tau \in \mathscr{S}_{q+1} =$ the permutation.group of order $(q+1)!$ Let now $\Lambda^*(M) = \{\Lambda^q(M), d\}$ be the de Rham complex of differential forms on M. Given $\omega \in \Lambda^q(M)$, we define $\rho(\omega) \in C^q(M)$ by

$$\rho(\omega)\left(x^0, \ldots, x^q\right) = \chi_q\left(x^0, \ldots, x^q\right) \int_{s_q[x^0, \ldots, x^0]} \omega.$$

The vanishing of χ_q outside $\mathscr{U}^{q+1}$ gives an obvious meaning to the right hand side for any $\left(x^0, \ldots, x^q\right) \in M^{q+1}$. It is also clear that the class

$$\bar{\rho}(\omega) = \overline{\rho(\omega)} \in \bar{C}^q(M)$$

is independent of the choice of $\mathscr{B}$ and χ with the above properties. The map $\bar{\rho} : \Lambda^*(M) \to \bar{C}^*(M)$ thus defined is chain map.

$$\delta\overline{\rho(\omega)} = \bar{\rho}(d\omega).$$

Indeed, from the definition of the simplex $s_q[\mathbf{x}]$ it follows easily that its boundary $\partial s_q[\mathbf{x}]$ can be expressed as follows:

$$\partial s_q\left[x^0, \ldots, x^q\right] = \sum_{i=0}^q (-1)^i s_{q-1}\left[x^0, \ldots, x^{i-1}, x^{i+1}, \ldots, x^q\right]$$

thus, the claimed identity is a consequence of Stokes' theorem for chains.

The induced homomorphism in cohomology $\bar{\rho}^* : H^*_{dR}(M) \to \bar{H}^*(M)$ is an isomorphism. It is locally an isomorphism of presheaves therefore it is globally an isomorphism .

Universal differential forms

In order to find an explicit formula for a left inverse to $\bar{\rho}^*$, it will be helpful to bring into the discussion the universal complex of the Fréchet algebra $\mathscr{A} = C^\infty(M)$:

Universal defferential forms:

$\Omega^0(\mathscr{A}) = \mathscr{A}, \Omega^1(\mathscr{A}) = \mathrm{Ker}(\mathscr{A}\hat{\otimes}\mathscr{A} \xrightarrow{\text{multipilication}} \mathscr{A})$, which is, in an obvious way, a bimodule over $\mathscr{A}$, and $\Omega^q(\mathscr{A}) = \Omega^1(\mathscr{A})\hat{\otimes}\ldots\hat{\otimes}\Omega^1(\mathscr{A})$ (q times) for $q \geq 1$; it is equipped with a continuous coboundary homomorphism $\partial : \Omega^q(\mathscr{A}) \to \Omega^{q+1}(\mathscr{A})$, uniquely characterized by the equations

$$\partial f = 1 \otimes f - f \otimes 1, \quad \forall f \in \mathscr{A},$$
$$\partial\left(f^0 \partial f^1 \otimes \ldots \otimes \partial f^q\right) = \partial f^0 \otimes \partial f^1 \otimes \ldots \otimes \partial f^q, \forall f^0, f^1, \ldots, f^q \in \mathscr{A}.$$

There is a natural surjection v of $C^q(M) \cong \mathscr{A}\hat{\otimes}\ldots\hat{\otimes}\mathscr{A}$ $(q+1$ times) onto $\Omega^q(M)$, which sends an elementary tensor $f^0 \otimes f^1 \otimes \ldots \otimes f^q \in C^q(M)$ to $f^0 df^1 \otimes \ldots \otimes df^q \in \Omega^q(M)$. In particular, it sends

$$\delta\left(f^0 \otimes f^1 \otimes \ldots \otimes f^q\right) = \sum_{i=0}^{q+1} (-1)^i f^0 \otimes \ldots \otimes f^{i-1} \otimes 1 \otimes f^i \otimes \ldots \otimes \ldots \otimes f^q$$

to

$$d\left(f^0 df^1 \otimes \ldots \otimes df^q\right) = df^0 \otimes df^1 \otimes \ldots \otimes df^q.$$

