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Master thesis

Quantum Field Theories with Aristotelian Symmetry

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Abstract

In field theories conserved dipole moment can arise from global dipole symmetry, which is interpreted and as a higher moment generalization of global phase symmetry $U(1)$. In this thesis we will first see various symmetries, like dipole symmetry, spatial rotations etc., and will derive the conserved currents and the algebra they satisfy. Afterwards, we will study the gauge symmetries of these transformations in the background of Aristotelian Geometry, which describes the geometry of absolute time and space. Equipped with the aforementioned elements, we will examine the behavior of quantum anomalies with guide tools like the Wess – Zumino consistency conditions.

Περίληψη

Σε θεωρίες πεδίου η διατηρούμενη διπολική ροπή μπορεί να προκύψει από καθολική διπολική συμμετρία, που ερμηνεύεται και ως υψηλότερης ροπής γενίκευση της καθολικής συμμετρίας φάσης $U(1)$. Σε αυτή την εργασία θα δούμε αρχικά διάφορες συμμετρίες, όπως η διπολική, οι χωρικές στροφές κλπ., και θα εξάγουμε τα διατηρούμενα ρεύματα και την άλγεβρα που ικανοποιούν. Στη συνέχεια, θα μελετήσουμε τις συμμετρίες βαθμίδας των μετασχηματισμών αυτών σε υπόβαθρο Αριστοτέλειας Γεωμετρίας, η οποία περιγράφει την γεωμετρία του απόλυτου χρόνου και χώρου. Εφοδιασμένοι με τα προαναφερθέντα στοιχεία, θα εξετάσουμε την συμπεριφορά των κβαντικών ανωμαλιών με οδηγό εργαλείων όπως οι συνθήκες συνέπειας Wess – Zumino.

Contents

Chapter 1

Introduction to Dipole Symmetry

1.1 Noether's Theorem

In this thesis the central object of our attention will be *fractons*, quasiparticles that represent a new hypothetical phase of matter . Fractons are characterized by their peculiar quantum behavior of having restricted mobility. For further elaboration on the physical properties of fractons see [1, 2, 3, 4]. The features of this emergent topological quasiparticle can have physical implications to a wide spectrum of research areas, from elasticity [5, 6, 7, 8, 9], hydrodynamics [10, 11, 12, 13, 14, 15], phase transitions [16, 17, 18, 19, 20, 21, 22] and quantum information [2, 23, 24, 25] to quantum field theory [26, 27, 28, 29, 30, 31], gravitation [32, 33, 34, 35, 36, 37, 38, 39, 40] and holography [41, 42, 43, 44, 45].

It can be seen that in some theories this limited mobility of isolated fracton particles is equivalent to the conservation of their dipole moment. To see this heuristically, note that for a point particle at $\vec{x}(t)$, with charge q and dipole moment $\vec{d}(t) = q\vec{x}(t)$, conservation of dipole moment $\dot{\vec{d}}(t) = 0$ is the same as $\dot{\vec{x}}(t) = 0$. For more details on this see [46, 47, 48, 49] and for a broad review on fractons see [50, 51, 52]. To study the quantum features of fractons, we will need to encode the *conservation of dipole moment* in a field-theoretic manner. This is done through the use of Noether's Theorem that connects conserved quantities with corresponding symmetries. For more details on Noether's Theorem see [53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63].

This chapter will be heavily based on [64], focusing on a generic complex scalar field $\phi = \phi(x)$, with $x \equiv (t, \vec{x})$, and some real Lagrangian (density)

$$\mathcal{L} = \mathcal{L}(x, \phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, c.c.),$$

where $c.c.$ means the complex conjugate of the preceding terms inside the parenthesis. Let us review the basic theoretical concepts we will need to move forward. We call a variation of the field $\phi(x)$ a class of functions $\phi_\epsilon(x)$ such that $\phi_{\epsilon=0}(x) = \phi(x)$ and we will work to first order in ϵ, sometimes called "infinitesimal" order, so we write

$$\phi_\epsilon(x) = \phi(x) + \epsilon\delta\phi(x) + \mathcal{O}(\epsilon^2)$$

by denoting

$$\delta\phi(x) \equiv \left.\frac{\partial\phi_\epsilon(x)}{\partial\epsilon}\right|_{\epsilon=0}.$$

We would like to write other ϵ-dependent quantities in a similar manner. For a variation $\phi_\epsilon(x)$ of the field, the Lagrangian

$$\mathcal{L}(x) \equiv \mathcal{L}(x, \phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), c.c.)$$

is also varied to

$$\mathcal{L}_\epsilon(x) \equiv \mathcal{L}(x, \phi_\epsilon(x), \partial_\mu\phi_\epsilon(x), \partial_\mu\partial_\nu\phi_\epsilon(x), c.c.)$$

and written as

$$\mathcal{L}_\epsilon(x) = \mathcal{L}(x) + \epsilon\delta\mathcal{L}(x) + \mathcal{O}(\epsilon^2)$$

with the infinitesimal change, to first order in ϵ, being represented again by

$$\delta\mathcal{L}(x) \equiv \left.\frac{\partial\mathcal{L}_\epsilon(x)}{\partial\epsilon}\right|_{\epsilon=0}.$$

For a real Lagrangian $\mathcal{L} = \mathcal{L}(x, \phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, c.c.)$, using[1]

$$\left(\frac{\partial\mathcal{L}}{\partial\phi}\right)^* = \frac{\partial\mathcal{L}}{\partial\phi^*}, \quad \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)^* = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}, \quad \text{etc.,}$$

changes to first order in ϵ by

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\phi\right)}\partial_\mu\left(\delta\phi\right) + \frac{\partial\mathcal{L}}{\partial\left(\partial_\mu\partial_\nu\phi\right)}\partial_\mu\partial_\nu\left(\delta\phi\right) + c.c. \tag{1.1}$$

[1]Let $\mathcal{L}(z, w)$ be analytic for both $z, w \in \mathbb{C}$. Define $\tilde{\mathcal{L}}(\operatorname{Re} z, \operatorname{Im} z) = \mathcal{L}(z, z^*)\ \ \forall z \in \mathbb{C}$ (see Section 6.1 of [58]). Then we can show that

$$\frac{\partial\mathcal{L}(z,z^*)}{\partial z} = \frac{1}{2}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\operatorname{Re} z} - i\frac{\partial\tilde{\mathcal{L}}}{\partial\operatorname{Im} z}\right)$$
$$\frac{\partial\mathcal{L}(z,z^*)}{\partial z^*} = \frac{1}{2}\left(\frac{\partial\tilde{\mathcal{L}}}{\partial\operatorname{Re} z} + i\frac{\partial\tilde{\mathcal{L}}}{\partial\operatorname{Im} z}\right)$$

If $\tilde{\mathcal{L}} \in \mathbb{R}$, then

$$\left(\frac{\partial\mathcal{L}(z,z^*)}{\partial z}\right)^* = \frac{\partial\mathcal{L}(z,z^*)}{\partial z^*}.$$

Then, through the relation[2]

$$\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi\right)}=\frac{\partial \mathcal{L}}{\partial\left(\partial_\nu \partial_\mu \phi\right)},$$

it can be seen that

$$\begin{aligned} \delta \mathcal{L}= & {\left[\frac{\partial \mathcal{L}}{\partial \phi}-\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}+\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi\right)}\right] \delta \phi } \\ & +\partial_\mu\left[\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)} \delta \phi+\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi\right)} \partial_\nu(\delta \phi)-\partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi\right)} \delta \phi\right]+c.c. \end{aligned} \tag{1.2}$$

Similarly to the above, the action functional

$$S[\phi] \equiv S[\operatorname{Re} \phi, \operatorname{Re} \phi] \equiv S\left[\phi, \phi^*\right]=\int_R d^{d+1} x \, \mathcal{L}(x)\left[\phi, \phi^*\right],$$

of the complex scalar ϕ under an arbitrary variation $\phi_\epsilon(x)$ of the field, also changes from $S \equiv S[\phi, \phi^*]$ to $S_\epsilon \equiv S_\epsilon[\phi, \phi^*] \equiv S[\phi_\epsilon, \phi_\epsilon^*]$ and we write

$$S_\epsilon=S+\epsilon \delta S+\mathcal{O}\left(\epsilon^2\right),$$

with infinitesimal change

$$\delta S \equiv\left.\frac{\partial S_\epsilon}{\partial \epsilon}\right|_{\epsilon=0}.$$

Using the concept of the functional derivative (see [54, 57, 53, 55, 65]) we can easily see that

$$\delta S=\int d^{d+1} x\left(\frac{\delta S}{\delta \phi(x)} \delta \phi(x)+\frac{\delta S}{\delta \phi^*(x)} \delta \phi^*(x)\right)=\int d^{d+1} x \, \delta \mathcal{L}(x).$$

Taking variations $\delta\phi$ that vanish on the boundary of the region of integration R and using 1.1 we get[3]

$$\begin{aligned} \frac{\delta S}{\delta \phi} & =\frac{\partial \mathcal{L}}{\partial \phi}-\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi\right)}+\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi\right)} \\ \frac{\delta S}{\delta \phi^*} & =\frac{\partial \mathcal{L}}{\partial \phi^*}-\partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi^*\right)}+\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \partial_\nu \phi^*\right)} \end{aligned} \tag{1.3}$$

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[2] Let $\bar{\mathcal{L}}=\bar{\mathcal{L}}(A_{\mu\nu})$ and $A_{(\mu\nu)}=\frac{1}{2}\left(A_{\mu\nu}+A_{\nu\mu}\right)$. If we define $\mathcal{L}(A_{\mu\nu})=\bar{\mathcal{L}}(A_{(\mu\nu)})$, we can show that

$$\frac{\partial \mathcal{L}}{\partial A_{\mu\nu}}=\frac{\partial \mathcal{L}}{\partial A_{\nu\mu}}.$$

[3] For complex scalar fields ϕ, since the Lagrangian depends on both the field ϕ and its complex conjugate ϕ^* separately, to calculate the functional derivative of the action, we need to choose two types of variations of ϕ. The first type should be such that $\delta\phi=\delta\phi^*$, while the second should be chosen such that $\delta\phi=-\delta\phi^*$. The logic is similar to that in footnote 2 and we can see that everything works well as if we had two totally independent fields ϕ and ϕ^* in the Lagrangian $\mathcal{L}$. This means that ϕ and ϕ^* can actually be treated as independent fields as far as matters of the action and the Lagrangian are concerned. For more details, see Section 6.1 of [58].

The fields ϕ that extremize the action, meaning $\delta S = 0$ for any variation $\delta\phi$ or, equivalently,

$$\frac{\delta S[\phi,\phi^*]}{\delta\phi} = \frac{\delta S[\phi,\phi^*]}{\delta\phi^*} = 0,$$

are usually said to be *on-shell* and they are *off-shell* otherwise. So for on-shell fields ϕ we get the following equations

$$\begin{aligned}\frac{\delta S}{\delta\phi} &= \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)} = 0\\ \frac{\delta S}{\delta\phi^*} &= \frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi^*)} = 0\end{aligned}$$

known by everyone as Euler-Lagrange equations or equations of motion.

In using Noether's Theorem to link symmetries to conserved currents, we will work with specific transformations where the spacetime point x and the field ϕ change simultaneously as one transformation. In particular, we take $x' = x'(x)$ and $\phi' = \phi'(x)$, where the function ϕ' is usually declared in the form $\phi'(x') = \mathcal{T}[\phi](x)$ with $\mathcal{T}[\phi]$ some functional of ϕ. We will also assume that the inverses $x(x')$ and $\mathcal{T}^{-1}[\phi]$ exist[4]. Obviously, to get the transformation $\phi'(x)$ we use the inverse transformation $x = x(x')$ and obtain $\phi'(x') = \mathcal{T}[\phi](x(x'))$. We usually see the total transformation being denoted as $x \to x'$, $\phi(x) \to \phi'(x')$. For a Lagrangian $\mathcal{L} = \mathcal{L}(x, \phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, c.c.)$ the transformation $x \to x'$, $\phi(x) \to \phi'(x')$ induces a corresponding transformation to the Lagrangian $\mathcal{L}(x) \to \mathcal{L}'(x')$ given by

$$\mathcal{L}'(x', \phi'(x'), \partial_\mu\phi'(x'), \partial_\mu\partial_\nu\phi'(x'), c.c.) \equiv J(x,x')\,\mathcal{L}(x, \phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), c.c.) \tag{1.4}$$

with

$$J(x,x') = \left|\det\left(\frac{\partial x}{\partial x'}\right)\right|$$

the Jacobian of the transformation $x = x(x')$. To identify the transformed Lagrangian $\mathcal{L}'$ explicitly we write the above expression with respect to x' and ϕ'. This is obviously done using the inverses $x(x')$ and $\mathcal{T}^{-1}[\phi]$. Trying to write ϕ in terms of ϕ' we will find that

$$\phi(x) = \mathcal{T}^{-1}[\phi' \circ x'](x).$$

The transformed action is defined as

$$S'[\phi', \phi'^*] \equiv \int_{R'} d^{d+1}x'\,\mathcal{L}'(x')[\phi', \phi'^*],$$

where $R' = x'(R)$, and we can see that

$$S'[\phi', \phi'^*] = S[\phi, \phi^*].$$

[4]This means that $x'(x(x')) = x' \quad \forall x'$ and $\mathcal{T}^{-1}[\mathcal{T}[\phi]] = \phi \quad \forall\phi$.

This means that if ϕ is a field that extremizes the action S with Lagrangian $\mathcal{L}$, then the corresponding transformed field ϕ' will extremize the action S' with Lagrangian $\mathcal{L}'$. We will call the transformation $x \to x'$, $\phi(x) \to \phi'(x')$ a *symmetry* of the action S or the Lagrangian $\mathcal{L}$ (or even a symmetry of our "physical theory") if the functional form of S' is "the same as that of S" meaning

$$S'[\phi', \phi'^*] = S[\phi', \phi'^*] \equiv \int_{R'} d^{d+1}x' \, \mathcal{L}(x')[\phi', \phi'^*]$$

for any original region R, which reduces to the condition

$$\mathcal{L}' = \mathcal{L}.$$

We see that for a symmetry transformation if ϕ satisfies the Euler-Lagrange equations with Lagrangian $\mathcal{L}$, then ϕ' will satisfy the same equations of motion with the same Lagrangian $\mathcal{L}$.

A continuous transformation $x \to x'$, $\phi(x) \to \phi'(x')$ will have an *infinitesimal version* of the form $x'_\epsilon(x)$, $\phi'_\epsilon(x') = \mathcal{T}_\epsilon[\phi](x_\epsilon(x'))$, where the ϵ generates the "infinitesimal" part of the transformation. Working again to infinitesimal order we have

$$x'_\epsilon(x) = x + \epsilon\xi(x) + \mathcal{O}(\epsilon^2), \tag{1.5}$$

where $\xi(x) \equiv \partial x'_\epsilon(x)/\partial\epsilon|_{\epsilon=0}$ (for the above equation note that, by definition, $x'_{\epsilon=0}(x) = x$). For the inverse transformation $x_\epsilon(x')$, we can easily show that

$$x_\epsilon(x') = x' - \epsilon\xi(x') + \mathcal{O}(\epsilon^2).$$

In addition, if we write

$$\mathcal{T}_\epsilon[\phi](x) = \phi(x) + \epsilon\mathcal{K}[\phi](x) + \mathcal{O}(\epsilon^2)$$

(note again $\mathcal{T}_{\epsilon=0}[\phi](x) = \phi(x)$), we can find that

$$\mathcal{T}_\epsilon^{-1}[\phi](x) = \phi(x) - \epsilon\mathcal{K}[\phi](x) + \mathcal{O}(\epsilon^2).$$

For the transformed field ϕ'_ϵ, we write

$$\phi'_\epsilon(x) = \phi(x) + \epsilon\delta\phi(x) + \mathcal{O}(\epsilon^2), \tag{1.6}$$

where, of course,

$$\delta\phi(x) \equiv \partial\phi'_\epsilon(x)/\partial\epsilon|_{\epsilon=0} = -\xi^\mu(x)\partial_\mu\phi(x) + \mathcal{K}[\phi](x) \tag{1.7}$$

(again, by definition, $\phi'_{\epsilon=0}(x) = \phi(x)$), while for the inverse transformation, $\phi_\epsilon(x) = \mathcal{T}_\epsilon^{-1}[\phi' \circ x'_\epsilon](x)$ for any arbitrary function $\phi'(x)$, we can prove that

$$\phi_\epsilon(x) = \phi'(x) + \epsilon\left[\xi^\mu(x)\partial_\mu\phi'(x) - \mathcal{K}[\phi'](x)\right] + \mathcal{O}(\epsilon^2).$$

In a similar way to the above equations, since $\phi'_\epsilon(x)$ is a variation of the original field $\phi(x)$, for $\mathcal{L}_\epsilon(x) \equiv \mathcal{L}(x, \phi'_\epsilon(x), \partial_\mu\phi'_\epsilon(x), \partial_\mu\partial_\nu\phi'_\epsilon(x), c.c.)$ we will again write

$$\mathcal{L}_\epsilon(x) = \mathcal{L}(x) + \epsilon\delta\mathcal{L}(x) + \mathcal{O}(\epsilon^2)$$

with $\delta\mathcal{L}(x)$ being given by 1.2, where we set 1.7 in $\delta\phi(x)$. We also have for the transformed action

$$S_\epsilon \equiv \int_{R'_\epsilon} d^{d+1}x'\, \mathcal{L}_\epsilon(x'),$$

where $R'_\epsilon = x'_\epsilon(R)$, the expansion

$$S_\epsilon = S + \epsilon\delta S + \mathcal{O}(\epsilon^2)$$

with

$$\delta S = \int_R d^{d+1}x\, \left[\partial_\mu(\xi^\mu(x)\mathcal{L}(x)) + \delta\mathcal{L}(x)\right].$$

Using the relation

$$\det(I + \epsilon A) = 1 + \epsilon\,\mathrm{tr}(A) + \mathcal{O}(\epsilon^2),$$

which is true for any $n \times n$ matrix A, we get

$$J(x, x') = \left|\det\left(\frac{\partial x}{\partial x'}\right)\right| = 1 - \epsilon\partial'_\mu\xi^\mu(x') + \mathcal{O}(\epsilon^2)$$

for small enough ϵ. This equation together with the definition 1.4 gives us for $\mathcal{L}'_\epsilon(x) \equiv \mathcal{L}'_\epsilon(x, \phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), c.c.)$ the expansion

$$\mathcal{L}'_\epsilon(x) = \mathcal{L}(x) - \epsilon[\partial_\mu(\xi^\mu(x)\mathcal{L}(x)) + \delta\mathcal{L}(x)] + \mathcal{O}(\epsilon^2).$$

If our transformation is a symmetry, then

$$\mathcal{L}'_\epsilon(x) = \mathcal{L}(x) \quad \forall\epsilon$$

or, an equivalent condition,

$$\delta\mathcal{L} = -\partial_\mu(\xi^\mu\mathcal{L}). \tag{1.8}$$

For a symmetry we also have $S_\epsilon = S$, so clearly $\delta S = 0$ for any region R, which is the same as condition 1.8. Now we define the Noether current of a symmetry transformation by

$$j^\mu \equiv \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\partial_\nu(\delta\phi) - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi + c.c.\right] + \xi^\mu\mathcal{L}. \tag{1.9}$$

Using 1.3 and definition 1.9, equation 1.2 becomes

$$\delta\mathcal{L} = \left(\frac{\delta S}{\delta\phi}\delta\phi + c.c.\right) + \partial_\mu j^\mu - \partial_\mu(\xi^\mu\mathcal{L}). \tag{1.10}$$

For a symmetry transformation, which satisfies 1.8, the above gives us the following relationship

$$\partial_\mu j^\mu = -\frac{\delta S}{\delta\phi}\delta\phi + c.c. \tag{1.11}$$

Hence, for symmetry transformations of on-shell fields ϕ equation 1.11 proves the conservation of the Noether current

$$\partial_\mu j^\mu = 0 \quad \text{on-shell.} \tag{1.12}$$

Note that the conservation law derived in this section, comprising the well-known Noether's Theorem, holds for fields that are on-shell only. Also the Noether current can actually be defined up to a constant multiplicative factor. This is important, because we will define the corresponding Noether charge of a symmetry

$$Q = \int d^dx\, j^0,$$

which is determined up to a factor too and is conserved on-shell

$$\frac{dQ}{dt} = 0 \quad \text{on-shell}$$

as well. So the "algebra" between charges of different symmetries will be specified up to multiplicative factors as well.

There is an alternative way to prove Noether's Theorem, that is of great practical and conceptual interest in many applications and will be important to us later. We start by taking the transformation $x'_\epsilon(x)$ and $\phi'_\epsilon(x') = \mathcal{T}_\epsilon[\phi](x_\epsilon(x'))$, which we call *global* transformation, and create a new *local* transformation, by the substitution $\epsilon \to \epsilon\,\eta(x)$, where $\eta = \eta(x)$ is a "well-behaved" real function of x. In particular, the transformations $x'_\epsilon(x)$ and $\mathcal{T}_\epsilon[\phi](x)$ become

$$x'^{\text{local}}_\epsilon(x) \equiv x'_{\epsilon\,\eta(x)}(x)$$
$$\mathcal{T}^{\text{local}}_\epsilon[\phi](x) \equiv \mathcal{T}_{\epsilon\,\eta(x)}[\phi](x).$$

Now the substitution $\epsilon \to \epsilon\,\eta(x)$ induces the resulting substitutions $\xi(x) \to \eta(x)\,\xi(x)$ and $\mathcal{K}[\phi](x) \to \eta(x)\,\mathcal{K}[\phi](x)$, so $\delta\phi(x) \to \eta(x)\,\delta\phi(x)$. The distinction between global and local transformations will be important. We will also need to define the auxiliary quantity

$$j^{\mu\nu} = -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi + c.c. \tag{1.13}$$

which is obviously symmetric, i.e. $j^{\mu\nu} = j^{\nu\mu}$. For a global transformation the infinitesimal change (to first order) of the Lagrangian was given by 1.2, according to which we have for the local transformation

$$\delta\mathcal{L}_{\text{local}} = \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\right]\delta\phi_{\text{local}}$$
$$+ \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi_{\text{local}} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\partial_\nu(\delta\phi_{\text{local}}) - \partial_\nu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu\phi)}\delta\phi_{\text{local}}\right] + c.c.$$

and since $\delta\phi_{\text{local}} = \eta\,\delta\phi$ gives us

$$\delta\mathcal{L}_{\text{local}} = \eta\,\delta\mathcal{L} + \partial_\mu\eta\,(j^\mu - \xi^\mu\mathcal{L}) - \partial_\mu(\partial_\nu\eta\,j^{\mu\nu})$$

which holds for any function $\eta = \eta(x)$. Hence, for "appropriate" choices of $\eta(x)$, meaning such that it vanishes on the boundary of the region of integration R (localized on a small region), we get

$$\int_R d^{d+1}x\,\delta\mathcal{L}_{\text{local}} = \int_R d^{d+1}x\,\eta\,[\delta\mathcal{L} + \partial_\mu(\xi^\mu\mathcal{L} - j^\mu)]$$

We also know that

$$\delta S_{\text{local}} = \int_R d^{d+1}x\;[\partial_\mu(\xi^\mu_{\text{local}}\mathcal{L}) + \delta\mathcal{L}_{\text{local}}]$$

with $\xi_{\text{local}} = \eta\,\xi$, so

$$\delta S_{\text{local}} = \int_R d^{d+1}x\,\delta\mathcal{L}_{\text{local}} = \int_R d^{d+1}x\,\eta\,[\delta\mathcal{L} + \partial_\mu(\xi^\mu\mathcal{L} - j^\mu)]\,.$$

Thus we conclude that for the transformed action $S_{\epsilon=\epsilon(x)}$, where we converted ϵ from a constant to a function of x, the following identity holds

$$\left.\frac{\delta S_\epsilon}{\delta\epsilon}\right|_{\epsilon=0} = -\partial_\mu j^\mu + \delta\mathcal{L} + \partial_\mu(\xi^\mu\mathcal{L}).$$

According to 1.10 we also have

$$\left.\frac{\delta S_\epsilon}{\delta\epsilon}\right|_{\epsilon=0} = \frac{\delta S}{\delta\phi}\delta\phi + c.c.$$

Now if the action is symmetric with respect to the original global symmetry, i.e. $S_\epsilon = S \quad \forall\epsilon =$ constant, then relationship 1.8 is true and we get

$$\left.\frac{\delta S_\epsilon}{\delta\epsilon}\right|_{\epsilon=0} = -\partial_\mu j^\mu.$$

The above equation clearly reproduces the previous results 1.11 and 1.12.

1.2 Noether currents

It is time to put Noether's Theorem to use by calculating the Noether currents of various symmetries. We will study symmetry transformations for which there is a complex function $f = f(x)$ such that

$$\phi'_\epsilon(x'_\epsilon) = e^{\epsilon f(x)}\phi(x). \tag{1.14}$$

From eqs 1.5, 1.6 and 1.14 we get

$$\delta\phi(x) = -\xi^\mu(x)\partial_\mu\phi(x) + f(x)\phi(x)$$

Also those specific symmetries we impose on our Lagrangian will obey the equation

$$J(x', x) = 1,$$

so

$$\mathcal{L}'(x') = \mathcal{L}(x),$$

We will take our Lagrangian to be of the form

$$\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.) \tag{1.15}$$

where $\dot{\phi} \equiv \partial_t\phi$. Then the (Noether) currents for our Lagrangian will be

$$j^0 = \left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\delta\phi + c.c.\right] + \xi^0\mathcal{L} \tag{1.16a}$$

$$j^i = \left[\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\phi)}\partial_j(\delta\phi) - \partial_j\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\phi)}\delta\phi + c.c.\right] + \xi^i\mathcal{L}. \tag{1.16b}$$

The approach followed here is, of course, not manifestly covariant at the moment. For instance, the current $j^\mu = (j^0, j^i)$ is not a tensor quantity under general coordinate transformations anymore. This and similar issues will be remedied in Chapter 3 by putting our theory in a more general framework. The conservation law of the current $j^\mu = (j^0, j^i)$ is again

$$\partial_\mu j^\mu = \partial_t j^0 + \partial_i j^i = 0 \quad \text{on-shell.}$$

Note that the expression $\partial_\mu j^\mu$ is again non-covariant, but we will keep using it purely for notational convenience. That is we will continue to use the regular tensor calculus terminology and conventions as a book keeping device.

First is time translation

$$t \to t' = t + c, \quad x^i \to x'^i = x^i, \quad \phi'(x') = \phi(x).$$

We get the ϵ-form, or "infinitesimal" form, of the above transformation by setting $c = \epsilon$, giving us

$$t \to t' = t + \epsilon, \quad x^i \to x'^i = x^i, \quad \phi'(x') = \phi(x).$$

From the infinitesimal form of time translation we obtain for this symmetry

$$\xi^0 = 1, \quad \xi^i = 0, \quad f = 0.$$

For space translation

$$t \to t' = t, \quad x^i \to x'^i = x^i + a^i, \quad \phi'(x') = \phi(x)$$

we set $a^i = \epsilon\delta^i_k$, representing space translation in the k-direction, and get the infinitesimal form

$$t \to t' = t, \quad x^i \to x'^i = x^i + \epsilon\delta^i_k, \quad \phi'(x') = \phi(x),$$

which gives us

$$\xi^0 = 0, \quad \xi^i = \delta^i_k, \quad f = 0.$$

Putting it all together as spacetime translation we get

$$\xi^\mu_{(\nu)} = \delta^\mu_\nu, \quad f = 0$$

with $\nu = 0$ representing translation in the "time"-direction and $\nu = k$ representing translation in the spatial k-direction. The currents we get from spacetime translations form the energy-momemtum tensor ${T^\mu}_\nu$, where ${T^\mu}_0$ is the time translation current and ${T^\mu}_k$ is the current for space translation in the k-direction. Explicitly, we see that the energy-momentum tensor is

$$\begin{aligned}
{T^0}_\nu &= -\left[\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\partial_\nu\phi + c.c.\right] + \delta^0_\nu\mathcal{L} \\
{T^i}_\nu &= -\left[\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\phi)}\partial_j\partial_\nu\phi - \partial_j\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_j\phi)}\partial_\nu\phi + c.c.\right] + \delta^i_\nu\mathcal{L}
\end{aligned}$$

and its conservation is written as

$$\partial_\mu {T^\mu}_\nu = 0 \quad \text{on-shell.}$$

The next symmetry we look at is spatial rotation

$$t \to t' = t, \quad x^i \to x'^i = {R^i}_j x^i, \quad \phi'(x') = \phi(x).$$

To obtain the infinitesimal form of spatial rotation, we do the following steps. We write ${R^i}_j = {R^i}_j(\vec{a})$ for some vector $\vec{a} \in \mathbb{R}^d$, using the fact that rotation matrices belong to the Lie group $SO(d)$. Then we take a curve $\vec{a}(\epsilon) = \epsilon\vec{v} + \mathcal{O}(\epsilon^2)$ (connected to the identity point) and get

$${R^i}_j(\epsilon) \equiv {R^i}_j(\vec{a}(\epsilon)) = {R^i}_j(\epsilon\vec{v} + \mathcal{O}(\epsilon^2)) = {R^i}_j(\vec{0}) + \epsilon v^k\partial_k {R^i}_j(\vec{0}) + \mathcal{O}(\epsilon^2)$$

or just, in a simpler notation,

$${R^i}_j(\epsilon) = \delta^i_j - \epsilon{\Omega^i}_j + \mathcal{O}(\epsilon^2).$$

Since for any rotation matrix $R \in SO(d)$ it holds by definition that $R^T R = I$, or in index notation

$${R_i}^k \delta_{kl} {R^l}_j = \delta_{ij} \quad \Rightarrow \quad {R^k}_i {R^l}_j \delta_{kl} = \delta_{ij},$$

by setting $R = R(\epsilon)$ we find

$$\begin{aligned} R^k{}_i(\epsilon) R^l{}_j(\epsilon)\delta_{kl} = \delta_{ij} \quad &\Rightarrow \quad (\delta^k_i - \epsilon\Omega^k{}_i + \mathcal{O}(\epsilon^2))\,(\delta^l_j - \epsilon\Omega^l{}_j + \mathcal{O}(\epsilon^2))\,\delta_{kl} = \delta_{ij} \\ &\Rightarrow \quad \delta_{ij} - \epsilon(\Omega_{ij} + \Omega_{ji}) + \mathcal{O}(\epsilon^2) = \delta_{ij} \\ &\Rightarrow \quad \Omega_{(ij)} = 0. \end{aligned}$$

Note that we lowered the index of Ω using δ_{ij}, i.e. $\Omega_{ij} = \delta_{ik}\Omega^k{}_j$. In general, latin indices will be raised and lowered using δ^{ij} and δ_{ij}. The reason for this lies in the fact that our geometry of space for fractons is essentially Newtonian/Eucledian, a matter that will be developed in greater detail later. Now, given that the Ω's are antisymmetric, to get the infinitesimal form of a rotation in the k-l plane, we set $(\Omega^{kl})_{ij} = 2\delta^k_{[i}\delta^l_{j]} = \delta^k_i\delta^l_j - \delta^k_j\delta^l_i$, so our rotation takes becomes

$$\begin{aligned} (R^{kl})^i{}_j(\epsilon) &= \delta^i_j - \epsilon(\Omega^{kl})^i{}_j + \mathcal{O}(\epsilon^2) = \delta^i_j - \epsilon\delta^{im}(\Omega^{kl})_{mj} + \mathcal{O}(\epsilon^2) \\ &= \delta^i_j - \epsilon\delta^{im}(\delta^k_m\delta^l_j - \delta^k_j\delta^l_m) + \mathcal{O}(\epsilon^2) \\ &= \delta^i_j - \epsilon(\delta^{ki}\delta^l_j - \delta^k_j\delta^{li}) + \mathcal{O}(\epsilon^2) \end{aligned}$$

or

$$(R^{kl})^i{}_j(\epsilon) = \delta^i_j + \epsilon(\delta^k_j\delta^{li} - \delta^{ki}\delta^l_j) + \mathcal{O}(\epsilon^2)$$

Hence, the infinitesimal form of spatial rotation is

$$t \to t' = t, \quad x^i \to x'^i = x^i + \epsilon(x^k\delta^{li} - x^l\delta^{ki}) + \mathcal{O}(\epsilon^2), \quad \phi'(x') = \phi(x),$$

from which follows that

$$\xi^0 = 0, \quad \xi^i = (x^k\delta^{li} - x^l\delta^{ki}), \quad f = 0.$$

The current for spatial rotation in the k-l plane is calculated to be

$$\begin{aligned} J^0{}_{kl} &= x_k T^0_{\ l} - x_l T^0_{\ k} \\ J^i{}_{kl} &= x_k T^i_{\ l} - x_l T^i_{\ k} + \left[\partial_k\phi\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_l\phi)} - \partial_l\phi\frac{\partial\mathcal{L}}{\partial(\partial_i\partial_k\phi)} + c.c.\right] \end{aligned}$$

with conservation law

$$\partial_\mu J^\mu{}_{kl} = 0 \quad \text{on-shell.}$$

Now we study phase rotation

$$\phi'(x) = e^{i\alpha}\phi(x)$$

or, written in a similar way to the previous symmetries,

$$t \to t' = t, \quad x^i \to x'^i = x^i, \quad \phi'(x') = e^{i\alpha}\phi(x).$$

The phase rotation transformation is also called $U(1)$ transformation, because phase rotation is essentially spanned from the elements of the Lie group $U(1)$. To identify the infinitesimal form of the $U(1)$ transformation, we set $\alpha = \epsilon$ and get

$$t \to t' = t, \quad x^i \to x'^i = x^i, \quad \phi'(x') = e^{i\epsilon}\phi(x)$$

from which is apparent that

$$\xi^0 = 0, \quad \xi^i = 0, \quad f = i.$$

The $U(1)$ current is shown to be

$$\begin{aligned} J^0 &= i\phi\,\frac{\partial \mathcal{L}}{\partial \dot{\phi}} + c.c. \\ J^i &= i\phi\,\frac{\partial \mathcal{L}}{\partial(\partial_i \phi)} + i\partial_j\phi\,\frac{\partial \mathcal{L}}{\partial(\partial_i\partial_j\phi)} - i\phi\,\partial_j\frac{\partial \mathcal{L}}{\partial(\partial_i\partial_j\phi)} + c.c. \end{aligned} \tag{1.17}$$

with conservation law

$$\partial_\mu J^\mu = 0 \quad \text{on-shell.}$$

For later use, we need the following definition (see 1.13 where $\delta\phi = i\phi$)

$$\tilde{J}^{ij} = -i\phi\frac{\partial \mathcal{L}}{\partial(\partial_i\partial_j\phi)} + c.c. \tag{1.18}$$

with the property $\tilde{J}^{ij} = \tilde{J}^{ji}$.

Finally, we focus our attention to a more exotic kind of symmetry, the one that characterizes fractons in a fundamental level and so the "star of the show". The dipole transformation is defined by

$$\phi'(x) = e^{i\beta_i x^i}\phi(x)$$

or, using the familiar notation,

$$t \to t' = t, \quad x^i \to x'^i = x^i, \quad \phi'(x') = e^{i\beta_i x^i}\phi(x),$$

where, of course, $\beta_i \equiv \delta_{ij}\beta^j$ for some vector $\vec{\beta} \in \mathbb{R}^d$. After setting $\beta^i = \epsilon\delta^{ki}$ or, equivalently, $\beta_i = \epsilon\delta_i^k$, we get the infinitesimal form of dipole symmetry for the k-direction

$$t \to t' = t, \quad x^i \to x'^i = x^i, \quad \phi'(x') = e^{i\epsilon x^k}\phi(x),$$

from which results that

$$\xi^0 = 0, \quad \xi^i = 0, \quad f = ix^k.$$

Using the defined quantity 1.18, we can easily obtain the Noether current for dipole symmetry

$$\begin{aligned} J^{k0} &= x^k J^0 \\ J^{ki} &= x^k J^i - \tilde{J}^{ki}. \end{aligned} \tag{1.19}$$

with conservation of dipole current

$$\partial_\mu J^{k\mu} = 0 \quad \text{on-shell,}$$

Chapter 2

Theories with Dipole Symmetry

2.1 Lagrangians with Dipole Symmetry

We are very familiar with theories that are invariant under spacetime translation, spatial rotation (even spacetime/Lorentz rotation) and also $U(1)$ (phase) rotation but dipole symmetric theories are anything but abundant in the literature. So the purpose of this section will be to identify a class of real (as always) Lagrangians that exhibit the desired behavior of dipole symmetry. Again in this chapter we will lean closely to the approach laid out in [64] and will, of course, continue to study Lagrangians of the form 1.15.

We start by considering a Lagrangian that has $U(1)$ and dipole symmetry like in Section 1.2. For both of these symmetries we found that $\xi^\mu = 0$, so $\delta\mathcal{L} = 0$ and $\delta\phi = f\phi$. Putting these in 1.1 (mind the form 1.15) we have the (off-shell) relation

$$f\phi\frac{\partial\mathcal{L}}{\partial\phi} + \partial_t(f\phi)\frac{\partial\mathcal{L}}{\partial\dot{\phi}} + \partial_i(f\phi)\frac{\partial\mathcal{L}}{\partial\left(\partial_i\phi\right)} + \partial_i\partial_j(f\phi)\frac{\partial\mathcal{L}}{\partial\left(\partial_i\partial_j\phi\right)} + c.c. = 0.$$

For $U(1)$ symmetry, $f = i$ and we get

$$i\phi\frac{\partial\mathcal{L}}{\partial\phi} + i\dot{\phi}\frac{\partial\mathcal{L}}{\partial\dot{\phi}} + i\partial_i\phi\frac{\partial\mathcal{L}}{\partial\left(\partial_i\phi\right)} + i\partial_i\partial_j\phi\frac{\partial\mathcal{L}}{\partial\left(\partial_i\partial_j\phi\right)} + c.c. = 0, \tag{2.1}$$

while, for dipole symmetry, $f = ix^k$ and we find

$$i\phi\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)} + 2i\partial_j\phi\frac{\partial\mathcal{L}}{\partial\left(\partial_i\partial_j\phi\right)} + c.c. = 0. \tag{2.2}$$

Now using 1.17 and 1.19 we get (off-shell)

$$\partial_\mu J^{k\mu} = x^k\partial_\mu J^\mu + J^k - \partial_i\tilde{J}^{ki}.$$

Obviously, given that on-shell we have the conservation laws

$$\partial_\mu J^{k\mu} = \partial_\mu J^\mu = 0 \quad \text{on-shell},$$

we find the on-shell relationship

$$J^k = \partial_i \tilde{J}^{ki} \quad \text{on-shell}.$$

We can do even better than this. It can be seen through a bit of algebra that (off-shell)

$$J^k - \partial_i \tilde{J}^{ki} = i\phi \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)} + 2i\partial_i\phi \frac{\partial \mathcal{L}}{\partial\left(\partial_k \partial_i \phi\right)} + c.c.,$$

so for Lagrangians with dipole symmetry it is true that

$$J^k = \partial_i \tilde{J}^{ki}, \tag{2.3}$$

which now holds off-shell as well. We conclude that a Lagrangian that has $U(1)$ symmetry satisfies 2.1, while a Lagrangian with dipole symmetry satisfies 2.2, which can also be written more compactly as in 2.3.