On the other hand, by the universality of $\{\Omega^*(\mathscr{A}), \partial\}$, there exists a canonical morphism of complexes $\mu : \Omega^*(\mathscr{A}) \to \Lambda^*(M)$ such that

$$\mu\left(f^0 \partial f^1 \otimes \ldots \otimes \partial f^q\right) = f^0 df^1 \wedge \ldots \wedge df^q.$$

Composing it with the morphism $v : C^*(M) \to \Omega^*(\mathscr{A})$, we obtain a morphism of complexes $\lambda = \mu^\circ v : C^*(M) \to \Lambda^*(M)$, which evidently vanishes on $C_0^*(M)$. Thus, it induces a morphism $\bar{\lambda} : \bar{C}^*(M) \to \Lambda^*(M)$, characterized by:

$$\bar{\lambda}\left(f^0 \otimes f^1 \otimes \ldots \otimes f^q\right)^- = f^0 df^1 \wedge \ldots \wedge df^q.$$

For an arbitrary cochain $\bar{\varphi} \in \bar{C}^a(M)$ one has therefore:

$$\bar{\lambda}(\bar{\varphi})_x\left(v^1, \ldots, v^q\right) = \frac{1}{q!} \sum_{\tau \in \mathscr{S}_4} \operatorname{sgn}(\tau) \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_q} \varphi\left(x, \exp_x \varepsilon_1 v^{\tau(1)}, \ldots, \exp_x \varepsilon_q v^{\tau(q)}\right)\Bigg|_{\varepsilon_i = 0}$$

As a refinement of the analytic index of an elliptic symbol, we shall construct for each even-dimensional Alexander-Spanier cohomology class (with compact support) on a C^∞ manifold M a localized index map from the K-group $K^0\left(T^*M, T^*M - M\right)$ to $\mathbb{C}$. When M is compact, $K^0\left(T^*M, T^*M - M\right)$ is the K-theory with compact support $K_c^0\left(T^*M\right)$ and the ordinary index map corresponds to the unit class $[1] \in \bar{H}^0(M)$.

$A : C_c^\infty\left(M, E \otimes |\Lambda|^{1/2}(M)\right) \to C^\infty\left(M, F \otimes |\Lambda|^{1/2}(M)\right)$,the distributional kernel $A(x,y)$ of $A \in \Psi^r(M; E, F)$ is a section of the bundle $\operatorname{Hom}(E,F) \otimes |\Lambda|^{1/2}(M \times M)$, i.e.

$$A(x,y) \in \operatorname{Hom}\left(E_y, F_x\right) \otimes |\Lambda|^{1/2} T_x M \otimes |\Lambda|^{1/2} T_y M, \forall (x,y) \in M \times M.$$

2.4.2 Localized indices

Let $A^0, \ldots, A^q \in \Psi^\infty(M; E)$ with at least one of them in $\Psi^{-\infty}(M; E)$; we define the distribution $\operatorname{tr}\left(A^0, \ldots, A^q\right)$ on M^{q+1} by the formula

$$\operatorname{tr}\left(A^0, \ldots, A^q\right)(\varphi) = (-1)^q \int \operatorname{tr}\left(A^0\left(x^0, x^1\right) \ldots A^q\left(x^q, x^0\right)\right) \varphi\left(x^0, \ldots, x^q\right), \forall \varphi \in C_c^\infty\left(M^{q+1}\right)$$

For a fixed $\varphi \in C_c^\infty\left(M^{q+1}\right)$, we also set

$$\tau(\varphi)\left(A^0, \ldots, A^q\right) = \operatorname{tr}\left(A^0, \ldots, A^q\right)(\varphi), \forall A^j \in \Psi^\infty(M; E).$$