We proceed by taking our Lagrangian to be a polynomial of the field and its derivatives. Most commonly, this is realized in the form

$$\mathcal{L} = \mathcal{K} - \mathcal{V}$$

with a kinetic term $\mathcal{K} = \mathcal{K}(\phi, \dot{\phi}, c.c.)$ and an interaction term $\mathcal{V} = \mathcal{V}(\phi, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$ containing no time derivatives of the field. The interaction term is usually of the form

$$\mathcal{V} = \mathcal{V}^{(0)} + \mathcal{V}^{(2)} + \mathcal{V}^{(4)} + \cdots,$$

where $\mathcal{V}^{(0)}$ is a real (obviously) function of $\phi^*\phi = |\phi|^2$, meaning

$$\mathcal{V}^{(0)} = \mathcal{V}^{(0)}(\phi^*\phi),$$

and the $\mathcal{V}^{(n)}$'s contain all the n-th order in spatial derivatives terms of the Lagrangian with no time derivatives in them. Next we should note that for theories with first order in time equations of motion (like the Schrödinger equation) the kinetic term is written as $\mathcal{K} = i\phi^*\dot{\phi} + c.c.$, while for theories second order in time (remember the Klein-Gordon equation) the kinetic term is $\mathcal{K} = \dot{\phi}^*\dot{\phi}$. It is clear that both $\mathcal{K}$ and $\mathcal{V}^{(0)}$ are by themselves real with $U(1)$ and dipole symmetry.

Remembering that we demand, of course, our Lagrangian to be real and $U(1)$ symmetric, a fairly general choice for $\mathcal{V}^{(2)}$ is

$$\mathcal{V}^{(2)} = a\phi^{*2}\partial^i\phi\partial_i\phi + b\phi^*\partial^i\partial_i\phi + c\partial^i\phi^*\partial_i\phi + c.c., \tag{2.4}$$

where a, b and c are complex functions of $\phi^*\phi$. If we expand the above expression fully we get

$$\mathcal{V}^{(2)} = (a\phi^{*2}\partial^i\phi\partial_i\phi + a^*\phi^2\partial^i\phi^*\partial_i\phi^*) + (b\phi^*\partial^i\partial_i\phi + b^*\phi\partial^i\partial_i\phi^*) + d\partial^i\phi^*\partial_i\phi,$$

where d is a real function of $\phi^*\phi$. Now, given that we want $\mathcal{V}^{(2)}$ to be dipole symmetric, we essentially impose on it the condition 2.2. Putting the above equation in 2.2 by setting $\mathcal{L} = \mathcal{V}^{(2)}$ we find the following constraint

$$2ia|\phi|^2\phi^*\partial_i\phi + 2ib\phi^*\partial_i\phi + id\partial_i\phi^*\phi + c.c. = 0.$$

By expanding this equation we can see that it can be rewritten in an equivalent way as

$$(2ia|\phi|^2 + 2ib - id)\phi^*\partial_i\phi + c.c. = 0.$$

We can extract a solution to this constraint equation easily by taking the first term to vanish, which gives us

$$b = -a|\phi|^2 + \frac{d}{2},$$

so

$$\mathcal{V}^{(2)} = [a\phi^{*2}(\partial^i\phi\partial_i\phi - \phi\partial^i\partial_i\phi) + c.c.] + d(\partial^i\phi^*\partial_i\phi + \frac{1}{2}\phi^*\partial^i\partial_i\phi + \frac{1}{2}\phi\partial^i\partial_i\phi^*).$$

For this solution, after defining the new quantities

$$X_{ij} \equiv \partial_i\phi\partial_j\phi - \phi\partial_i\partial_j\phi$$
$$Y_{ij} \equiv \partial_i\phi^*\partial_j\phi + \phi^*\partial_i\partial_j\phi$$

and renaming $d \to 2b$, the Lagrangian term 2.4 becomes

$$\mathcal{V}^{(2)} = (a\phi^{*2}X^i{}_i + c.c.) + b(Y^i{}_i + c.c.)$$

with a a complex and b a real function of $\phi^*\phi$. We can, indeed, check that the term $\mathcal{V}^{(2)}$ is dipole symmetric. By doing a (finite) dipole transformation

$$\phi \to e^{i\beta_i x^i}\phi$$

we find

$$X_{ij} \to e^{i2\beta_i x^i}X_{ij}$$
$$Y_{ij} \to Y_{ij} + i\beta_j(\phi^*\partial_i\phi + c.c.),$$

which leads to

$$\mathcal{V}^{(2)} \to \mathcal{V}^{(2)}.$$

Using similar techniques to the above we could find a general form for the Lagrangian term $\mathcal{V}^{(4)}$. However, we will not dedicate our efforts to finding a general choice for $\mathcal{V}^{(4)}$, but we will use the helpful mathematical object

$$X_{ij} = \partial_i\phi\partial_j\phi - \phi\partial_i\partial_j\phi$$

we have already defined and its transformation under a dipole transformation

$$X_{ij} \to e^{i2\beta_i x^i} X_{ij}$$

to construct a dipole symmetric term that is 4th order in spatial derivatives (and no time derivatives)

$$\mathcal{V}^{(4)} = \kappa (X^{ij})^* X_{ij} + \lambda (X^i{}_i)^* X^j{}_j$$

with κ and λ real constants.

If we take $\mathcal{K} = i\phi^*\dot{\phi} + c.c.$, $\mathcal{V}^{(0)} = m^2|\phi|^2$, $\mathcal{V}^{(2)} = 0$ and $\mathcal{V}^{(4)} = \kappa(X^{ij})^*X_{ij} + \lambda(X^i{}_i)^*X^j{}_j$ we get a Lagrangian of the form

$$\mathcal{L} = (i\phi^*\dot{\phi} + c.c.) - m^2|\phi|^2 - \kappa(X^{ij})^*X_{ij} - \lambda(X^i{}_i)^*X^j{}_j$$

and for $\mathcal{K} = \dot{\phi}^*\dot{\phi}$

$$\mathcal{L} = \dot{\phi}^*\dot{\phi} - m^2|\phi|^2 - \kappa(X^{ij})^*X_{ij} - \lambda(X^i{}_i)^*X^j{}_j,$$

where m is the mass parameter of our scalar field ϕ. Similar expressions have appeared for field theory descriptions of fractons in [50, 66, 13, 67, 64].

2.2 Symmetry Algebra

In this section we will calculate the symmetry algebra of our symmetries. Given a Lagrangian $\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$ we can define the conjugate momenta

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \quad \text{and} \quad \pi^* \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*}$$

and we can see that the conjugate momenta are functions of the form

$$\pi = \pi(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$$
$$\pi^* = \pi^*(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$$

as well. By assuming the ability to invert these functions with respect to the "velocity" fields $\dot{\phi}$ and $\dot{\phi}^*$ and getting relations of the form

$$\dot{\phi} = \dot{\phi}(\phi, \partial_i\phi, \partial_i\partial_j\phi, \pi, c.c.) \tag{2.5}$$
$$\dot{\phi}^* = \dot{\phi}^*(\phi, \partial_i\phi, \partial_i\partial_j\phi, \pi, c.c.) \tag{2.6}$$

we are ready to perform a Legendre transformation to our Lagrangian (field density) $\mathcal{L}$ and get its corresponding Hamiltonian (field density) $\mathcal{H}$ defined as

$$\mathcal{H} = (\pi\dot{\phi} + \pi^*\dot{\phi}^*) - \mathcal{L}.$$

Note that here π and π^* play the role of independent variables, while $\dot{\phi}$ and $\dot{\phi}^*$ have become the dependent variables (functions) of 2.5. We have also set the functions $\dot{\phi}$ and $\dot{\phi}^*$ in the Lagrangian $\mathcal{L}$ and made it a function of the form

$$\mathcal{L} = \mathcal{L}(\phi, \partial_i\phi, \partial_i\partial_j\phi, \pi, c.c.),$$

so the Hamiltonian is actually defined as

$$\mathcal{H} = \mathcal{H}(\phi, \partial_i\phi, \partial_i\partial_j\phi, \pi, c.c.).$$

From the entire procedure it is now apparent why our initial Lagrangian did not depend on terms like $\partial_i\dot{\phi}$ and $\ddot{\phi}$, i.e. derivatives of $\dot{\phi}$.

We will now try to extend the usual Poisson brackets definition of particle mechanics to a suitable definition for fields. See also [54, 53, 68]. In particle mechanics we worked with the phase space coordinates $\vec{q} = \{q^i\}$ and $\vec{p} = \{p_i\}$ with $i = 1, \ldots, n$, so for functions $f = f(\vec{q}, \vec{p})$ and $g = g(\vec{q}, \vec{p})$ we defined their Poisson bracket as

$$\{f, g\}(\vec{q}, \vec{p}) = \sum_{i=1}^{n} \left(\frac{\partial f(\vec{q}, \vec{p})}{\partial q^i} \frac{\partial g(\vec{q}, \vec{p})}{\partial p_i} - \frac{\partial f(\vec{q}, \vec{p})}{\partial p_i} \frac{\partial g(\vec{q}, \vec{p})}{\partial q^i} \right).$$

In field theory the discrete finite index $i \in \{1, \ldots, n\} \subset \mathbb{N}$ is replaced by the continuous space index $\vec{x} \in \mathbb{R}^d$ and another discrete finite index $r \in \{1, \ldots, k\} \subset \mathbb{N}$, i.e. $i \to \{\vec{x}, r\}$, and the new quantities of interest are now $\vec{\phi} = \{\phi^r(\vec{x})\}$ and $\vec{\pi} = \{\pi_r(\vec{x})\}$. So for functionals $F = F[\vec{\phi}, \vec{\pi}]$ and $G = G[\vec{\phi}, \vec{\pi}]$ we define

$$\{F, G\}[\vec{\phi}, \vec{\pi}] = \int d^d x \sum_{r=1}^{k} \left(\frac{\delta F[\vec{\phi}, \vec{\pi}]}{\delta \phi^r(\vec{x})} \frac{\delta G[\vec{\phi}, \vec{\pi}]}{\delta \pi_r(\vec{x})} - \frac{\delta F[\vec{\phi}, \vec{\pi}]}{\delta \pi_r(\vec{x})} \frac{\partial G[\vec{\phi}, \vec{\pi}]}{\delta \phi^r(\vec{x})} \right).$$

Since

$$\frac{\delta \phi^r(\vec{x})}{\delta \phi^s(\vec{y})} = \delta^r_s \delta(\vec{x} - \vec{y}) \quad \text{and} \quad \frac{\delta \phi^r(\vec{x})}{\delta \pi^s(\vec{y})} = 0,$$

we can show for any functional $F = F[\vec{\phi}, \vec{\pi}]$ that, using a simplified obvious notation,

$$\{\phi^r(\vec{x}), F\} = \frac{\delta F}{\delta \pi_r(\vec{x})}.$$

Similarly, we can see that

$$\{\pi_r(\vec{x}), F\} = -\frac{\delta F}{\delta \phi^r(\vec{x})}.$$

If we had functionals of the form

$$F[\vec{\phi},\vec{\pi}] = \int d^d y\, \mathcal{F}(\vec{y})[\vec{\phi},\vec{\pi}],$$

then it is true that

$$\frac{\delta F[\vec{\phi},\vec{\pi}]}{\partial\phi^r(\vec{x})} = \int d^d y\, \frac{\delta\mathcal{F}(\vec{y})[\vec{\phi},\vec{\pi}]}{\delta\phi^r(\vec{x})}$$

(and similarly for $\pi_r(\vec{x})$). For F and G of the above form we can prove that their Poisson brackets become

$$\{F,G\}[\vec{\phi},\vec{\pi}] = \int d^d x \int d^d y\, \{\mathcal{F}(\vec{x}),\mathcal{G}(\vec{y})\}[\vec{\phi,\pi}].$$

In using the preceding equation we often have to also make use of the following easily proven relationships

$$\{\phi^r(\vec{x}),\pi_s(\vec{y})\} = \delta^r_s\delta(\vec{x}-\vec{y}), \quad \{\phi^r(\vec{x}),\phi^s(\vec{y})\} = 0, \quad \{\pi_r(\vec{x}),\pi_s(\vec{y})\} = 0.$$

In our dynamical complex scalar field theory with dynamical variables $\phi(t)(\vec{x}) \equiv \phi(t,\vec{x}) = \phi(x)$, $\phi^*(t)(\vec{x}) \equiv \phi^*(t,\vec{x}) = \phi^*(x)$, $\pi(t)(\vec{x}) \equiv \pi(t,\vec{x}) = \pi(x)$ and $\pi^*(t)(\vec{x}) \equiv \pi^*(t,\vec{x})$, in order to use these time-dependent functions of space in the above Poisson bracket we write

$$\begin{aligned}\{F,G\}[\phi(t),\pi(t),c.c.] = \int d^d x\, &\left(\frac{\delta F[\phi(t),\pi(t),c.c.]}{\delta\phi^r(t,\vec{x})}\frac{\delta G[\phi(t),\pi(t),c.c.]}{\delta\pi_r(t,\vec{x})} - \right.\\ &\left. - \frac{\delta F[\phi(t),\pi(t),c.c.]}{\delta\pi_r(t,\vec{x})}\frac{\partial G[\phi(t),\pi(t),c.c.]}{\delta\phi^r(t,\vec{x})}\right) + \\ + \int d^d x\, &\left(\frac{\delta F[\phi(t),\pi(t),c.c.]}{\delta\phi^{*r}(t,\vec{x})}\frac{\delta G[\phi(t),\pi(t),c.c.]}{\delta\pi^*_r(t,\vec{x})} - \right.\\ &\left. - \frac{\delta F[\phi(t),\pi(t),c.c.]}{\delta\pi^*_r(t,\vec{x})}\frac{\partial G[\phi(t),\pi(t),c.c.]}{\delta\phi^{*r}(t,\vec{x})}\right).\end{aligned}$$

In this formalism, it is apparent that the following equations hold (simplifying the notation)

$$\begin{aligned}\{\phi(t,\vec{x}),F\} &= \frac{\delta F}{\delta\pi(t,\vec{x})} &\qquad \{\pi(t,\vec{x}),F\} &= -\frac{\delta F}{\delta\phi(t,\vec{x})}\\ \{\phi^*(t,\vec{x}),F\} &= \frac{\delta F}{\delta\pi^*(t,\vec{x})} &\qquad \{\pi^*(t,\vec{x}),F\} &= -\frac{\delta F}{\delta\phi^*(t,\vec{x})}.\end{aligned}$$

Again for functionals of the form

$$F(\tau)[\phi(t),\pi(t),c.c.] = \int d^d y\, \mathcal{F}(\tau,\vec{y})[\phi(t),\pi(t),c.c.]$$

we get

$$\frac{\delta F(\tau)[\phi(t),\pi(t),c.c.]}{\delta\phi(t,\vec{x})} = \int d^d y\,\frac{\delta\mathcal{F}(\tau,\vec{y})[\phi(t),\pi(t),c.c.]}{\delta\phi(t,\vec{x})}$$

(similarly for π, ϕ^* and π^*), hence for functionals $F(\tau)$ and $G(\tau)$ of that form their Poisson brackets become

$$\{F(\tau),G(\tau)\}[\phi(t),\pi(t),c.c.] = \int d^d x \int d^d y\,\{\mathcal{F}(\tau,\vec{x}),\mathcal{G}(\tau,\vec{y})\}[\phi(t),\pi(t),c.c.].$$

For the above calculation we will need the relationships

$$\begin{aligned}
\{\phi(t,\vec{x}),\pi(t,\vec{y})\} &= \{\phi^*(t,\vec{x}),\pi^*(t,\vec{y})\} = \delta(\vec{x}-\vec{y}),\\
\{\phi(t,\vec{x}),\phi^*(t,\vec{y})\} &= \{\pi(t,\vec{x}),\pi^*(t,\vec{y})\} =\\
=\{\phi(t,\vec{x}),\pi^*(t,\vec{y})\} &= \{\phi^*(t,\vec{x}),\pi(t,\vec{y})\} = 0.
\end{aligned}$$

Before proceeding, we need to take a look at the Lagrangian $L(t)[\phi(t),\dot{\phi}(t),c.c.] = \int d^d x\mathcal{L}(t,\vec{x})[\phi(t),\dot{\phi}(t),c.c.]$ and the Hamiltonian $H(t)[\phi(t),\pi(t),c.c.] = \int d^d x\,\mathcal{H}[\phi(t),\pi(t),c.c.]$. We can see that (simplified notation again)

$$\pi(t,\vec{x}) = \frac{\delta L(t)}{\delta\dot{\phi}(t,\vec{x})} \quad \text{and} \quad \pi^*(t,\vec{x}) = \frac{\delta L(t)}{\delta\dot{\phi}^*(t,\vec{x})}.$$

For on-shell fields ϕ the Euler-Lagrange equations can be rewritten as

$$\dot{\pi}(t,\vec{x}) = \frac{\delta L(t)}{\delta\phi(t,\vec{x})} \quad \text{and} \quad \dot{\pi}^*(t,\vec{x}) = \frac{\delta L(t)}{\delta\phi^*(t,\vec{x})} \quad \text{on-shell.}$$

Now we vary the Hamiltonian

$$H = \int d^d x\,\mathcal{H} = \int d^d x\,(\pi\dot{\phi}+\pi^*\dot{\phi}^*) - L$$

and get

$$\begin{aligned}
\delta H = &\int d^d x\ (\delta\pi\dot{\phi}+\pi\delta\dot{\phi}+\delta\pi^*\dot{\phi}^*+\pi^*\delta\dot{\phi}^*) -\\
&- \int d^d x\,\left(\frac{\delta L}{\delta\phi}\delta\phi + \frac{\delta L}{\delta\phi^*}\delta\phi^* + \frac{\delta L}{\delta\dot{\phi}}\delta\dot{\phi} + \frac{\delta L}{\delta\dot{\phi}^*}\delta\dot{\phi}^*\right),
\end{aligned}$$

or

$$\delta H = \int d^d x\ \left[\left(-\frac{\delta L}{\delta\phi}\delta\phi + \dot{\phi}\delta\pi\right) + \left(-\frac{\delta L}{\delta\phi^*}\delta\phi^* + \dot{\phi}^*\delta\pi^*\right)\right].$$

This means that

$$\begin{aligned}
\dot{\phi}(t,\vec{x}) &= \frac{\delta H(t)}{\delta\pi(t,\vec{x})} & \frac{\delta L(t)}{\delta\phi(t,\vec{x})} &= -\frac{\delta H(t)}{\delta\phi(t,\vec{x})}\\
\dot{\phi}^*(t,\vec{x}) &= \frac{\delta H(t)}{\delta\pi^*(t,\vec{x})} & \frac{\delta L(t)}{\delta\phi^*(t,\vec{x})} &= -\frac{\delta H(t)}{\delta\phi^*(t,\vec{x})}.
\end{aligned}$$

For on-shell fields ϕ we get

$$\dot{\pi}(t,\vec{x}) = -\frac{\delta H(t)}{\delta\phi(t,\vec{x})} \quad \text{and} \quad \dot{\pi}^*(t,\vec{x}) = -\frac{\delta H(t)}{\delta\phi^*(t,\vec{x})} \quad \text{on-shell.}$$

Thus the Euler-Lagrange equations for on-shell fields ϕ in the Lagrangian formalism become Hamilton's equations of motion in the Hamiltonian formalism

$$\begin{aligned} \dot{\phi}(t,\vec{x}) &= \frac{\delta H(t)}{\delta\pi(t,\vec{x})} & \dot{\pi}(t,\vec{x}) &= -\frac{\delta H(t)}{\delta\phi(t,\vec{x})} \\ \dot{\phi}^*(t,\vec{x}) &= \frac{\delta H(t)}{\delta\pi^*(t,\vec{x})} & \dot{\pi}^*(t,\vec{x}) &= -\frac{\delta H(t)}{\delta\phi^*(t,\vec{x})}. \end{aligned}$$

If we have a functional of the form $F(t) = F(t)[\phi(t), \pi(t), c.c.]$, then we can show that for on-shell fields ϕ

$$\dot{F}(t) = \{F(t), H(t)\} + \frac{\partial F(t)}{\partial t} \quad \text{on-shell.}$$

Looking back at the currents 1.16 we see that they are of the form $j^\mu = j^\mu(t,\vec{x})[\phi(t), \pi(t), c.c.]$, so the corresponding charge is

$$Q(t) = Q(t)[\phi(t), \pi(t), c.c.] = \int d^dx\, j^0(t,\vec{x})[\phi(t), \pi(t), c.c.].$$

In particular, from 1.7 and 1.16a we have

$$Q(t) = \int d^dx\, \left[(\pi\delta\phi + \pi^*\delta\phi^*) + \xi^0\mathcal{L}\right]$$

with

$$\delta\phi = -\xi^0\dot{\phi} - \xi^i\partial_i\phi + \mathcal{K}[\phi],$$

so we can prove that (again see [54, 53, 68])

$$\{\phi(t,\vec{x}), Q(t)\} = \delta\phi(t,\vec{x}), \qquad \{\phi^*(t,\vec{x}), Q(t)\} = \delta\phi^*(t,\vec{x}).$$

Again we have the on-shell equation

$$\dot{Q}(t) = \{Q(t), H(t)\} + \frac{\partial Q(t)}{\partial t} \quad \text{on-shell.} \tag{2.7}$$

For the transformations we studied in Section 1.2 we had $\xi^\mu = \xi^\mu(\vec{x})$ and $f = f(\vec{x})$, which means that $\delta\phi = \delta\phi(\vec{x})[\phi(t), \pi(t), c.c]$. Also for our Lagrangians $\mathcal{L} = \mathcal{L}(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$ it is apparently true that $\mathcal{L}(t,\vec{x}) = \mathcal{L}(\vec{x})[\phi(t), \pi(t), c.c]$. Putting these together we conclude that the charges of those transformations are of the form $Q(t) = Q[\phi(t), \pi(t), c.c.]$ and thus $\partial Q(t)/\partial t = 0$. Now given the fact that the transformations we studied in Section 1.2 are all symmetries, then $\dot{Q}(t) = 0$ on-shell. These facts hold true for our Hamiltonian as well, i.e. $H(t) = H[\phi(t), \pi(t), c.c.]$, so $\partial H(t)/\partial t = 0$, and from 2.7 we can deduce that $\dot{H}(t) = 0$ on-shell. This similarity is expected, since, as we will see below, the Hamiltonian is, up to a constant multiplicative factor, the charge of time-translation, a symmetry of our Lagrangian. Summarizing we have for our transformations

$$\frac{\partial Q(t)}{\partial t} = \frac{\partial H(t)}{\partial t} = 0$$

and because they are symmetries they satisfy the on-shell relations

$$\dot{Q}(t) = \dot{H}(t) = 0 \quad \text{on-shell.}$$

Hence, from 2.7 we find that

$$\{Q(t), H(t)\} = 0 \quad \text{on-shell}$$

for the charge of any of our symmetries.

Since, in general, $Q(t) = Q(t)[\phi(t), \pi(t), c.c.]$, charges of transformations are eligible to put in Poisson brackets. From Section 1.2 we see that the charges of our studied symmetries are, up to a multiplicative factor,

$$H = \int d^d x \, \mathcal{H} \tag{2.8a}$$

$$P_i = \int d^d x \, \mathcal{P}_i \tag{2.8b}$$

$$M_{ij} = -\int d^d x \, (x_i \mathcal{P}_j - x_j \mathcal{P}_i) \tag{2.8c}$$

$$Q^{(0)} = \int d^d x \, J^0 \tag{2.8d}$$

$$D_i = \int d^d x \, x_i J^0 \tag{2.8e}$$

with $U(1)$ charge density

$$J^0 = i(\phi\pi - \phi^* \pi^*)$$

and momentum density

$$\mathcal{P}_i \equiv i(\partial_i \phi \pi + \partial_i \phi^* \pi^*).$$

In equations 2.8a to 2.8e we have the energy charge for time translation, the momentum charge for spatial translation, the angular momentum charge for spatial rotation, the $U(1)$ charge for $U(1)$ phase rotation and the dipole charge for the dipole transformation, respectively. These charges Q are conserved for on-shell fields ϕ, i.e.

$$Q = \text{constant} \quad \text{on-shell,}$$

since they are charges of symmetries. Additionally, we know that the on-shell Poisson bracket of the Hamiltonian H with any of the above symmetry charges Q vanishes, meaning

$$\{Q, H\} = 0 \quad \text{on-shell}.$$

Now we are ready to find the Poisson bracket algebra of our charges. After many calculations we finally find

$$\{M_{ij}, M_{kl}\} = i(\delta_{ik}M_{jl} - \delta_{jk}M_{il} - \delta_{il}M_{jk} + \delta_{jl}M_{ik}) \tag{2.9a}$$

$$\{M_{ij}, P_k\} = i(\delta_{ik}P_j - \delta_{jk}P_i) \tag{2.9b}$$

$$\{M_{ij}, D_k\} = i(\delta_{ik}D_j - \delta_{jk}D_i) \tag{2.9c}$$

$$\{P_i, D_j\} = i\delta_{ij}Q^{(0)} \tag{2.9d}$$

with the rest vanishing. Actually, as already noted above, the Poisson brackets of H with the other charges vanish only on-shell. For specific Hamiltonians this could even be true off-shell. An example would be to take $\mathcal{L} = \dot{\phi}^*\dot{\phi} - \kappa(X^{ij})^*X_{ij} - \lambda(X^i{}_i)^*X^j{}_j$ as our Lagrangian, which would give the Hamiltonian $\mathcal{H} = \pi^*\pi + \kappa(X^{ij})^*X_{ij} + \lambda(X^i{}_i)^*X^j{}_j$.

For a continuous transformation $x \to x'$, $\phi(x) \to \phi'(x')$ with *infinitesimal version* $x'_\epsilon(x)$, $\phi'_\epsilon(x') = \mathcal{T}_\epsilon[\phi](x_\epsilon(x'))$ we had for the transformed field ϕ'_ϵ

$$\phi'_\epsilon(x) = \phi(x) + \epsilon\delta\phi(x) + \mathcal{O}(\epsilon^2), \tag{2.10}$$

where

$$\delta\phi(x) = -\xi^\mu(x)\partial_\mu\phi(x) + \mathcal{K}[\phi](x). \tag{2.11}$$

We can define the generator of the transformation as the operator δ that when it acts on fields ϕ gives

$$\delta(\phi) = \delta\phi = -\xi^\mu\partial_\mu\phi + \mathcal{K}[\phi],$$

meaning

$$\delta = -\xi^\mu\partial_\mu + \mathcal{K}.$$

Actually, the generator is again defined up to a constant multiplicative factor, as is the charge. For more information on transformation generators see [69, 70, 71, 56, 72, 68, 63, 59, 58, 57]. For more mathematical exposition of Lie Groups and Lie Algebras see [73, 74, 75, 76].

We can now do for the generators the same thing we did for our symmetry transformations. We will denote the generator of a transformation as we did for its corresponding charge. Going again back to Section 1.2 we find that the generators of our studied symmetry transformations are

$$H = i\partial_t \tag{2.12a}$$

$$P_i = i\partial_i \tag{2.12b}$$

$$M_{ij} = -(x_iP_j - x_jP_i) \tag{2.12c}$$

$$Q^{(0)} = 1 \tag{2.12d}$$

$$D_i = x_i = x_iQ^{(0)} \tag{2.12e}$$

and the commutator algebra of our generators is

$$\boxed{\begin{aligned} [M_{ij}, M_{kl}] &= i(\delta_{ik} M_{jl} - \delta_{jk} M_{il} - \delta_{il} M_{jk} + \delta_{jl} M_{ik}) && \text{(2.13a)} \\ [M_{ij}, P_k] &= i(\delta_{ik} P_j - \delta_{jk} P_i) && \text{(2.13b)} \\ [M_{ij}, D_k] &= i(\delta_{ik} D_j - \delta_{jk} D_i) && \text{(2.13c)} \\ [P_i, D_j] &= i\delta_{ij} Q^{(0)} && \text{(2.13d)} \end{aligned}}$$

with the rest vanishing. Notice that the generators and the charges satisfy the same *symmetry algebra*, as was expected given the mutual mathematical correspondence to each other. This time, however, the vanishing of the generator H with the rest of the generators happens in general, not only on-shell. Actually, there is no explicit reference to the fields ϕ in the form of the generators. The only dependence of the generators on the fields ϕ is that the later constitute their domain of definition as functions and that the functional form of the generators originates in how these fields were transformed infinitesimally. For these reasons it is also clear why our generators are independent of time, something true for charges only on-shell. We are starting to see how going away from a field-centric description to a field independent description is characterized by many advantages. For instance, with generators we managed to find the symmetry algebra of our fracton theory without any reference to the Lagrangian or Hamiltonian formalism. We only needed to know the required symmetries of our theory. Also the calculations needed in using generators are much less cumbersome with much less background details than with charges.

2.3 Gauging procedure

We will now introduce the basic concepts needed to pass from a description based on dynamical matter fields ϕ to a description based on new auxiliary non-dynamical fields, usually called *background fields*. This is done through the very commonly used technique of the gauging procedure. For more information on gauging see [72, 57, 63, 58]. We will start by describing the gauging procedure for our original Lagrangian $\mathcal{L}^{(0)} = \mathcal{L}^{(0)}(\phi, \dot{\phi}, \partial_i\phi, \partial_i\partial_j\phi, c.c.)$ with corresponding original action

$$S^{(0)} = S^{(0)}[\phi, \phi^*] = \int d^{d+1}x \, \mathcal{L}^{(0)}(x)[\phi, \phi^*].$$

We take our original action $S^{(0)}$ to be symmetric under a global $U(1)$ phase rotation

$$\phi'(x) = e^{-i\alpha}\phi(x).$$

This means that the action $S^{(0)} = S^{(0)}[\phi, \phi^*]$ is invariant under $U(1)$ transformation, so

$$\mathcal{L}^{(0)}(\phi'(x), \dot{\phi}'(x), \partial_i\phi'(x), \partial_i\partial_j\phi'(x), c.c.) = \mathcal{L}^{(0)}(\phi(x), \dot{\phi}(x), \partial_i\phi(x), \partial_i\partial_j\phi(x), c.c.),$$

meaning $\mathcal{L}^{(0)}$ is invariant under $U(1)$ too. In the infinitesimal version

$$\phi'_\epsilon(x) = e^{-i\epsilon}\phi(x) = \phi(x) - i\epsilon\phi(x) + \mathcal{O}(\epsilon^2)$$

we get $\delta\mathcal{L}^{(0)} = 0$, so $\delta S^{(0)} = 0$. If we transform our field ϕ with a local $U(1)$ phase rotation

$$\phi'(x) = e^{-i\alpha(x)}\phi(x),$$

then our Lagrangian and action are not necessarily symmetric, so

$$\mathcal{L}^{(0)}(\phi'(x), \dot{\phi}'(x), \partial_i\phi'(x), \partial_i\partial_j\phi'(x), c.c.) \neq \mathcal{L}^{(0)}(\phi(x), \dot{\phi}(x), \partial_i\phi(x), \partial_i\partial_j\phi(x), c.c.).$$

According to Sections 1.1 and 1.2, in the infinitesimal form

$$\phi'_\epsilon(x) = e^{i\epsilon\Lambda(x)}\phi(x) = \phi(x) - i\epsilon\Lambda(x)\phi(x) + \mathcal{O}(\epsilon^2)$$

we find

$$\delta\mathcal{L}^{(0)} = -\partial_\mu\Lambda J^\mu + \partial_i(\partial_j\Lambda\tilde{J}^{ij})$$

or

$$\delta\mathcal{L}^{(0)} = -J^0\partial_t\Lambda - \left(J^i - \partial_j\tilde{J}^{ji}\right)\partial_i\Lambda + \tilde{J}^{ij}\partial_i\partial_j\Lambda. \tag{2.14}$$

To make our action symmetric under a local $U(1)$ phase rotation, we should also make our Lagrangian invariant under it. To achieve this we must create a new Lagrangian $\mathcal{L}$ with additional fields, the gauge fields, that also transform under a local $U(1)$ phase rotation in such a way so that the new Lagrangian is invariant. This means that our goal is to add new terms to our original Lagrangian $\mathcal{L}^{(0)}$ such that the final Lagrangian $\mathcal{L}$ satisfies $\delta\mathcal{L} = 0$ under a local $U(1)$ phase rotation. From equation 2.14 we see that our first step should be to add a term that counteracts the non-vanishing objects in $\delta\mathcal{L}^{(0)}$. We take this term to be first order in the new gauge fields and denote it $\mathcal{L}^{(1)}$. In particular, we define it as

$$\mathcal{L}^{(1)} = J^0 A_t + \left(J^i - \partial_j\tilde{J}^{ji}\right)A_i + \frac{1}{2}\tilde{J}^{ij}a_{ij}$$

and say that the "currents" J^0, $J^i - \partial_j\tilde{J}^{ji}$ and $\tilde{J}^{ij}$ are coupled with the gauge fields A_t, A_i and a_{ij}, respectively. Here the gauge field a_{ij} is symmetric, i.e. $a_{ij} = a_{ji}$. Under a local $U(1)$ transformation we have

$$\delta\mathcal{L}^{(1)} = J^0\delta A_t + \left(J^i - \partial_j\tilde{J}^{ji}\right)\delta A_i + \frac{1}{2}\tilde{J}^{ij}\delta a_{ij} + \delta_\phi\mathcal{L}^{(1)},$$

where $\delta_\phi\mathcal{L}^{(1)}$ is the infinitesimal change to $\mathcal{L}^{(1)}$ due to the change induced to the fields ϕ. To counteract $\delta\mathcal{L}^{(0)}$ in 2.14 we demand the gauge fields transform as

$$A_t \to A_t + \partial_t\Lambda, \quad A_i \to A_i + \partial_i\Lambda \quad \text{and} \quad a_{ij} \to a_{ij} - 2\partial_i\partial_j\Lambda$$

with gauge parameter $\Lambda = \Lambda(x)$, which gives

$$\delta\mathcal{L}^{(1)} = -\delta\mathcal{L}^{(0)} + \delta_\phi\mathcal{L}^{(1)},$$

so

$$\delta(\mathcal{L}^{(0)} + \mathcal{L}^{(1)}) = \delta\mathcal{L}^{(0)} + \delta\mathcal{L}^{(1)} = \delta_\phi \mathcal{L}^{(1)}.$$

We managed to construct a new Lagrangian $\mathcal{L}^{(0)} + \mathcal{L}^{(1)}$ that is still not invariant under local $U(1)$ phase rotations. The next step is to add to our Lagrangian another term $\mathcal{L}^{(2)}$ that is second order in the gauge fields. Using this iterative gauging procedure we will eventually end up with our final Lagrangian $\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \cdots + \mathcal{L}^{(n)}$ that will be invariant under local $U(1)$ phase rotations, i.e. $\delta\mathcal{L} = 0$ for any $\Lambda = \Lambda(x)$. A similar discussion of this procedure can be found in Section 3.3 of [57]. For more applications of gauge theory on fractons see [66, 77, 78, 79].