(2.1) Lemma. (i) Let $\varphi \in C_{\lambda,cc}^q(M)$. Then $\tau(\varphi) \in C_\lambda^q\left(\Psi^{-\infty}(M; E)\right) =$ the space of q-dimensional cyclic cochains of the algebra $\Psi^{-\infty}(M; E)$, i.e.

$$\tau(\varphi)\left(A^1, \ldots, A^q, A^0\right) = (-1)^q \tau(\varphi)\left(A^0, \ldots, A^{q-1}, A^q\right) \quad \forall A^j \in \Psi^{-\infty}(M; E).$$

(ii) $\tau : C_{\lambda,c}^*(M) \to C_\lambda^*\left(\Psi^{-\infty}(M; E)\right)$ is a homomorphism of complexes, i.e.

$$\tau(\delta\varphi) = b\tau(\varphi)$$

where b is the coboundary of the cyclic cohomology complex [8]. (iii) If $q > 0, A^j \in \Psi^{-\infty}(M; E)$ and $f^j \in DO^0(M; E), j = 0, \ldots, q$, one has

$$\tau(\varphi)\left(A^0 + f^0, \ldots, A^q + f^q\right) = \tau(\varphi)\left(A^0, \ldots, A^q\right)$$

construct a pairing of the above projections with arbitrary Alexander-Spanier cocycles on M, which will recapture the stable information carried by the symbols.

cohomology with compact supports. Consider a cocycle $\varphi \in Z_{\lambda,cc}^q(M)$, that is $\varphi \in C_{\lambda,cc}^q(M)$ and $\delta\varphi \in C_g^{q+1}(M)$. Let L be an invertible lift of the symbol $\tilde{a}$ such that $L(x,y) = 0$ outside a "small" neighborhood of the diagonal, the "size" of which depends on where $\delta\varphi$ vanishes, in a way which will be obvious from the context. Such a lift can be manufactured, for example, by localizing the support of A and B in the above construction. Denoting as before,

$$P_L = L \begin{pmatrix} I_E & 0 \\ 0 & 0 \end{pmatrix} L^{-1}, \quad R_L = P_L - \begin{pmatrix} 0 & 0 \\ 0 & I_F \end{pmatrix} \in \Psi^{-\infty}$$

we define

$$\operatorname{Ind}_{\varphi}(a)=\tau(\varphi)\left(R_{L}, \ldots, R_{L}\right)=(-1)^{q} \int_{M^{q+1}} \operatorname{tr}\left(R_{L}\left(x^{0}, x^{1}\right) \ldots R_{L}\left(x^{q}, x^{0}\right)\right) \varphi\left(x^{0}, \ldots, x^{q}\right)$$

One has to check that the definition makes sense, i.e. that the right hand side is independent of the lift L. If q is odd, in view of Lemma (2.1) (i), this is obvious; it is also uninteresting, since it gives $\operatorname{Ind}_{\varphi}(a)=0$. So, we shall assume from now on that q is even.

$$\tau(\varphi)\left(P_{1}, \ldots, P_{1}\right)-\tau(\varphi)\left(P_{0}, \ldots, P_{0}\right)=(q+1) \int_{0}^{1} \tau(\delta \varphi)\left(T_{s}, P_{s}, \ldots, P_{s}\right) d s$$

where $T_{s}=\left(1-2 P_{s}\right) \frac{d}{d s} P_{s}$.
Proof. We notice that $\frac{d}{d s} P_{s}=\left[T_{s}, P_{s}\right]$ and therefore,

$$\begin{aligned}
\frac{d}{d s} \tau(\varphi)\left(P_{s}, \ldots P_{s}\right) & =\sum_{0}^{a} \tau(\varphi)\left(P_{s}, \ldots,\left[T_{s}, P_{s}\right], \ldots, P_{s}\right) \\
& =(q+1) \tau(\varphi)\left(\left[T_{s}, P_{s}\right], P_{s}, \ldots, P_{s}\right) .
\end{aligned}$$