Let us denote by Φ all the matter fields of a theory (like $\Phi = (\phi, \phi^*)$) and by A all of its gauge fields (like $A = (A_t, A_i, a_{ij})$). What we have managed with gauging our theory is adding to the original action $S^{(0)}[\Phi]$ a new term $\tilde{S}[\Phi; A]$ such that the total action $S[\Phi; A] = S^{(0)}[\Phi] + \tilde{S}[\Phi; A]$ is invariant under any local $U(1)$ phase rotation. Here, since $U(1)$ phase rotation is an internal symmetry of our physical system, meaning $x' = x$ and $\xi^\mu = 0$, we have

$$0 = \delta S = \int d^{d+1}x \left(\frac{\delta S}{\delta \Phi_k} \delta\Phi_k + \frac{\delta S}{\delta A_l} \delta A_l \right) = \delta_\Phi S + \delta_A S,$$

hence

$$\delta_\Phi S[\Phi; A] = -\delta_A S[\Phi; A]. \tag{2.15}$$

If we choose fields Φ that are on-shell with respect to the total action S, meaning

$$\frac{\delta S}{\delta \Phi_k} = 0$$

with the gauge fields A left arbitrary, we get

$$\delta_\Phi S[\Phi; A] = 0 \quad \text{on-shell},$$

so from 2.15 we find the on-shell equation

$$\delta_A S[\Phi; A] = \int d^{d+1}x \, \frac{\delta S}{\delta A_l} \delta A_l = 0 \quad \text{on-shell},$$

where the gauge fields A are free to be chosen by us and the δA are defined for our specific local symmetry transformation. This way get a new current-like quantities that are conserved on-shell. We can even get back our original currents by setting $A = 0$ in the above, which gives

$$\delta_A S[\Phi; A]|_{A=0} = \delta_A S^{(1)}[\Phi; A]|_{A=0} = \int d^{d+1}x \, \delta_A \mathcal{L}^{(1)} = -\int d^{d+1}x \, \delta\mathcal{L}^{(0)} = 0 \quad \text{on-shell}.$$

In the above example we have

$$\begin{aligned}\delta_A S[\Phi; A]|_{A=0} &= \int d^{d+1}x\,(J^0\partial_t\Lambda + (J^i - \partial_j\tilde{J}^{ji})\,\partial_i\Lambda - \tilde{J}^{ij}\partial_i\partial_j\Lambda)\\ &= \int d^{d+1}x\,(-\partial_t J^0\Lambda - \partial_i(J^i - \partial_j\tilde{J}^{ji})\,\Lambda - \partial_i\partial_j\tilde{J}^{ij}\Lambda)\\ &= -\int d^{d+1}x\,\partial_\mu J^\mu\Lambda = 0 \quad \text{on-shell}\end{aligned}$$

for any $\Lambda = \Lambda(x)$ and we get back the conservation law

$$\partial_\mu J^\mu = 0 \quad \text{on-shell.}$$

We see that by gauging our theory, we essentially transfer the dynamical properties of the original theory to the new gauge fields. We can use these gauge fields to get back our original conservation laws and that is why gauge fields are also called *source fields*. They are not part of the dynamics of the gauged action, since they are freely chosen when we take the matter fields to be on-shell, which makes gauge fields function as *background fields* as well. We can use this trick to encode any kind of conservation law to the gauge fields and their gauge transformation. For instance, for the dipole symmetric theories we study, the conservation of the dipole current

$$\partial_\mu J^{i\mu} = 0 \quad \text{on-shell}$$

can be written as

$$J^i = \partial_j\tilde{J}^{ji} \quad \text{on-shell.} \tag{2.16}$$

To represent this conservation law in our gauged action we can define the first order Lagrangian term as

$$\mathcal{L}^{(1)} = J^0 A_t + J^i A_i + \frac{1}{2}\tilde{J}^{ij}a_{ij}$$

with new gauge fields A_t, A_i and a_{ij}. In the local $U(1)$ phase rotation we will have

$$\delta\mathcal{L}^{(1)} = J^0\delta A_t + J^i\delta A_i + \frac{1}{2}\tilde{J}^{ij}\delta a_{ij} + \delta_\phi\mathcal{L}^{(1)}$$

with gauge transformations

$$A_t \to A_t + \partial_t\Lambda, \quad A_i \to A_i + \partial_i\Lambda + \psi_i \quad \text{and} \quad a_{ij} \to a_{ij} + \partial_i\psi_j + \partial_j\psi_i,$$

where we introduced an additional Stückelberg field $\psi_i = \psi_i(x)$. For information on Stueckelberg fields see [80, 81]. The extra gauge parameter ψ_i exists to reveal relation 2.16 through the gauging procedure and for that reason it is called *dipole shift*. In more detail we have

$$\begin{aligned}\delta_A S[\Phi; A]|_{A=0} &= \int d^{d+1}x\,[J^0\partial_t\Lambda + J^i(\partial_i\Lambda + \psi_i) + \tilde{J}^{ij}\partial_i\psi_j]\\ &= \int d^{d+1}x\,(-\partial_t J^0\Lambda - \partial_i J^i\Lambda + J^i\psi_i - \partial_j\tilde{J}^{ji}\psi_i)\\ &= \int d^{d+1}x\,[-\partial_\mu J^\mu\Lambda + (J^i - \partial_j\tilde{J}^{ji})\psi_i] = 0 \quad \text{on-shell}\end{aligned}$$

for any $\Lambda = \Lambda(x)$ and $\psi_i = \psi_i(x)$ and we regain our $U(1)$ and dipole conservation laws

$$\partial_\mu J^\mu = 0 \quad \text{on-shell}$$
$$J^i = \partial_j \tilde{J}^{ji} \quad \text{on-shell.}$$

Chapter 3

Aristotelian Geometry

3.1 Second order formulation

Now we are finally ready to tackle the matter of writing our dipole symmetric theories in a covariant fashion. The geometrical background of these theories should reflect the almost friction-like behavior of fractons. It turns out [82, 83, 84, 77, 67, 64] that for physical systems, like fractons, with spacetime translation symmetry and spatial rotation symmetry, but no boost symmetry, the most ideal geometric construction is the Aristotelian geometry, a term first introduced by Roger Penrose in [85] and later mentioned in [86] and in his famous book [87]. Relevant background on the mathematics of differential geometry can be found in [88, 89]. You can also look at [90] or the unofficial notes made for this course [91]. For more information on Aristotelian geometries and Newton-Cartan geometries in general see [92], Chapter 5 of [93], Chapter 12 of [94], [95, 96, 97, 98, 99, 100]. The contents of this chapter will be influenced substantially by the work in [67].

We start by introducing the basic features of an Aristotelian geometry. First we take a $(d+1)$-dimensional manifold $\mathcal{M}$ and equip it with a nowhere-vanishing 1-form (field) n_μ called the *clock form*. We also equip our manifold with a nowhere-vanishing rank $(0,2)$ symmetric tensor $h_{\mu\nu}$ called *spatial metric*. This tensor is degenerate (non-invertible) with a 1-dimensional kernel spanned by a nowhere-vanishing vector v^μ, i.e.

$$v^\mu h_{\mu\nu} = 0.$$

We normalize v^μ such that

$$v^\mu n_\mu = 1.$$

Using the above quantities, we define the *spatial inverse metric* $h^{\mu\nu}$, a rank (2,0) symmetric tensor that is degenerate with a 1-dimensional kernel spanned by n_μ, i.e.

$$n_\mu h^{\mu\nu} = 0.$$

The tensor $h^{\mu\nu}$ is the inverse of $h_{\mu\nu}$ in the following sense

$$h^{\mu\kappa}h_{\kappa\nu} = \delta^{\mu}_{\nu} - v^{\mu}n_{\nu}.$$

Before proceeding forward, we should try to interpret the physical meaning of the mathematical objects we have just defined. In the geometry of Aristotelian spacetime, represented by the $(d+1)$-dimensional manifold $\mathcal{M}$, the lack of spacetime boosts, like Galilean and Lorentz boosts, creates a clear distinction between space and time coordinates. This distinction is produced by the clock form n_μ, the spatial metric $h_{\mu\nu}$ and the relationship between them. Specifically, the clock form n_μ exists to extract the "time part" of a tensorial quantity through index contractions. For example, if we have a vector X^μ, then we say that this vector point to the future if $X^\mu n_\mu > 0$ and the past if $X^\mu n_\mu < 0$. However, when $X^\mu n_\mu = 0$, then we say that our vector X^μ is purely *spatial*. For a discussion of this matter see Lectures 9 and 13 from [101] or from the unofficial lecture notes for the course [102]. You can also look at Chapter 8 of [93] or Chapter 6 of [103]. The vector v^μ plays a similar role to n_μ, since, by equation $v^\mu n_\mu = 1$, it can be interpreted as the velocity of a reference frame at rest, i.e. moving purely in the "time direction", and will be called *rest velocity*. This velocity of an observer can equally extract the "time component" of a tensorial quantity through appropriate index contractions. Now for an arbitrary rank (r,s) tensor $T^{\cdots\mu\cdots}{}_{\cdots\nu\cdots}$, if

$$n_\mu T^{\cdots\mu\cdots}{}_{\cdots\nu\cdots} = 0,$$

then we will say that this tensor is spatial in the μ index. Similarly, if

$$v^\nu T^{\cdots\mu\cdots}{}_{\cdots\nu\cdots} = 0,$$

then the tensor $T^{\cdots\mu\cdots}{}_{\cdots\nu\cdots}$ is spatial in the ν index. Finally, if a tensor is spatial in all of its indices, then it is called purely spatial or just spatial. From these definitions, it is evident that $h_{\mu\nu}$ and $h^{\mu\nu}$ are indeed spatial tensors, as their name suggests.

We have understood how space and time end up becoming distinguishable from each other at the framework of a more classical geometry like the Aristotelian spacetime. There is, nonetheless, a certain missing conceptual ingredient we will need to provide to get to this nature of absolute space and absolute time. Given that we have separated space and time using the clock form n_μ and the rest velocity v^μ, we could interpret our spacetime as a collection of "space slices" (again see [101, 102, 92, 93, 103]). This family of space-like hypersurfaces represents "space" at different "instants of time" and is the appropriate structure for which $h_{\mu\nu}$ and $h^{\mu\nu}$ become proper metric and inverse metric respectively. The spatial metric $h_{\mu\nu}$ and the spatial inverse metric $h^{\mu\nu}$ can even be extended to spacetime objects by combining them with the clock form n_μ and the rest velocity v^μ. In particular, we could define symmetric tensors

$$\gamma_{\mu\nu} \equiv n_\mu n_\nu + h_{\mu\nu}$$

and

$$\gamma^{\mu\nu} \equiv v^\mu v^\nu + h^{\mu\nu},$$

from which follows that

$$\gamma^{\mu\kappa}\gamma_{\kappa\nu} = \delta^\mu_\nu.$$

Due to the fact that both $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$ are symmetric invertible tensors, we could consider $\gamma_{\mu\nu}$ as the spacetime metric of our overall $(d+1)$-dimensional Aristotelian manifold $\mathcal{M}$ and $\gamma^{\mu\nu}$ as its inverse metric. Although trying to unify space and time in an Aristotelian geometry by use of the above derived tensors seems counterproductive, it is only a temporary trick we need to make in order to be able to define integration in our Aristotelian manifold. Thus, for conceptual reasons, we will avoid mention of the words metric and inverse metric for $\gamma_{\mu\nu}$ and $\gamma^{\mu\nu}$, but we will use them when it is a mathematical necessity. We will also make use of the standard notation $\gamma \equiv \det\gamma$, where $\det\gamma$ means the determinant of $\gamma_{\mu\nu}$.

The final piece to our geometric construction is the determination of a suitable connection. The appropriate choice is different from the usual Levi-Civita connection of General Relativity, as expected from the above peculiar components of Aristotelian geometry. The main feature we require for our Aristotelian connection is it being clock form and spatial metric compatible, i.e. $\nabla_\kappa n_\mu = 0$ and $\nabla_\kappa h^{\mu\nu} = 0$. A connection that satisfies this property is given by the following connection coefficients

$$\Gamma^\kappa_{\mu\nu} = v^\kappa \partial_\mu n_\nu + \frac{1}{2} h^{\kappa\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}),$$

where we use the convention

$$\nabla_{\frac{\partial}{\partial x^\mu}} \left(\frac{\partial}{\partial x^\nu}\right) \equiv \Gamma^\kappa_{\mu\nu} \frac{\partial}{\partial x^\kappa}.$$

Note that to show the results presented in this chapter, we will have to make extensive use of the defining equations of Aristotelian geometry and their consequences. For example, the equation $v^\kappa n_\kappa = 1$, gives us $\partial_\mu v^\kappa n_\kappa = -v^\kappa \partial_\mu n_\kappa$. Other derived equations of the same nature are $\partial_\mu h^{\kappa\lambda} n_\lambda = -h^{\kappa\lambda}\partial_\mu n_\lambda$ and $\partial_\mu h_{\kappa\lambda} v^\lambda = -h_{\kappa\lambda}\partial_\mu v^\lambda$. Of course, there exist similar equations for any kind of derivative operator, including the Lie derivative and the covariant derivative. Now, for the Aristotelian connection defined above, the corresponding covariant derivative can, indeed, be shown to satisfy

$$\nabla_\kappa n_\mu = 0$$

and

$$\nabla_\kappa h^{\mu\nu} = 0.$$

Given this Aristotelian covariant derivative we can also prove through calculations that

$$h_{\mu\kappa}\nabla_\nu v^\kappa = \frac{1}{2}\pounds_v h_{\mu\nu}$$

and

$$\nabla_\kappa h_{\mu\nu} = -n_{(\mu}\pounds_v h_{\nu)\kappa}.$$

By defining the antisymmetric tensor

$$F^n_{\mu\nu} \equiv \partial_\mu n_\nu - \partial_\nu n_\mu,$$

the torsion (tensor) $T^\kappa{}_{\mu\nu} = 2\Gamma^\kappa_{[\mu\nu]}$ can be written as

$$T^\kappa{}_{\mu\nu} = v^\kappa F^n_{\mu\nu}.$$

Finally, using the identity

$$\frac{\partial \det M}{\partial M_{\kappa\lambda}} = (M^{-1})^{\kappa\lambda} \det M, \tag{3.1}$$

holding true for any (real) invertible matrix M[1], we obtain

$$\Gamma^\kappa_{\kappa\mu} + F^n_{\mu\kappa} v^\kappa = \frac{1}{\sqrt{\gamma}}\partial_\mu\sqrt{\gamma}.$$

The curvature (tensor) of our Aristotelian geometry has the usual form

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\kappa_{\nu\lambda} - \partial_\nu\Gamma^\kappa_{\mu\lambda} + \Gamma^\kappa_{\mu\rho}\Gamma^\rho_{\nu\lambda} - \Gamma^\kappa_{\nu\rho}\Gamma^\rho_{\mu\lambda}$$

and we can easily see that $n_\kappa R^\kappa{}_{\lambda\mu\nu} = 0$. We notice that, contrary to the Levi-Civita connection of General Relativity, the Aristotelian connection is not spacetime metric compatible and not torsion-free. There is actually no way to make Aristotelian geometry torsionless given the constraint that it is clock form compatible. The way we expressed mathematically Aristotelian geometry in this section is sometimes called second order formulation.

3.2 Coupling to background sources

Having described the basic mathematical elements of Aristotelian geometry, we are finally ready to introduce the required machinery needed to describe the fundamental behavior of any non-relativistic fracton field theory. This is done though the gauging trick we studied in Section 2.3, where we showed how the conservation laws of a physical theory can be encoded to background source gauge fields.

[1] To prove this, first note that

$$\det(I + \epsilon A) = 1 + \epsilon \operatorname{tr} A + \mathcal{O}(\epsilon^2).$$

Then the desired equation follows by the definition of partial differentiation.

Before proceeding further, we must take the time to explain how diffeomorphism invariance is taken care of with gauging. Imagine we had a field theory on a flat spacetime with metric $\eta_{\mu\nu}$, a set of fields Φ, a Lagrangian

$$\mathcal{L}^{(0)} = \mathcal{L}^{(0)}(\Phi, \partial_\mu \Phi, \partial_\mu \partial_\nu \Phi; \eta_{\mu\nu})$$

and an action

$$S^{(0)}[\Phi] = \int d^{d+1}x \, \mathcal{L}^{(0)}(x)[\Phi; \eta].$$

Our current theory is, obviously, not covariant or, equivalently, not diffeomorphism invariant. We can make our theory diffeomorphism invariant by substituting the flat metric $\eta_{\mu\nu}$ with an arbitrary metric $g_{\mu\nu}$. We must also convert the partial derivative ∂_μ acting on the (in general) tensor fields Φ to the Levi-Civita covariant derivative $\widetilde{\nabla}_\mu$. After these changes we will have a new Lagrangian

$$\mathcal{L} = \mathcal{L}(\Phi, \widetilde{\nabla}_\mu \Phi, \widetilde{\nabla}_\mu \widetilde{\nabla}_\nu \Phi; g_{\mu\nu})$$

and a new action

$$S[\Phi; g] = \int d^{d+1}x \, \sqrt{|g|} \mathcal{L}(x)[\Phi; g],$$

where $g \equiv \det g$. Note that there is $g_{\mu\nu}$ dependence even in the Levi-Civita covariant derivative. This new gauged action is diffeomorphism invariant and the metric $g_{\mu\nu}$ plays the role of the gauge field. The conserved current of diffeomorphism invariance is a generalized energy-momentum tensor $T^{\mu\nu}$ that, in many cases, coincides with the canonical energy-momentum tensor of Noether's Theorem by just gauge fixing $g_{\mu\nu} = \eta_{\mu\nu}$.

To derive the conservation law we take an infinitesimal diffeomorphism $x'^\mu_\epsilon(x) = x^\mu + \epsilon \xi^\mu + \mathcal{O}(\epsilon^2)$ that is the identity everywhere except for a small region inside the region of integration in the action. This means that the pulledback action $S[x'^* \Phi; x'^* g]$ will have the same region of integration as $S[\Phi; g]$. As the action is diffeomorphism invariant, i.e.

$$S[x'^* \Phi; x'^* g] = S[\Phi; g],$$

we find to infinitesimal order that

$$0 = \delta S = \int d^{d+1}x \left(\frac{\delta S}{\delta \Phi_k} \delta \Phi_k + \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \right).$$

We take the matter fields Φ to be on-shell, so

$$\delta S = \int d^{d+1}x \, \frac{\delta S}{\delta g_{\mu\nu}} \delta g_{\mu\nu} \quad \text{on-shell.}$$

But under this infinitesimal pullback we have

$$\delta g_{\mu\nu} = \pounds_\xi g_{\mu\nu}$$

and we can show that

$$\pounds_\xi g_{\mu\nu} = 2\widetilde{\nabla}_{(\mu}\xi_{\nu)},$$

which gives us

$$\begin{aligned}\delta S &= \int d^{d+1}x\, \frac{\delta S}{\delta g_{\mu\nu}} 2\widetilde{\nabla}_{(\mu}\xi_{\nu)} \\ &= \int d^{d+1}x \sqrt{|g|}\, \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}} \widetilde{\nabla}_\mu \xi_\nu \quad \text{on-shell}\end{aligned}$$

or

$$\delta S = \int d^{d+1}x \sqrt{|g|}\, T^{\mu\nu} \widetilde{\nabla}_\mu \xi_\nu \quad \text{on-shell}$$

with

$$T^{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g_{\mu\nu}}$$

the generalized energy-stress tensor (up to a multiplicative constant factor), which obviously is symmetric $T^{\mu\nu} = T^{\nu\mu}$. Now diffeomorphism invariance gives us

$$\begin{aligned}\delta S &= \int d^{d+1}x \sqrt{|g|}\, T^{\mu\nu} \widetilde{\nabla}_\mu \xi_\nu \\ &= -\int d^{d+1}x \sqrt{|g|}\, \widetilde{\nabla}_\mu T^{\mu\nu} \xi_\nu = 0 \quad \text{on-shell}\end{aligned}$$

for any $\xi^\mu = \xi^\mu(x)$, which results in the generalized conservation law

$$\widetilde{\nabla}_\mu T^{\mu\nu} = 0 \quad \text{on-shell.}$$

It should be noted that in curved spacetime a conservation law must be written with respect to a covariant derivative, the generalization of the common derivative notion of flat spacetime. The conservation of the energy-momentum tensor $T^{\mu\nu}$ is the result of diffeomorphism invariance of our theory, a symmetry that appeared only after introducing the gauge field $g_{\mu\nu}$. The metric $g_{\mu\nu}$ is thus the background source of the energy-momentum tensor $T^{\mu\nu}$. In fact, from

$$\delta_g S = \int d^{d+1}x \sqrt{|g|}\, T^{\mu\nu} \delta g_{\mu\nu}$$

we can clearly see that the energy-momentum tensor $T^{\mu\nu}$ is coupled to the metric $g_{\mu\nu}$. For more information on this definition of the energy-momentum tensor $T^{\mu\nu}$ and the function of the metric $g_{\mu\nu}$ as its background source and as a gauge field for diffeomorphism invariance see [104, 93, 103, 105, 56, 106, 107, 108, 109, 110].

We are now ready to tackle the problem of gauging our Aristotelian Geometry with the appropriate background source fields. Firstly, as we are interested in theories invariant to spacetime translation and spatial rotation, we must couple the corresponding currents to a metric-like quantity that achieves diffeormorphism invariance for the resultant theory. This role will be fulfilled by the clock form n_μ and the spatial metric $h_{\mu\nu}$ [82, 83, 84]. Next, we need to introduce the gauge field for the $U(1)$ symmetry, which, of course, is A_μ, a gauge field that had appeared in the non-covariant case too. The gauge transformation corresponding to the $U(1)$ symmetry should be covariant now, so we take

$$A_\mu \to A_\mu + \nabla_\mu \Lambda.$$

Looking back to the non-covariant case again, we see that for dipole symmetry we need two kinds of background fields, the A_μ and a spatial symmetric tensor $a_{\mu\nu}$, i.e. a tensor that obeys $a_{\mu\nu} = a_{\nu\mu}$ and $v^\mu a_{\mu\nu} = 0$. We will also need a spatial 1-form ψ_μ, meaning $v^\mu \psi_\mu = 0$, and its "spatial covariant derivative". The covariant derivative of a spatial tensor is not actually spatial itself. To make a spatial tensor out of a non-spatial one we can use the spatial tensor

$$h^\mu_\nu \equiv h^{\mu\kappa} h_{\kappa\nu} = \delta^\mu_\nu - v^\mu n_\nu,$$

which is spatial because $v^\nu h^\mu_\nu = 0$ and $n_\mu h^\mu_\nu = 0$. The h^μ_ν functions like a *spatial projector*, since contracting it with any tensor $T^{\mu\cdots}{}_{\nu\ldots}$ gives a spatial tensor

$$\widetilde{T}^{\mu\cdots}{}_{\nu\ldots} \equiv h^\mu_{\mu'} \cdots h^{\nu'}_\nu \cdots T^{\mu'\cdots}{}_{\nu'\ldots}$$

satisfying

$$v^\nu \widetilde{T}^{\mu\cdots}{}_{\nu\ldots} = 0 \quad \text{and} \quad n_\mu \widetilde{T}^{\mu\cdots}{}_{\nu\ldots} = 0.$$

If T is already spatial, then the action of the spatial projector h^μ_ν on T leaves is invariant, meaning $\widetilde{T} = T$. This means that from the spatial 1-form ψ_μ with non-spatial $\nabla_\mu \psi_\nu$ we can get its spatial covariant derivative $h^{\mu'}_\mu h^{\nu'}_\nu \nabla_{\mu'} \psi_{\nu'}$. Then the dipole shift transformation will be written as

$$A_\mu \to A_\mu + \psi_\mu, \quad a_{\mu\nu} \to a_{\mu\nu} + h^{\mu'}_\mu h^{\nu'}_\nu (\nabla_{\mu'} \psi_{\nu'} + \nabla_{\nu'} \psi_{\mu'}).$$

It should be emphasized that of all the gauge fields introduced only A_μ changes under a $U(1)$ transformation, while A_μ and $a_{\mu\nu}$ are the only ones that change under a dipole shift.

Instead of the dipole gauge field $a_{\mu\nu}$ that is affected only by a dipole shift transformation, we will actually use a modified version that also reflects the dipole behavior. This proxy quantity will include A_μ, but should stay $U(1)$ invariant, so we combine both $a_{\mu\nu}$ and the $U(1)$ invariant field strength

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

into the modified dipole gauge field

$$A^\mu{}_\nu \equiv n_\nu v^\kappa F_{\kappa\lambda} h^{\lambda\mu} + \frac{1}{2}(h^\kappa_\nu F_{\kappa\lambda} h^{\lambda\mu} + a_{\nu\kappa} h^{\kappa\mu}),$$

where $n_\mu A^\mu{}_\nu = 0$. If we define $\psi^\mu \equiv h^{\mu\nu}\psi_\nu$, then $n_\mu\psi^\mu = 0$ and $\psi_\mu = h_{\mu\nu}\psi^\nu$. We also have $h^\mu_\kappa h^{\kappa\nu} = h^{\mu\nu}$ and $h^\kappa_\mu h_{\kappa\nu} = h_{\mu\nu}$. Under a dipole shift transformation the field strength changes as

$$F_{\mu\nu} \to F_{\mu\nu} + \partial_\mu\psi_\nu - \partial_\nu\psi_\mu = F_{\mu\nu} + \nabla_\mu\psi_\nu - \nabla_\nu\psi_\mu,$$

since $\psi_\kappa T^\kappa{}_{\mu\nu} = 0$. Also under dipole shift we find $A^\mu{}_\nu \to A^\mu{}_\nu + \nabla_\nu\psi^\mu + n_\nu\psi_\kappa\nabla_\lambda v^\kappa h^{\lambda\mu}$. We can show that

$$\nabla_\nu v^\mu = -v^\kappa\nabla_\nu h_{\kappa\lambda}h^{\lambda\mu} = h_{\nu\kappa}\nabla_\lambda v^\kappa h^{\lambda\mu}.$$

Using the above relationship we find that under dipole shift $A^\mu{}_\nu$ changes as

$$A^\mu{}_\nu \to A^\mu{}_\nu + \nabla_\nu\psi^\mu + n_\nu\psi^\kappa\nabla_\kappa v^\mu.$$

As already mentioned the modified dipole gauge field $A^\mu{}_\nu$ is invariant under $U(1)$ transformations. We can also define a modified dipole field strength

$$F^\kappa{}_{\mu\nu} \equiv \nabla_\mu A^\kappa{}_\nu - \nabla_\nu A^\kappa{}_\mu + F^n_{\mu\nu}v^\lambda A^\kappa{}_\lambda + 2n_{[\mu}A^\lambda{}_{\nu]}\nabla_\lambda v^\kappa,$$

which satisfies $n_\kappa F^\kappa{}_{\mu\nu} = 0$ and is $U(1)$ invariant, but under a dipole shift changes as

$$F^\kappa{}_{\mu\nu} \to F^\kappa{}_{\mu\nu} + \left(R^\kappa{}_{\lambda\mu\nu} + F^n_{\mu\nu}\nabla_\lambda v^\kappa - 2n_{[\mu}\nabla_{\nu]}\nabla_\lambda v^\kappa\right)\psi^\lambda.$$

As someone might have already noticed, it is not possible to construct a quantity out of the gauge fields A_μ and $a_{\mu\nu}$, the only gauge fields that change under a dipole shift, that is dipole shift invariant. This restriction will show up in our work later.

3.3 First order formulation

In the previous sections we studied the second order formulation of Aristotelian geometry. We will now develop Aristotelian geometry in its first order formulation using the language of orthonormal bases, also known as vielbeins. For an introduction to vielbeins see [104, 89, 93]. We will use greek indices μ, latin indices a and, occasionally, barred latin indices $\bar{a}$ with range $\mu = 0, 1, \ldots, d$, $a = 1, \ldots, d$ and $\bar{a} = 0, 1, \ldots, d$, respectively. We first start by looking at the spatial metric $h_{\mu\nu}$ and its spatial inverse metric $h^{\mu\nu}$. There exist local spatial 1-form vielbein fields e^a and local spatial vector vielbein fields e_a such that the metric h satisfies $h = \delta_{ab}\, e^a \otimes e^b$ (in a local neighborhood of any point in the manifold) and the inverse metric $\tilde{h}$ satisfies $\tilde{h} = \delta^{ab}\, e_a \otimes e_b$ (in a local neighborhood of any point in the manifold). In a coordinate basis $\partial_\mu \equiv \partial/\partial x^\mu$ these equations become

$$h_{\mu\nu} = \delta_{ab}e^a_\mu e^b_\nu \quad \text{and} \quad h^{\mu\nu} = \delta^{ab}e^\mu_a e^\nu_b.$$

The vielbeins e^a and e_a with coordinate components e^a_μ and e^μ_a (in some manifold chart), given they are spatial, must obey the equations

$$v^\mu e^a_\mu = 0 \quad \text{and} \quad n_\mu e^\mu_a = 0.$$

Also, since e^a is the duel spatial basis of e_a, we have $e^a(e_b) = \delta^a_b$ or

$$e^a_\mu e^\mu_b = \delta^a_b.$$

We should not forget that the 1-form n and the vector v were chosen with the normalization $n(v) = 1$ or

$$n_\mu v^\mu = 1$$

in mind. If we now define $e^{\bar{a}=0} \equiv n$ and $e_{\bar{a}=0} \equiv v$, we will have the 1-forms $e^{\bar{a}}$ and vectors $e_{\bar{a}}$ ($\bar{a} = 0, 1, \ldots, d$) with (coordinate) components $e^{\bar{a}}_\mu$ and $e^\mu_{\bar{a}}$. Using these objects the above equations take the compact form $e^{\bar{a}}_\mu e^\mu_{\bar{b}} = \delta^{\bar{a}}_{\bar{b}}$. The existence of a right inverse implies the existence of a left inverse and vice verse, and these inverses are equal to each other and unique. This statement produces the partner equation $e^\mu_{\bar{a}} e^{\bar{a}}_\nu = \delta^\mu_\nu$ that can be written as

$$v^\mu n_\nu + e^\mu_a e^a_\nu = \delta^\mu_\nu$$

or, using the spatial projector h^μ_ν, as

$$h^\mu_\nu = e^\mu_a e^a_\nu.$$

The spacetime metric $\gamma_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu}$ or $\gamma = n \otimes n + h = \delta_{\bar{a}\bar{b}}\, e^{\bar{a}} \otimes e^{\bar{b}}$ can be written in coordinate components as

$$\gamma_{\mu\nu} = \delta_{\bar{a}\bar{b}} e^{\bar{a}}_\mu e^{\bar{b}}_\nu,$$

which together with the equations $e^{\bar{a}}_\mu e^\mu_{\bar{b}} = \delta^{\bar{a}}_{\bar{b}}$ and $e^\mu_{\bar{a}} e^{\bar{a}}_\nu = \delta^\mu_\nu$ leads us to the realization that $e^{\bar{a}}$ and $e_{\bar{a}}$ are regular spacetime vielbeins of Aristotelian geometry.

When we want to translate a tensor T from greek coordinate indices μ to barred latin vielbein indices and vice versa, we need to use $e^{\bar{a}}_\mu$ and $e^\mu_{\bar{a}}$. For instance, we will have $T^{\bar{a}\cdots}{}_{\nu\cdots} = e^{\bar{a}}_\mu T^{\mu\cdots}{}_{\nu\cdots}$, $T^{\mu\cdots}{}_{\bar{b}\cdots} = e^\nu_{\bar{b}} T^{\mu\cdots}{}_{\nu\cdots}$ and $T^{\mu\cdots}{}_{\bar{b}\cdots} = e^\mu_{\bar{a}} T^{\bar{a}\cdots}{}_{\bar{b}\cdots}$, $T^{\bar{a}\cdots}{}_{\nu\cdots} = e^{\bar{b}}_\nu T^{\bar{a}\cdots}{}_{\bar{b}\cdots}$. In particular, we have $T^{\bar{a}=0\cdots}{}_{\nu\cdots} = e^{\bar{a}=0}_\mu T^{\mu\cdots}{}_{\nu\cdots} = n_\mu T^{\mu\cdots}{}_{\nu\cdots}$ and $T^{\mu\cdots}{}_{\bar{b}=0} = e^\nu_{\bar{b}=0} T^{\mu\cdots}{}_{\nu\cdots} = v^\nu T^{\mu\cdots}{}_{\nu\cdots}$. But if a tensor $T^{\mu\cdots}{}_{\nu\cdots}$ is spatial, then as we know it satisfies

$$n_\mu T^{\mu\cdots}{}_{\nu\cdots} = 0 \quad \text{and} \quad v^\nu T^{\mu\cdots}{}_{\nu\cdots} = 0,$$

which can be written as

$$T^{\bar{a}=0\cdots}{}_{\nu\cdots} = 0 \quad \text{and} \quad T^{\mu\cdots}{}_{\bar{b}=0} = 0.$$

Notice that for spatial tensors $T^{\bar{a}\cdots}{}_{\bar{b}\cdots}$ only the pure latin index components $T^{a\cdots}{}_{b\cdots}$ are non-vanishing. This means that we can declare spatial tensors in vielbein language by just writing $T^{a\cdots}{}_{b\cdots}$. As an example, the spatial metric $h_{\mu\nu}$, the spatial inverse metric $h^{\mu\nu}$ and the spatial projector h^μ_ν all obey $h^{\bar{a}=0,\nu} = h^{\bar{a}=0,\bar{b}=0} = h_{\bar{a}=0,\nu} = h_{\bar{a}=0,\bar{b}=0} = h^{\bar{a}=0}_\nu = h^\mu_{\bar{a}=0} = h^{\bar{a}=0}_{\bar{b}=0} = 0$. In pure latin indices we get

$$h^{a\nu} = e^a_\mu h^{\mu\nu} = e^{a\nu} \qquad h^{ab} = e^a_\mu e^b_\nu h^{\mu\nu} = \delta^{ab}$$

$$h_{a\nu} = e^\mu_a h_{\mu\nu} = e_{a\nu} \qquad h_{ab} = e^\mu_a e^\nu_b h_{\mu\nu} = \delta_{ab}$$

$$h^a_\nu = e^a_\mu h^\mu_\nu = e^a_\nu \qquad h^\mu_b = e^\nu_b h^\mu_\nu = e^\mu_b,$$

where $e^{a\nu} \equiv \delta^{a\bar{b}} e^{\nu}_{\bar{b}} = \delta^{ab} e^{\nu}_{b}$ and $e_{a\nu} \equiv \delta_{a\bar{b}} e^{\bar{b}}_{\nu} = \delta_{ab} e^{b}_{\nu}$. In fact, since $\gamma_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}}$ and $\gamma^{\bar{a}\bar{b}} = \delta^{\bar{a}\bar{b}}$, all (pure) latin indices will be raised and lowered using δ^{ab} and δ_{ab}, respectively, as can be seen from $T^{\cdots}{}_{\cdots}{}^{a}{}_{\cdots} \equiv \delta^{a\bar{b}} T^{\cdots}{}_{\cdots\bar{b}\cdots} = \delta^{ab} T^{\cdots}{}_{\cdots b\cdots}$ and $T^{\cdots}{}_{a}{}^{\cdots}{}_{\cdots} \equiv \delta_{a\bar{b}} T^{\cdots\bar{b}\cdots}{}_{\cdots} = \delta_{ab} T^{\cdots b\cdots}{}_{\cdots}$.

To calculate covariant derivatives in the vielbein language we make extensive use of the spin connection $\omega^{\bar{a}}{}_{\bar{b}\mu}$, the vielbein equivalent of the connection coefficients $\Gamma^{\kappa}_{\mu\nu}$ in the coordinate basis language. The spin connection $\omega^{\bar{a}}{}_{\bar{b}\mu}$ is defined implicitly through the following relation

$$\nabla_{\mu} e_{\bar{b}} = \omega^{\bar{a}}{}_{\bar{b}\mu} e_{\bar{a}}, \tag{3.2}$$

from which we can prove that

$$\nabla_{\mu} e^{\bar{a}} = -\omega^{\bar{a}}{}_{\bar{b}\mu} e^{\bar{b}}.$$

Using the above and following the usual procedure for calculating the covariant derivative in coordinate indices, we can show that in mixed coordinate-vielbein indices the covariant derivative of a tensor $T^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots}$ becomes

$$\begin{aligned}\nabla_{\kappa} T^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots} \equiv& \nabla_{\kappa}\left(T^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots}\right) \equiv \left(\nabla_{\kappa} T\right)^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots} \\ =& \partial_{\kappa} T^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots} + \Gamma^{\mu}_{\kappa\mu'} T^{\mu'\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}\cdots} + \cdots + \omega^{\bar{a}}{}_{\bar{a}'\kappa} T^{\mu\cdots\bar{a}'\cdots}{}_{\nu\cdots\bar{b}\cdots} + \cdots \\ &- \Gamma^{\nu'}_{\kappa\nu} T^{\mu\cdots\bar{a}\cdots}{}_{\nu'\cdots\bar{b}\cdots} - \cdots - \omega^{\bar{b}'}{}_{\bar{b}\kappa} T^{\mu\cdots\bar{a}\cdots}{}_{\nu\cdots\bar{b}'\cdots} - \cdots .\end{aligned}$$

For a spatial tensor T we find for pure latin indices

$$\begin{aligned}\nabla_{\kappa} T^{\mu\cdots a\cdots}{}_{\nu\cdots b\cdots} =& \partial_{\kappa} T^{\mu\cdots a\cdots}{}_{\nu\cdots b\cdots} + \Gamma^{\mu}_{\kappa\mu'} T^{\mu'\cdots a\cdots}{}_{\nu\cdots b\cdots} + \cdots + \omega^{a}{}_{a'\kappa} T^{\mu\cdots a'\cdots}{}_{\nu\cdots b\cdots} + \cdots \\ &- \Gamma^{\nu'}_{\kappa\nu} T^{\mu\cdots a\cdots}{}_{\nu'\cdots b\cdots} - \cdots - \omega^{b'}{}_{b\kappa} T^{\mu\cdots a\cdots}{}_{\nu\cdots b'\cdots} - \cdots .\end{aligned}$$

When we change orthonormal basis, meaning going from an initial choice of vielbeins $e^{\bar{a}}$ and $e_{\bar{a}}$ to another one $e'^{\bar{a}'}$ and $e'_{\bar{a}'}$, we can write this as $e^{\bar{a}} \to e'^{\bar{a}'} = \Lambda^{\bar{a}'}{}_{\bar{a}} e^{\bar{a}}$ and $e_{\bar{a}} \to e'_{\bar{a}'} = \Lambda'^{\bar{a}}{}_{\bar{a}'} e_{\bar{a}}$, with $\Lambda^{\bar{a}}{}_{\bar{b}} = \Lambda^{\bar{a}}{}_{\bar{b}}(x)$ and $\Lambda'^{\bar{a}}{}_{\bar{b}} = \Lambda'^{\bar{a}}{}_{\bar{b}}(x)$ being local smooth functions of spacetime. This functions are similar to the regular Lorentz transformations of flat Minkowski spacetime. In general, they must satisfy $\Lambda^{\bar{a}}{}_{\bar{\kappa}} \Lambda'^{\bar{\kappa}}{}_{\bar{b}} = \delta^{\bar{a}}_{\bar{b}}$ and the Lorentz-like relation $\Lambda^{\bar{k}}{}_{\bar{a}} \Lambda^{\bar{l}}{}_{\bar{b}} \delta_{\bar{k}\bar{l}} = \delta_{\bar{a}\bar{b}}$, given that $\gamma_{\bar{a}\bar{b}} = \delta_{\bar{a}\bar{b}}$. From the definition 3.2 we can find that the spin connection $\omega^{\bar{a}}{}_{\bar{b}\mu}$ changes under $e^{\bar{a}} \to e'^{\bar{a}'} = \Lambda^{\bar{a}'}{}_{\bar{a}} e^{\bar{a}}$ (and, of course, $e_{\bar{a}} \to e'_{\bar{a}'} = \Lambda'^{\bar{a}}{}_{\bar{a}'} e_{\bar{a}}$) as $\omega^{\bar{a}}{}_{\bar{b}\mu} \to \omega'^{\bar{a}'}{}_{\bar{b}'\mu} = \Lambda^{\bar{a}'}{}_{\bar{a}} \Lambda'^{\bar{b}}{}_{\bar{b}'} \omega^{\bar{a}}{}_{\bar{b}\mu} - \partial_{\mu} \Lambda^{\bar{a}'}{}_{\bar{k}} \Lambda'^{\bar{k}}{}_{\bar{b}'}$. It is also worth mentioning that under a change of coordinate basis $\omega^{\bar{a}}{}_{\bar{b}\mu}$ changes as $\omega^{\bar{a}}{}_{\bar{b}\mu} \to \omega^{\bar{a}}{}_{\bar{b}\mu'} = \dfrac{\partial x^{\mu}}{\partial x'^{\mu'}} \omega^{\bar{a}}{}_{\bar{b}\mu}$, so the spin connection is an 1-form with respect to its greek index. We will be interested exclusively to spatial rotations, i.e. transformations $\Lambda^{\bar{a}}{}_{\bar{b}}$ such that $\Lambda^{\bar{a}=0}{}_{\bar{b}} = \delta^{0}_{\bar{b}}$, $\Lambda^{\bar{a}}{}_{\bar{b}=0} = \delta^{\bar{a}}_{0}$ and $\Lambda^{a}{}_{b} = R^{a}{}_{b}$ with $R \in SO(d)$ satisfying $R^{k}{}_{a} R^{l}{}_{b} \delta_{kl} = \delta_{ab}$ and $\det R = 1$, and having $R' \in SO(d)$ as its inverse. Under this spatial rotations only pure latin indices will change. Specifically, for the vielbeins we will have

$$e^{a} \to e'^{a} = R^{a}{}_{b} e^{b} \quad \text{and} \quad e_{a} \to e'_{a} = R'^{b}{}_{a} e_{b},$$

while for the spin connection we find

$$\omega^a{}_{b\mu} \to \omega'^a{}_{b\mu} = R^a{}_k R'^l{}_b \omega^k{}_{l\mu} - \partial_\mu R^a{}_k R'^k{}_b.$$