This last expression is easily recognized to coincide with $(q+1) b \tau(\varphi)\left(T_{s}, P_{s}, \ldots, P_{s}\right)$, which by Lemma (2.1) (ii) is in turn equal to $(q+1) \tau(\delta \varphi)\left(T_{s}, P_{s}, \ldots, P_{s}\right)$.

$$\operatorname{Ind}_{\bar{\varphi}}(a)=\operatorname{Ind}(a)$$

and the definition is unambiguous. The next lemma shows that the localized index map thus defined actually depends only on the cohomology class $[\bar{\varphi}]$. (2.3) Lemma. If $\psi \in C_{\alpha, c c}^{q-1}(M)$, then $\operatorname{Ind} \delta \psi \psi(a)=0$. Proof. With $P=P_{L}$ and L as above, one has:

$$\begin{aligned}
\operatorname{Ind}_{\delta \psi}(a) & =\tau(\delta \psi)(P, \ldots, P)=b \tau(\psi)(P, \ldots, P) \\
& =\tau(\psi)(P, \ldots, P)=-\tau(\psi)(P, \ldots, P)
\end{aligned}$$

(2.4) Theorem. For any $[\bar{\varphi}] \in \bar{H}_{c}^{e v}(M)$ the map Ind $d_{[\bar{\varphi}]}$ from elliptic symbols to $\mathbb{C}$ induces a homomorphism $\operatorname{Ind}_{[\bar{\varphi}]}: K^{0}\left(T^{*} M, T^{*} M-M\right) \rightarrow \mathbb{C}$.

$A_{s} B_{s}-I \in \Psi^{-\infty}, B_{s} A_{s}-I \in \Psi^{-\infty}$ and each A_{s} (resp. B_{s}) is supported in a sufficiently small neighborhood of the diagonal. Denote by L_{s} the lifting of $\tilde{a}_{s}$ manufactured from A_{s} and B_{s}, and by P_{s} the corresponding idempotent. Again, we may assume $q>0$, and then the claim follows by applying Lemma (2.2) to the path $\left\{P_{s}\right\}$.

The localized index maps thus defined can be easily transferred to elliptic operators. Namely, if $[\bar{\varphi}] \in \bar{H}_{c}^{e v}(M)$ and $A \in \Psi^{\prime}(M ; E, F)^{-1}$, we define

$$\operatorname{Ind}_{[\bar{\varphi}]} A=\operatorname{Ind}_{[\bar{\Phi}]}(a),$$

where $a \in P s y^{0}(M ; E, F)$ is uniquely determined by the condition $a\left|S^{*} M=\sigma_{p r}(A)\right| S^{*} M$. As in the case of the ordinary index. Since for the computation of a $[\bar{\varphi}]$-index we can always pick a representative φ with compact support, there will be no loss of generality in restricting our attention to compact

2.4.3 Localized index for groupoids

The localized index for Groupoids As stated earlier the obvious generalization of Alexander Spanier cohomology for groupoids is differentiable groupoid cohomology To define a localized index we first need wy some form of transversal density. (For example if the groupoid is the holonomy groupoid of a foliation the localized pairing is going to roughly give a continuous function on the leaf space, if we integrate this with some form of transversal density (or measure on the leap space) we get a number. Consider a lie groupoid $G \rightrightarrows M$ with lie algebroid $A \longrightarrow M$. Consider the following line bundle over M ,the bundle of "transversal densities")

$$L=\bigwedge^{top} T^{*} M \otimes \bigwedge^{top} A$$

It is easy to see why this should be transversal densities l, given a section of $\bigwedge^{\text{top}} A^{*}$ and pairiving it with a section of L we get a top form on M, exactly what a transversal dénsity would do for foliations.