It can be shown that the spin connection $\omega^{\bar{a}}{}_{\bar{b}\mu}$ is related to the connection coefficients $\Gamma^\kappa_{\mu\nu}$ by the equation

$$\omega^{\bar{a}}{}_{\bar{b}\mu} = e^{\bar{a}}_\kappa \partial_\mu e^\kappa_{\bar{b}} + e^{\bar{a}}_\kappa \Gamma^\kappa_{\mu\lambda} e^\lambda_{\bar{b}}. \tag{3.3}$$

For the Aristotelian connection coefficients

$$\Gamma^\kappa_{\mu\nu} = v^\kappa \partial_\mu n_\nu + \frac{1}{2} h^{\kappa\lambda}(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\lambda\mu} - \partial_\lambda h_{\mu\nu}),$$

relation 3.3 gives us

$$\omega^{\bar{a}=0}{}_{\bar{b}=0,\mu} = \omega^{\bar{a}=0}{}_{b\mu} = 0, \quad \omega^a{}_{\bar{b}=0,\mu} = \frac{1}{2} e^{a\kappa} \pounds_v h_{\kappa\mu}$$

and

$$\omega^a{}_{b\mu} = \frac{1}{2}\left(e^a_\kappa \partial_\mu e^\kappa_b - e^b_\kappa \partial_\mu e^\kappa_a\right) - e^\kappa_{[a} e^\lambda_{b]} \partial_\kappa h_{\lambda\mu},$$

from which we see that

$$\omega^{ab}{}_\mu = -\omega^{ba}{}_\mu.$$

We can now prove for the spatial tensors $h_{\mu\nu}$, $h^{\mu\nu}$ and h^μ_ν the following

$$\nabla_\mu h_{ab} = \nabla_\mu h^{ab} = \nabla_\mu h^\nu_a = 0 \quad \text{and} \quad \nabla_\mu h^a_\nu = -\frac{1}{2} n_\nu e^{a\kappa} \pounds_v h_{\kappa\mu}.$$

We will now continue our analysis by identifying the proper background sources our Aristotelian geometry should be coupled to in the first order formulation. In the second order formulation we had n_μ and $h_{\mu\nu}$ representing diffeomorphism invariance. Now in the first order formulation diffeomorphism invariance will be represented by n_μ and e^a_μ. For the vielbeins we also have the spatial rotation transformation

$$e^a \to R^a{}_b e^b.$$

The $U(1)$ symmetry is again imposed by the gauge field A_μ (the only one changing under $U(1)$ transformations) with gauge transformation

$$A_\mu \to A_\mu + \nabla_\mu \Lambda$$

and Λ the $U(1)$ gauge parameter. The dipole gauge field $a_{\mu\nu}$ is spatial and symmetric, so

$$a_{\mu\nu} = a_{ab} e^a_\mu e^b_\nu$$

or

$$a_{ab} = a_{\mu\nu} e^\mu_a e^\nu_b$$

with $a_{ab} = a_{ba}$ symmetric. The dipole shift parameter ψ_μ is spatial too, so

$$\psi_\mu = e^a_\mu \psi_a$$

or

$$\psi_a = e^\mu_a \psi_\mu.$$

We can show that the $U(1)$ gauge field A_μ and the (spatial) dipole gauge field a_{ab} change under a dipole shift transformation as

$$A_\mu \to A_\mu + e^a_\mu \psi_a, \quad a_{ab} \to a_{ab} + e^\mu_a \nabla_\mu \psi_b + e^\mu_b \nabla_\mu \psi_a.$$

In the vielbein language the modified dipole gauge field $A^\mu{}_\nu$, which satisfies $n_\mu A^\mu{}_\nu = 0$, is spatial only in the μ index, so

$$A^\mu{}_\nu = e^\mu_a A^a{}_\nu$$

or

$$A^a{}_\mu = e^a_\kappa A^\kappa{}_\mu,$$

which gives us

$$A^a{}_\mu = n_\mu v^\kappa F_{\kappa\lambda} e^{\lambda a} + \frac{1}{2}(h^\kappa_\mu F_{\kappa\lambda} e^{\lambda a} + a^{ak} e_{k\mu}).$$

Under a dipole shift we can prove the modified dipole gauge field $A^a{}_\mu$ changes as

$$A^a{}_\mu \to A^a{}_\mu + \nabla_\mu \psi^a + \frac{1}{2} n_\mu e^{a\kappa} \pounds_v h_{\kappa\lambda} e^\lambda_k \psi^k.$$

The modified field strength $F^\kappa{}_{\mu\nu}$, which obeys $n_\kappa F^\kappa{}_{\mu\nu} = 0$, is spatial only in the κ index, so

$$F^\kappa{}_{\mu\nu} = e^\kappa_a F^a{}_{\mu\nu}$$

or

$$F^a{}_{\mu\nu} = e^a_\kappa F^\kappa{}_{\mu\nu},$$

which gives us

$$F^a{}_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + \omega^a{}_{k\mu} A^k{}_\nu - \omega^a{}_{k\nu} A^k{}_\mu + n_{[\mu} A^b{}_{\nu]} e^\kappa_b \pounds_v h_{\kappa\lambda} e^{\lambda a}.$$

Chapter 4

Aristotelian Anomalies

4.1 Symmetry and Conservation Laws

4.1.1 Second order formulation

The main objective of this chapter is to identify the quantum anomalies that characterize a theory with Aristotelian geometry, such as the fracton theories we study in this thesis. For that purpose we will focus on the symmetries of our theory imposed by the corresponding background source gauge fields we have introduced so far. For more information on the techniques used in this chapter see [56, 111, 112, 113, 114, 115, 116, 117]. The notation and methodology of this chapter will be again heavily based on [67]. Let us start with the second order background sources n_μ, $h_{\mu\nu}$, A_μ and $A^\mu{}_\nu$. First we have our action $S[\Phi; n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu]$ with Φ any matter fields, like the complex scalar fields we studied previously. We then form the generating functional, also known as partition function, given by the functional integral

$$Z[n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu] \equiv \int \mathcal{D}\Phi \, e^{iS[\Phi; n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu]}$$

and define the connected generating functional

$$W[n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu] \equiv -i \log Z[n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu].$$

The currents coupled to the background fields are defined by the variation of W as

$$\delta W = \int d^{d+1}x \sqrt{\gamma} \left[-\epsilon^\mu \delta n_\mu + \left(v^{(\mu} \pi^{\nu)} + \frac{1}{2} \tau^{\mu\nu} \right) \delta h_{\mu\nu} + J^\mu \delta A_\mu + J^\mu{}_\lambda \delta A^\lambda{}_\mu \right] \quad (4.1)$$

with $\gamma = \det(\gamma_{\mu\nu}) = \det(n_\mu n_\nu + h_{\mu\nu})$. In the above we have the energy current ϵ^μ, momentum current π^μ, stress tensor $\tau^{\mu\nu}$, $U(1)$ current J^μ and dipole current $J^\mu{}_\lambda$. We can raise or lower greek indices by using $h_{\mu\nu}$ and $h^{\mu\nu}$, e.g. we can define $\pi_\mu \equiv h_{\mu\nu}\pi^\nu$, $J^{\mu\nu} \equiv h^{\nu\kappa} J^\mu{}_\kappa$. We take π^μ, $\tau^{\mu\nu}$ and $J^{\mu\nu}$ to be spatial tensors, and $\tau^{\mu\nu}$ and $J^{\mu\nu}$ to be symmetric.

We are considering dipole symmetric theories equipped with appropriate gauge fields such that the total action is invariant under diffeomorphism, $U(1)$ and dipole transformations. Under infinitesimal diffeomorphism transformations $x'(x) = x + \epsilon\xi(x) + \mathcal{O}(\epsilon^2)$ we know that tensors T change as $T \to T + \epsilon\delta_\xi T + \mathcal{O}(\epsilon^2)$ with

$$\delta_\xi T = \pounds_\xi T,$$

where ξ^μ is the gauge parameter for infinitesimal diffeomorphism transformations. For that reason the Lie derivative of our background sources are of great importance. Some examples are the following

$$\begin{aligned} \pounds_\xi n_\mu &= \nabla_\mu(n_\kappa \xi^\kappa) + \xi^\kappa F^n_{\kappa\mu} \\ \pounds_\xi h_{\mu\nu} &= \xi^\kappa \nabla_\kappa h_{\mu\nu} + 2h_{\kappa(\mu}\nabla_{\nu)}\xi^\kappa \\ \pounds_\xi A_\mu &= \nabla_\mu(A_\kappa \xi^\kappa) + \xi^\kappa F_{\kappa\mu}. \end{aligned}$$

The gauge parameter for an infinitesimal $U(1)$ transformation is Λ, while for an infinitesimal dipole transformation the gauge parameter is ψ_μ. The total transformation can be denoted as $\hat{X} = (\xi^\mu, \Lambda, \psi_\mu)$ and we can see that the action of the total transformation $\hat{X}$ on the backround gauge fields is a sum of the action of the individual transformations it is comprised from, i.e.

$$\delta_{\hat{X}} = \delta_\xi + \delta_\Lambda + \delta_\psi.$$

In particular, for our background gauge fields n_μ, $h_{\mu\nu}$, A_μ and $A^\mu{}_\nu$ we have

$$\delta_{\hat{X}} n_\mu = \pounds_\xi n_\mu \tag{4.2a}$$

$$\delta_{\hat{X}} h_{\mu\nu} = \pounds_\xi h_{\mu\nu} \tag{4.2b}$$

$$\delta_{\hat{X}} A_\mu = \pounds_\xi A_\mu + \partial_\mu \Lambda + \psi_\mu \tag{4.2c}$$

$$\delta_{\hat{X}} A^\mu{}_\nu = \pounds_\xi A^\mu{}_\nu + \nabla_\nu \psi^\mu + n_\nu \psi^\kappa \nabla_\kappa v^\mu. \tag{4.2d}$$

When there are no quantum anomalies in our theory, the (connected) generating functional W is invariant under diffeomorphism, $U(1)$ and dipole transformations, i.e.

$$\delta_{\hat{X}} W = 0.$$

To find $\delta_{\hat{X}} W$ we just substitute equations 4.2 in 4.1 and from the symmetry condition $\delta_{\hat{X}} W = 0$ we find with integration by parts the conservation laws[1]

$$\nabla'_\mu \epsilon^\mu = -f_\mu v^\mu - \left(\tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}}\right) \nabla_\mu v^\kappa h_{\kappa\nu} \tag{4.3a}$$

$$\nabla'_\mu \left(v^\mu \pi^\nu + \tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}}\right) = f_\mu h^{\mu\nu} - h^{\nu\kappa} \nabla_\kappa v^\mu \pi_\mu \tag{4.3b}$$

$$\nabla'_\mu J^\mu = 0 \tag{4.3c}$$

$$\nabla'_\mu J^{\mu\nu} = h^\nu_\mu J^\mu, \tag{4.3d}$$

[1] A guide on how to prove equation 4.3d is laid out below.

where we made the following auxiliary definitions

$$\begin{aligned}
\nabla'_\mu &\equiv \nabla_\mu + F^n_{\mu\nu} v^\nu \\
f_\mu &\equiv -F^n_{\mu\nu}\epsilon^\nu - h_{\mu\kappa} A^\kappa{}_\lambda J^\lambda + F^\nu{}_{\mu\kappa} J^{\kappa\lambda} h_{\lambda\nu} - n_\mu A^\kappa{}_\lambda J^\lambda{}_\nu \nabla_\kappa v^\nu \\
\tau^{\mu\nu}_{\mathrm{d}} &\equiv -A^\mu{}_\kappa J^{\kappa\nu}.
\end{aligned}$$

It is worth looking closer to how equation 4.3d is derived. First we can show that

$$\delta_\psi W = \int d^{d+1}x \sqrt{\gamma}\, K^\nu \psi_\nu,$$

where

$$K^\nu \equiv J^\nu - \nabla'_\mu J^{\mu\nu} + h^{\nu\kappa} \nabla_\kappa v^\lambda J^\mu{}_\lambda n_\mu.$$

Using the fact that $J^{\mu\nu}$ is spatial and symmetric, K^ν is simplified to

$$K^\nu \equiv J^\nu - \nabla'_\mu J^{\mu\nu}.$$

Also ψ_μ is spatial satisfying the condition $v^\mu \psi_\mu = 0$, leading to

$$h^\mu_\nu K^\nu = 0.$$

Now we should notice that if for some tensor T we have

$$n_\mu T^{\mu\cdots}{}_{\cdots} = 0,$$

then

$$h^\mu_\nu T^{\nu\cdots}{}_{\cdots} = T^{\mu\cdots}{}_{\cdots}.$$

We can easily see that this holds true also for the following

$$h^\mu_\kappa \nabla_\nu T^{\kappa\cdots}{}_{\cdots} = \nabla_\nu T^{\mu\cdots}{}_{\cdots}$$

and

$$h^\mu_\kappa \nabla'_\nu T^{\kappa\cdots}{}_{\cdots} = \nabla'_\nu T^{\mu\cdots}{}_{\cdots}.$$

Putting everything together we reach our final result

$$\nabla'_\mu J^{\mu\nu} = h^\nu_\mu J^\mu.$$

4.1.2 First order formulation

Now we will approach the same problem from the first order formulation side, where the first order background sources are n_μ, e^a_μ, A_μ and $A^a{}_\mu$. The generating function will be of the form $W[n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu]$ with the currents being given by its infinitesimal variation

$$\delta W = \int d^{d+1}x \sqrt{\gamma}\, \left(-\epsilon^\mu \delta n_\mu + \tau^\mu{}_a \delta e^a_\mu + J^\mu \delta A_\mu + J^\mu{}_a \delta A^a{}_\mu\right) \tag{4.4}$$

with $\gamma = \det(\gamma_{\mu\nu}) = \det(n_\mu n_\nu + \delta_{ab} e^a_\mu e^b_\nu)$, $\tau^\mu{}_a \equiv (v^\mu \pi^\nu + \tau^{\mu\nu} + \tau^{\mu\nu}_{\mathrm{d}}) e_{\nu a}$ and $J^\mu{}_a = J^\mu{}_\nu e^\nu_a$.

We are considering dipole symmetric theories equipped with appropriate gauge fields such that the total action is invariant under diffeomorphism, spatial $SO(d)$, $U(1)$ and dipole transformations. Under infinitesimal spatial $SO(d)$ transformations $R^a{}_b(x) = \delta^a_b - \epsilon \Omega^a{}_b(x) + \mathcal{O}(\epsilon^2)$ with $\Omega_{ab} = -\Omega_{ba}$, only the background fields e^a_μ and e^μ_a will be affected. Specifically, we will have

$$\delta_\Omega e^a_\mu = -\Omega^a{}_b e^b_\mu \quad \text{and} \quad \delta_\Omega e^\mu_a = \Omega^b{}_a e^\mu_b,$$

where $\Omega^a{}_b$ is the gauge parameter for infinitesimal spatial $SO(d)$ transformations. The gauge parameter for an infinitesimal diffeomorphism transformation is ξ^μ, for an infinitesimal $U(1)$ transformation is Λ, while for an infinitesimal dipole transformation the gauge parameter is ψ_a. The gauge parameter ξ^μ is a vector, Λ a function and ψ_a a 1-form. The gauge parameter $\Omega^a{}_b$ will be taken to be a spatial $(1,1)$ tensor.

The total transformation can be denoted as $\hat{X} = (\xi^\mu, \Omega^a{}_b, \Lambda, \psi_a)$ and we can again see that the action of the total transformation $\hat{X}$ on the backround gauge fields is a sum of the action of the individual transformations it is comprised from, i.e.

$$\delta_{\hat{X}} = \delta_\xi + \delta_\Omega + \delta_\Lambda + \delta_\psi.$$

In particular, for our background gauge fields n_μ, e^a_μ, A_μ and $A^a{}_\mu$ we have[2]

$$\delta_{\hat{X}} n_\mu = \pounds_\xi n_\mu \tag{4.5a}$$
$$\delta_{\hat{X}} e^a_\mu = \pounds_\xi e^a_\mu - \Omega^a{}_b e^b_\mu \tag{4.5b}$$
$$\delta_{\hat{X}} A_\mu = \pounds_\xi A_\mu + \partial_\mu \Lambda + e^a_\mu \psi_a \tag{4.5c}$$
$$\delta_{\hat{X}} A^a{}_\mu = \pounds_\xi A^a{}_\mu - \Omega^a{}_b A^b{}_\mu + \nabla_\mu \psi^a + \frac{1}{2} n_\mu e^{a\kappa} \pounds_v h_{\kappa\lambda} e^\lambda_b \psi^b. \tag{4.5d}$$

Some miscellaneous variations we should note are

$$\begin{aligned} \delta_{\hat{X}} a_{ab} &= \pounds_\xi a_{ab} + \Omega^c{}_a a_{cb} + \Omega^c{}_b a_{ac} + e^\mu_a \nabla_\mu \psi_b + e^\mu_b \nabla_\mu \psi_a \\ \delta_{\hat{X}} \omega^a{}_{b\mu} &= \pounds_\xi \omega^a{}_{b\mu} + \nabla_\mu \Omega^a{}_b. \end{aligned}$$

The action of the Lie derivative $\pounds_\xi$ in this chapter must be interpreted only in the following way. When we have a tensor with mixed greek and latix indices $T^{\mu\cdots a\cdots}{}_{\nu\cdots b\cdots}$, then $\pounds_\xi T^{\mu\cdots a\cdots}{}_{\nu\cdots b\cdots}$ is the Lie derivative we get if we had fixed the latin indices and considered $T^{\mu\cdots a\cdots}{}_{\nu\cdots b\cdots}$ to be a tensor only with respect to its greek indices. For example, we consider e^a_μ and $A^a{}_\mu$ to be 1-forms, while a_{ab} are functions. The spin connection $\omega^a{}_{b\mu}$ was already a 1-form with respect to its greek index, but was not a tensor with respect to its

[2]Explanation on the use of the Lie derivative $\pounds_\xi$ will be given below.

latin indices. This is reflected on its peculiar transformation under spatial $SO(d)$ rotations $\delta_\Omega \omega^a{}_{b\mu} = \nabla_\mu \Omega^a{}_b$. Under these guidelines, it should be easy to see how the variation of the first order background fields 4.5 reproduces the variation of the second order background fields 4.2. We must also notice how the spatial $SO(d)$ rotation of the spatial vielbeins e^a_μ and e^μ_a is a symmetry hidden in the second order formulation. It only emerges in the first order formulation through the spatial vielbeins e^a_μ and e^μ_a.