Now we are going to prove that L carries a representation of E even though A and TM don't. A and TM carry a representation at the level of bisections. Consider a bisection β over $g \in G : \beta(s(g)) = g$. From the diffeomorphism sending $s(g) \to r(g)$ associated to β we get the differential $T_{s(g)}M \to T_{r(g)}M$.
The differential of $R_{g^{-1}}L_\beta : G_{s(g)} \to G_{r(g)}$ which sends $1_{s(g)}$ to $1_{r(g)}$ gives a linear map $A_{s(g)} \to A_{r(g)}$. By the description of these maps it is easy to see that we abs they commute with the anchor maps so we get a map of the complexes $A_\bullet \to T_\bullet M$
It is easy to see that this essentially is a representation of the 1 jets oF bisections groupoid J^1G on A and TM. Furthermore we have a representation of J^1G as chain maps to the chain complex of vector bundles over M consisting just of 2 nonzero terms, A in degree 0 and TM in degree 1 of course this representation doesn't descend to a representation of G but as we shall see it does so up to homotopy.
Towards this it suffices to prove that 2 elements over $g \in G$ in J^1G give homotopic chain maps .Obviously it will be enough to prove this over units 1_x and to assume that one of the two elements comes from the identity bisection. OF course the identity bisection id induces the identity map both on A_x and T_xM. Consider another bisection σ over 1_x.

The difference $\theta_x = T_x\sigma - T_x\text{id}$ gives of well defined map

$$\theta_x : T_xM \to (\ker ds)_{1_x} = A_x$$

The induced map of σ on T_xM is given by

$$T_{1_x}r \circ T_x\sigma = T_{1_x}r \circ (T_x\sigma - T_x\text{id}) + T_{1_x}r \circ T_x \text{ id } = \alpha_x \circ \theta_x + I$$

The induced map of σ on A_x is given by the differential at 1_x of

$$G_x \xrightarrow{(\sigma\circ r,\text{id})} G * G \to G_x$$

using the differential of multiplication (see [mackenzie]) this is found to be equal to :
(using the identity bisections)

$$X \to X + T_x\sigma \circ T_{1_x}r(X) - T_x(\text{id}) \circ T_{1_x}r(X) = \ = X + \theta_x \circ \alpha_x(X)$$

Therefore $e_x : T_xM \to A_x$ defines x homotopy of the induced maps. Now that we have shown that for every $g \in G$ we have or well defined map chain map up to homotopy we will get a well defined representation of G on the berezinian $\bigwedge^{top} T^*M \otimes \bigwedge^{top} A$ of the complex.
Because the above maps come from representations of J^1G they are linear isomorphisms.To show that two maps that are homotopic induce the same map on the berezinian we only need to prove that a chain automorphism that is homotopic to the identity induces the identity on the berezinian. In the case of interest to us $A_\bullet \to T_\bullet M$ this follows from standard linear algebra and is left to the reader.
There is a more general lemma we mention along with it's stragihtforward proof:
In general if we define the berezinian of a chain complex $\ldots \to E_i \to E_{i+1} \to \ldots$ to be

$$\bigotimes \bigwedge^{\text{top}} E_{2i} \otimes \bigotimes \bigwedge^{\text{top}} E^*_{2i+1}$$

then homotopic chain isomorphisms induce the same map on berezinians.
Proof: As before it suffices to consider the case where we have chain isomorphisms on the same chain complex and one of these is the identity. We need the following lemma: For a morphism of short exact sequences of vector spaces:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f & & \downarrow f_2 & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

We have that $\det(f) = \det(f_1)\det(f_2)$ this follows easily from the block decomposition of f.
Consider a chain map $\phi_\bullet$ on $E_\bullet$ that is homotopic to the identity.φ induces morphisms of the following short exact sequences

$$0 \to f_{i-1}(E_{i-1}) \to E_i \to E_i/F_{i-1}(E_{i-1}) \to 0$$

$$0 \to \ker f_i/f_{i-1}(E_{i-1}) \to E_i/f_{i-1}(E_{i-1}) \to E_i/\ker f_i \to 0$$

Using the above lemma on the above exact sequences together the fact that φ is a quasi isomorphism we easily get the result (that $\Pi \det(\varphi_{2i}) / \Pi \det(\varphi_{2i+1}) = 1$)
Now assume that L has a G invariant section Ω (which is the same as a holonomy invariant transversal measure)

using this we can define a trace on the convolution algebra $\mathcal{A}_G = \Gamma(G, r^* \bigwedge^{\text{top}} A^*)$ The following defines a trace:If Ω is represented as $\sum \omega \otimes \Lambda_A$ then:

$$\tau(a) = \int_M a(1_x)(\Lambda_{Ax})\omega_x$$

Proposition: τ defines a trace.
Proof:

$$a \circ b(\gamma)[\alpha_{r(\gamma)}] = \int_{G_s(\gamma)} a(\gamma h^{-1})[\alpha_{r(\gamma)}]b(h)[R_{\gamma^{-1}*}[-]] = \int_{G^r(\gamma)} a(h)[\alpha_{r(\gamma)}]b(h^{-1}\gamma)[R_{h*} \circ i_*[-]]$$

We have to prove that $\tau(a \circ b) = \tau(b \circ a)$ First we need to reduce to the case where a, b are supported in a small enough neighborhood of the diagonal.First suppose that a is supported in a neighborhood of γ and b in a neighborhood of γ^{-1} Then suppose that σ is a bisection over γ and that for η close to $1_{s(\gamma)}$,we denote a_0, b_0 the pullbacks by the translations given by the bisections(they are supported in a neighborhood U of 1_x).

$$b(L_\sigma\eta)[L_\sigma\alpha_{r(\eta)}] = a_0(\eta)[\alpha_{r(\eta)}] \tag{2.1}$$
$$a(R_{\sigma^{-1}}\eta)[\alpha_{r(\eta)}] = b_0(\eta)[\alpha_{r(\eta)}] \tag{2.2}$$

Now $\tau(a \circ b)$ is given by :

$$\tau(a \circ b) = \sum \int_M \int_{G_x} a(h^{-1})[\Lambda_{Ax}]b(h)\omega_x = \sum \int_G a(h^{-1})[\Lambda_{As(h)}]b(h) \wedge s^*\omega$$

Where b is extended arbitrarily to a form on G the wedge product with $s^*\omega$ which vanishes on $\ker(ds)$ is unambiguously defined.
For the expression of $\tau(b \circ a)$ we will use the pullback of the above by the inversion $i : G \to G$ and the final expression is going to look like

$$\sum \int_G b(h)[\Lambda_{Ar(h)}]i^*a(h) \wedge r^*\omega$$

Now in both cases we are going to pullback the above integrals by the left translation by the bisection σ :L_σ and we will actually get that $\tau(a \circ b) = \tau(a_0 \circ b_0)$ and $\tau(b \circ a) = \tau(b_0 \circ a_0)$:

$$\tau(a \circ b) = \sum \int_U a(R_{\sigma^{-1}}\eta^{-1})[\Lambda_{As(\eta)}]L_\sigma^*b(h) \wedge s^*\omega = \sum \int_U a_0(\eta^{-1})[\Lambda_{As(\eta)}]b_0(\eta) \wedge s^*\omega = \tau(a_0 \circ b_0)$$

$$\tau(b \circ a) = \sum \int_U b(L_\sigma\eta)[\Lambda_{Ar(L_\sigma\eta)}]L_\sigma^*i^*a(\eta) \wedge L_\sigma^*r^*\omega = \sum \int_U b(L_\sigma\eta)[\Lambda_{Ar(L_\sigma\eta)}]i^*R_{\sigma^{-1}}^*a(\eta) \wedge r^*\delta_\sigma^*\omega$$
$$= \sum \int_U b_0(\eta)[\Lambda_{Ar(\eta)}]i^*a(\eta) \wedge r^*\omega = \tau(b_0 \circ a_0)$$

Where the last equality follows from the translational invariance of Ω ,δ_σ denotes the diffeomorphism associated to σ. Using partitions of unity argument it suffices to prove the statement for sections supported in a neighborhood of a point in the diagonal. Around that point we are going to use a diffeomorphism $A \to U$ (from an open neighborhood of A) using the exponential map and local sections of A around x_0 Let $(x_1, x_2, ..., x_n, t_1, ..., t_k) \to \exp_x(t_1Y_1 + ... + t_kY_k)$ be such coordinates:
Also represent Ω as $Y_1 \wedge ... \wedge Y_k \otimes f dx_1...dx_n$ then:

$$\tau(a \circ b) = \int_U a(h^{-1})[\Lambda_{As(h)}]b(h)[R_{h^{-1}}^*[-]] \wedge s^*\omega =$$

$$= \int a(\exp_x(t \cdot Y)^{-1})[\Lambda_{Ax}]b(\exp_x(t \cdot Y))[R_{h^{-1}}^*[-]] \wedge s^*\omega =$$

$$= \int a(\exp_x(t \cdot Y)^{-1})[Y_1 \wedge ... \wedge Y_k|_x]b(\exp_x(t \cdot Y))[Y_1 \wedge ... \wedge Y_k]f dx_1...dx_n dt_1...dt_k$$

$\tau(b \circ a)$ though more tedious of a computation gives the same thing due to the translational invariance of Ω.
Before we define the localized index we need to note that $C^p(G)$ (and $C_M^p(G)$ carry a cyclic structure given by : $\tau_k\phi(g_1, ..., g_k) = \phi((g_1g_2...g_k)^{-1}, g_1, ..., g_{k-1})$ The corresponding cyclic cohomology is denoted by $HC^p(G)$. Given this trace the we can define a chain map of cocyclic modules that gives a map from lie groupoid cyclic cohomology to cyclic cohomology of the convolution algebra

$$\chi_p : C^p(G) \to \text{Hom}(\mathcal{A}_G^{\otimes(p+1)}, \mathbb{C})$$

$$\chi_p(\phi)(a_0 \otimes ... \otimes a_k) = \int_M \left(\int_{g_0 g_1 ... g_k = 1_x} \phi(g_1, ..., g_k) a_0(g_0) ... a_k(g_k) \right)$$

. As suggested by the connes moscovici index theorem we will take localized K theory of the algebra $\mathcal{A}$ and using the chern charachter as well as the isomorphism of lie algebroid cohomology with groupoid cohomology we will get a pairing

$$HC^*(A) \times K_0^{\mathrm{loc}}(A) \to \mathbb{C}$$

. As noted in the quantization procedure for symbols on groupoids the support of the kernel can be localized around the diagonal.In other words an elliptic pseudodifferential operatr on a groupoid defines (exactly as done in connes moscovici) a localized index class in $K_0^{\mathrm{loc}} A$ the pairing of this localized index class with a lie algebroid cohomology class can be given by a topological formula involving the symbol of the operator.This formula generalizes several index theorems ,for foliations for group actions etc. Details can be found in [38].

2.5 Appendix

Proof of exactness of: $0 \to C^*(B_M) \xrightarrow{\iota_*} C^*(\mathbb{T}M) \xrightarrow{ev_{0*}} C^*(\mathbf{T}M) \to 0$
(1) exactness at $C^*(\mathbb{T}M)$: First consider the situation locally and take an arbitrary section $f \in C_c(\mathbb{T}U))$ (where U is a convex open subset of $\mathbb{R}^n$) that is represented by kernels k, K as above. Now it is easy to see that the correspondence between symbols and kernels has an inverse in this case.Namely there is a family of symbols p_t such that $p_t(x, t\xi)$ and $p_0(x_0, \xi)$ represent f.They are explicitly given by:

$$p_t(x,\xi) = \int K(x, x+tZ, t) e^{i\langle Z,\xi\rangle} dZ = \int g(x+tZ, -Z, t) e^{i\langle Z,\xi\rangle} dZ$$

$$p_0(x,\xi) = \int k(x, -Z) e^{i\langle Z,\xi\rangle} dZ = \int g(x, -Z, 0) e^{i\langle Z,\xi\rangle} dZ$$

Where $g(x, Z, t)$ is compactly supported with respect to x, Z then it is easy to see that $p_t \to p_0$ in Sym^0 therefore Lemma 0 immediately gives that $||Op_{p_t(x,t\xi)} - Op_{p_0(x,t\xi)}||_{L_2 \to L_2} \to 0$ as $t \to 0$.
We also know from Lemma 2 that the class of $Op_{p_0(x,t\xi)}$ in $\mathcal{K}_\infty / \mathcal{K}_0$ has norm bounded by $\sup |p_0(x,\xi)|$. But the sup norm of p_0 is just $\sup_{x\in M} ||\pi_{(x,0)}(f)||$

All these suggest that the inequality

$$\sup_{x\in M} ||\pi_{(x,0)}(f)|| \geq \limsup_{t\to 0} ||\pi_{(x,t)}(f)||$$

is true , so having proved this locally in order to extend it globally we use subtle partition of unity arguments. For each $x \in M$ find a finite number of open neighborhoods $x \in V_x$ and $V_x \in U_x^i$ such that each U_x^i is an open chart mapped to a convex subset of $\mathbb{R}^n$ and the U_x^i cover M Then cover M by finite such open neighborhoods V_j and take a partition of unity ψ_j subordinate to them,use it to break down any element of $L_2(M)$.
Then we know that the open subsets $\mathbb{T}U_j^i$ of $\mathbb{T}M$ cover it's closed subspace $s^{-1}(\bar{V}_j)$ so take a partition of unity ϕ_j^i and ϕ_j on $\mathbb{T}M$ subordinate to $\mathbb{T}U_j^i$ and $\mathbb{T}M - s^{-1}(\bar{V}_j)$ Now for any $f \in C_c(\mathbb{T}M, \Omega^{1/2}(\ker ds) \otimes \Omega^{1/2}(\ker dt))$ break down $\pi(f)$:

$$\pi(f) = \sum \pi(\phi_j^i f)\psi_j + \pi(\phi_j f)\psi_j$$

Since $\pi(\phi_j f)\psi_j = 0$ and $\pi(\phi_j^i f)\psi_j$ reduces locally. We have proved that

$$C \sup_{x\in M} ||\pi_{(x,0)}(f)|| \geq \limsup_{t\to 0} ||\pi_{(x,t)}(f)||$$

globally. This inequality gets passed of course to $f \in C^*(\mathbb{T}M)$ and we can finally conclude that $\ker(ev_{0*}) = \mathcal{K}_0 = C^*(B_M)$.
(2)exactness at $C^*(\mathbf{T}M)$:
For an open coordinate chart consider the composite map:

$$C_c(T^*U) \to C^*(\mathbb{T}U) \xrightarrow{\text{extension by 0}} C^*(\mathbb{T}M) \xrightarrow{ev_{0*}} C^*(\mathbf{T}M) \cong C_0(T^*M)$$

It is equal to the map $C_c(T^*U) \xrightarrow{\text{extension by 0}} C_0(T^*M)$. Therefore the image of $C^*(\mathbb{T}M) \xrightarrow{ev_{0*}} C^*(\mathbf{T}M) \cong C_0(T^*M)$ contains the dense subspace $C_c(T^*M)$ and this together with the standard fact that $*-$morphisms of C^* algebras have closed range give that it is surjective.

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