When there are no quantum anomalies in our theory, the generating functional W is invariant under diffeomorphism, spatial $SO(d)$, $U(1)$ and dipole transformations, i.e.

$$\delta_{\hat{X}} W = 0.$$

To find $\delta_{\hat{X}} W$ we just substitute equations 4.5 in 4.4 and from the symmetry condition $\delta_{\hat{X}} W = 0$ we find with integration by parts the conservation laws

$$\nabla'_\mu \epsilon^\mu = -f_\mu v^\mu - \tau^\mu{}_a \nabla_\mu v^\nu e^a_\nu \tag{4.6a}$$

$$\nabla'_\mu \tau^\mu{}_a = e^\mu_a f_\mu - n_\mu \tau^\mu{}_b e^b_\nu e^\kappa_a \nabla_\kappa v^\nu \tag{4.6b}$$

$$\nabla'_\mu J^\mu = 0 \tag{4.6c}$$

$$\nabla'_\mu J^\mu{}_a = e_{a\mu} J^\mu. \tag{4.6d}$$

Invariance of the generating functional W under spatial $SO(d)$ rotations gives us $\tau^\mu{}_{[a} e_{b]\mu} = -J^\mu{}_{[a} A_{b]\mu}$, which holds identically. From this redundant conservation law we are reminded again of the hidden nature of the spatial $SO(d)$ rotation symmetry.

4.2 Aristotelian Symmetry Algebra

In this section we will derive how the symmetry transformations of our Aristotelian geometry form a Lie algebra. We will focus on the more fundamental first order formulation of Aristotelian geometry from now on. Specifically, we need to demonstrate that for any transformations $\hat{X}$ and $\hat{X}'$ they satisfy

$$[\delta_{\hat{X}'}, \delta_{\hat{X}}] = \delta_{[\hat{X}', \hat{X}]}, \tag{4.7}$$

having thus a closed algebra. However, the commutator

$$[\hat{X}', \hat{X}] = (\xi^\mu_{[\hat{X}',\hat{X}]}, \Omega^a{}_{b[\hat{X}',\hat{X}]}, \Lambda_{[\hat{X}',\hat{X}]}, \psi_{a[\hat{X}',\hat{X}]})$$

is not specified yet. We will define it as

$$\boxed{\begin{aligned} \xi^\mu_{[\hat{X}',\hat{X}]} &\equiv \pounds_{\xi'} \xi^\mu = [\xi', \xi]^\mu && (4.8a) \\ \Omega^a{}_{b[\hat{X}',\hat{X}]} &\equiv \pounds_{\xi'} \Omega^a{}_b - \pounds_\xi \Omega'^a{}_b + \Omega^a{}_c \Omega'^c{}_b - \Omega'^a{}_c \Omega^c{}_b && (4.8b) \\ \Lambda_{[\hat{X}',\hat{X}]} &\equiv \pounds_{\xi'} \Lambda - \pounds_\xi \Lambda' && (4.8c) \\ \psi_{a[\hat{X}',\hat{X}]} &\equiv \pounds_{\xi'} \psi_a - \pounds_\xi \psi'_a + \psi_b \Omega'^b{}_a - \psi'_b \Omega^b{}_a, && (4.8d) \end{aligned}}$$

where we also made use of the identity

$$\pounds_V U = [V, U]$$

for any vectors V and U. The action of the operators $\delta_{\hat{X}}$ on the background gauge fields n_μ, e^a_μ, A_μ and $A^a{}_\mu$ was already identified in Section 4.1.2. We will also need to see how the gauge transformations $\hat{X}$ act on the gauge parameters ξ^μ, $\Omega^a{}_b$, Λ and ψ_a themselves. To fulfill this necessity we will make the following definition

$$\boxed{\begin{aligned} \delta_{\hat{X}'}\xi^\mu &\equiv \pounds_{\xi'}\xi^\mu = [\xi', \xi]^\mu && (4.9a)\\ \delta_{\hat{X}'}\Omega^a{}_b &\equiv \pounds_{\xi'}\Omega^a{}_b - \pounds_\xi \Omega'^a{}_b + \Omega^a{}_c\Omega'^c{}_b - \Omega'^a{}_c\Omega^c{}_b && (4.9b)\\ \delta_{\hat{X}'}\Lambda &\equiv \pounds_{\xi'}\Lambda - \pounds_\xi\Lambda' && (4.9c)\\ \delta_{\hat{X}'}\psi_a &\equiv \pounds_{\xi'}\psi_a - \pounds_\xi\psi'_a + \psi_b\Omega'^b{}_a - \psi'_b\Omega^b{}_a. && (4.9d) \end{aligned}}$$

From the above definitions 4.8 and 4.9 we see that

$$\boxed{\begin{aligned} \xi^\mu_{[\hat{X}',\hat{X}]} &= \delta_{\hat{X}'}\xi^\mu \\ \Omega^a{}_{b[\hat{X}',\hat{X}]} &= \delta_{\hat{X}'}\Omega^a{}_b \\ \Lambda_{[\hat{X}',\hat{X}]} &= \delta_{\hat{X}'}\Lambda \\ \psi_{a[\hat{X}',\hat{X}]} &= \delta_{\hat{X}'}\psi_a \end{aligned}}$$

or

$$[\hat{X}', \hat{X}] = \delta_{\hat{X}'}\hat{X}.$$

We can write all these preliminary definitions in a more abstract but compact form. To achieve this goal, from now on we will also make frequent use of matrix notation by denoting tensors like $\Omega^a{}_c\Omega'^c{}_b$ and $\psi_b\Omega'^b{}_a$ as just $\Omega\Omega'$ and $\psi\Omega'$, respectively. Hence all of our definitions can be presented in a readable fashion as

$$\xi_{[\hat{X}',\hat{X}]} \equiv \delta_{\hat{X}'}\xi \equiv \pounds_{\xi'}\xi = [\xi', \xi] \tag{4.10a}$$

$$\Omega_{[\hat{X}',\hat{X}]} \equiv \delta_{\hat{X}'}\Omega \equiv \pounds_{\xi'}\Omega - \pounds_\xi\Omega' + [\Omega, \Omega'] \tag{4.10b}$$

$$\Lambda_{[\hat{X}',\hat{X}]} \equiv \delta_{\hat{X}'}\Lambda \equiv \pounds_{\xi'}\Lambda - \pounds_\xi\Lambda' \tag{4.10c}$$

$$\psi_{[\hat{X}',\hat{X}]} \equiv \delta_{\hat{X}'}\psi \equiv \pounds_{\xi'}\psi - \pounds_\xi\psi' + \psi\Omega' - \psi'\Omega, \tag{4.10d}$$

where $[\Omega, \Omega']$ represents the tensor $[\Omega, \Omega']^a{}_b \equiv (\Omega\Omega')^a{}_b - (\Omega'\Omega)^a{}_b \equiv \Omega^a{}_c\Omega'^c{}_b - \Omega'^a{}_c\Omega^c{}_b$.

We will now try to calculate the action of the commutator $[\delta_{\hat{X}''}, \delta_{\hat{X}'}]$ while acting on the gauge parameters ξ^μ, $\Omega^a{}_b$, Λ and ψ_a, in order to show that $[\delta_{\hat{X}''}, \delta_{\hat{X}'}] = \delta_{[\hat{X}'',\hat{X}']}$. For this purpose we will need the property

$$[\pounds_V, \pounds_U]T = \pounds_{[V,U]}T$$

for any tensor T and any vectors V and U, where, obviously, $[\pounds_V, \pounds_U]T = \pounds_V \pounds_U T - \pounds_U \pounds_V T$. Let us calculate $\delta_{\hat{X}''}\delta_{\hat{X}'}\xi^\mu$ first. We have in our compact notation

$$\begin{aligned}\delta_{\hat{X}''}\delta_{\hat{X}'}\xi = \delta_{\hat{X}''}\pounds_{\xi'}\xi = \pounds_{\delta_{\hat{X}''}\xi'}\xi + \pounds_{\xi'}\delta_{\hat{X}''}\xi = \pounds_{\pounds_{\xi''}\xi'}\xi + \pounds_{\xi'}\pounds_{\xi''}\xi \\ = \pounds_{[\xi'',\xi']}\xi + \pounds_{\xi'}\pounds_{\xi''}\xi = [\pounds_{\xi''}, \pounds_{\xi'}]\xi + \pounds_{\xi'}\pounds_{\xi''}\xi = \pounds_{\xi''}\pounds_{\xi'}\xi.\end{aligned}$$

Alternatively, we could just define $\tilde{\xi} \equiv \xi_{[\hat{X}',\hat{X}]}$ and write

$$\delta_{\hat{X}''}\delta_{\hat{X}'}\xi = \delta_{\hat{X}''}\xi_{[\hat{X}',\hat{X}]} = \delta_{\hat{X}''}\tilde{\xi} = \pounds_{\xi''}\tilde{\xi} = \pounds_{\xi''}\xi_{[\hat{X}',\hat{X}]} = \pounds_{\xi''}\pounds_{\xi'}\xi.$$

Since we now have that

$$\delta_{\hat{X}''}\delta_{\hat{X}'}\xi = \pounds_{\xi''}\pounds_{\xi'}\xi$$

we can form $[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\xi^\mu$ and find

$$[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\xi = \pounds_{\xi''}\pounds_{\xi'}\xi - \pounds_{\xi'}\pounds_{\xi''}\xi = [\pounds_{\xi''}, \pounds_{\xi'}]\xi = \pounds_{[\xi'',\xi']}\xi = \pounds_{\xi_{[\hat{X}'',\hat{X}']}}\xi = \delta_{[\hat{X}'',\hat{X}']}\xi$$

and we just showed that

$$[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\xi = \delta_{[\hat{X}'',\hat{X}']}\xi.$$

In calculating $\delta_{\hat{X}''}\delta_{\hat{X}'}\Lambda$ we have

$$\begin{aligned}\delta_{\hat{X}''}\delta_{\hat{X}'}\Lambda &= \delta_{\hat{X}''}\left(\pounds_{\xi'}\Lambda - \pounds_{\xi}\Lambda'\right) = \pounds_{\delta_{\hat{X}''}\xi'}\Lambda + \pounds_{\xi'}\delta_{\hat{X}''}\Lambda - \pounds_{\delta_{\hat{X}''}\xi}\Lambda' - \pounds_{\xi}\delta_{\hat{X}''}\Lambda' \\ &= \pounds_{[\xi'',\xi']}\Lambda + \pounds_{\xi'}(\pounds_{\xi''}\Lambda - \pounds_{\xi}\Lambda'') - \pounds_{[\xi'',\xi]}\Lambda' - \pounds_{\xi}(\pounds_{\xi''}\Lambda' - \pounds_{\xi'}\Lambda'') \\ &= [\pounds_{\xi''}, \pounds_{\xi'}]\Lambda + \pounds_{\xi'}\pounds_{\xi''}\Lambda - \pounds_{\xi'}\pounds_{\xi}\Lambda'' - [\pounds_{\xi''}, \pounds_{\xi}]\Lambda' - \pounds_{\xi}\pounds_{\xi''}\Lambda' + \pounds_{\xi}\pounds_{\xi'}\Lambda'' \\ &= \pounds_{\xi''}\pounds_{\xi'}\Lambda - \pounds_{\xi'}\pounds_{\xi}\Lambda'' - \pounds_{\xi''}\pounds_{\xi}\Lambda' + \pounds_{\xi}\pounds_{\xi'}\Lambda'' \\ &= \pounds_{\xi''}\pounds_{\xi'}\Lambda + \pounds_{\xi}\pounds_{\xi'}\Lambda'' \\ &+ [-\pounds_{\xi'}\pounds_{\xi}\Lambda'' + (\hat{X}' \leftrightarrow \hat{X}'')]\end{aligned}$$

In the last line of the above equation the symbol $(\hat{X}' \leftrightarrow \hat{X}'')$ in the overall bracket $[\cdots]$ means that we add all the other terms in the bracket $[\cdots]$ but with $\hat{X}'$ and $\hat{X}''$ exchanged, e.g. in this case $(\hat{X}' \leftrightarrow \hat{X}'') = -\pounds_{\xi''}\pounds_{\xi}\Lambda'$. Since

$$\begin{aligned}\delta_{\hat{X}''}\delta_{\hat{X}'}\Lambda = \pounds_{\xi''}\pounds_{\xi'}\Lambda + \pounds_{\xi}\pounds_{\xi'}\Lambda'' \\ + [-\pounds_{\xi'}\pounds_{\xi}\Lambda'' + (\hat{X}' \leftrightarrow \hat{X}'')],\end{aligned}$$

where the terms in the bracket $[\cdots]$ are symmetric under the exchange $\hat{X}' \leftrightarrow \hat{X}''$, we get

$$\begin{aligned}[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\Lambda &= \delta_{\hat{X}''}\delta_{\hat{X}'}\Lambda - \delta_{\hat{X}'}\delta_{\hat{X}''}\Lambda = (\pounds_{\xi''}\pounds_{\xi'}\Lambda + \pounds_{\xi}\pounds_{\xi'}\Lambda'') - (\pounds_{\xi'}\pounds_{\xi''}\Lambda + \pounds_{\xi}\pounds_{\xi''}\Lambda') \\ &= [\pounds_{\xi''}, \pounds_{\xi'}]\Lambda - \pounds_{\xi}(\pounds_{\xi''}\Lambda' - \pounds_{\xi'}\Lambda'') = \pounds_{[\xi'',\xi']}\Lambda - \pounds_{\xi}\Lambda_{[\hat{X}'',\hat{X}']} \\ &= \pounds_{\xi_{[\hat{X}'',\hat{X}']}}\Lambda - \pounds_{\xi}\Lambda_{[\hat{X}'',\hat{X}']} = \delta_{[\hat{X}'',\hat{X}']}\Lambda\end{aligned}$$

and we proved that

$$[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\Lambda = \delta_{[\hat{X}'',\hat{X}']}\Lambda.$$

We can follow the same procedure for $\Omega^a{}_b$ and ψ_a. In total, we get

$$\begin{aligned}
\delta_{\hat{X}''}\delta_{\hat{X}'}\xi &= \pounds_{\xi''}\pounds_{\xi'}\xi \\
\delta_{\hat{X}''}\delta_{\hat{X}'}\Omega &= \pounds_{\xi''}\pounds_{\xi'}\Omega + \pounds_{\xi}(\pounds_{\xi'}\Omega'' - \Omega'\Omega'') - \Omega(\pounds_{\xi'}\Omega'' - \Omega'\Omega'') \\
&\quad + (\pounds_{\xi'}\Omega'' + \Omega''\Omega')\Omega + [\Omega''\pounds_{\xi}\Omega' - \Omega''\Omega\Omega' \\
&\quad - \pounds_{\xi'}(\pounds_{\xi}\Omega'' - \Omega\Omega'' + \Omega''\Omega) + (\hat{X}' \leftrightarrow \hat{X}'')] \\
\delta_{\hat{X}''}\delta_{\hat{X}'}\Lambda &= \pounds_{\xi''}\pounds_{\xi'}\Lambda + \pounds_{\xi}\pounds_{\xi'}\Lambda'' + [-\pounds_{\xi'}\pounds_{\xi}\Lambda'' + (\hat{X}' \leftrightarrow \hat{X}'')] \\
\delta_{\hat{X}''}\delta_{\hat{X}'}\psi &= \pounds_{\xi''}\pounds_{\xi'}\psi + \pounds_{\xi}(\pounds_{\xi'}\psi'' + \psi''\Omega') - \psi(\pounds_{\xi'}\Omega'' - \Omega'\Omega'') \\
&\quad + (\pounds_{\xi'}\psi'' + \psi''\Omega')\Omega + [-\psi'\Omega\Omega'' - \pounds_{\xi}\psi'\Omega'' \\
&\quad + \pounds_{\xi'}(-\pounds_{\xi}\psi'' + \psi\Omega'' - \psi''\Omega) + (\hat{X}' \leftrightarrow \hat{X}'')].
\end{aligned}$$

From the above we can prove that

$$\begin{aligned}
[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\xi &= \delta_{[\hat{X}'', \hat{X}']}\xi \\
[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\Omega &= \delta_{[\hat{X}'', \hat{X}']}\Omega \\
[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\Lambda &= \delta_{[\hat{X}'', \hat{X}']}\Lambda \\
[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\psi &= \delta_{[\hat{X}'', \hat{X}']}\psi.
\end{aligned}$$

or

$$[\delta_{\hat{X}''}, \delta_{\hat{X}'}]\hat{X} = \delta_{[\hat{X}'', \hat{X}']}\hat{X}.$$

It is now easy to derive the Jacobi identity

$$[\hat{X}'', [\hat{X}', \hat{X}]] + [\hat{X}', [\hat{X}, \hat{X}'']] + [\hat{X}, [\hat{X}'', \hat{X}']] = 0.$$

Here are the simple steps

$$\begin{aligned}
[\hat{X}'', [\hat{X}', \hat{X}]] = -[[\hat{X}', \hat{X}], \hat{X}''] &= -\delta_{[\hat{X}', \hat{X}]}\hat{X}'' = -[\delta_{\hat{X}'}, \delta_{\hat{X}}]\hat{X}'' = \\
&= -\delta_{\hat{X}'}\delta_{\hat{X}}\hat{X}'' + \delta_{\hat{X}}\delta_{\hat{X}'}\hat{X}'' \\
&= -\delta_{\hat{X}'}[\hat{X}, \hat{X}''] + \delta_{\hat{X}}[\hat{X}', \hat{X}''] \\
&= -[\hat{X}', [\hat{X}, \hat{X}'']] + [\hat{X}, [\hat{X}', \hat{X}'']] \\
&= -[\hat{X}', [\hat{X}, \hat{X}'']] - [\hat{X}, [\hat{X}'', \hat{X}']].
\end{aligned}$$

To show that $[\delta_{\hat{X}'}, \delta_{\hat{X}}] = \delta_{[\hat{X}', \hat{X}]}$ we can calculate the action of the commutator $[\delta_{\hat{X}'}, \delta_{\hat{X}}]$ while acting on the background gauge fields n_μ, e^a_μ, A_μ and a_{ab}. In compact notation we will denote the background gauge fields as n, e, A and a. Also the compact notation for e^μ_a will be $\tilde{e}$ and for $(\Omega^T)_b{}^a = \Omega^a{}_b$ will be Ω^T, e.g. $\left(\Omega^T\tilde{e}\right)_a{}^\mu = (\Omega^T)_a{}^b e^\mu_b =$

$e^\mu_b \Omega^b{}_a = (\tilde{e}\Omega)^\mu{}_a$. After a lot of computation we arrive at the following compact relations

$$\begin{aligned}
\delta_{\hat{X}'}\delta_{\hat{X}} n &= \pounds_{\xi'}\pounds_\xi n \\
\delta_{\hat{X}'}\delta_{\hat{X}} e &= \pounds_{\xi'}\pounds_\xi e + (\pounds_\xi \Omega' + \Omega'\Omega)e \\
&\quad + [-\pounds_\xi(\Omega' e) + (\hat{X} \leftrightarrow \hat{X}')] \\
\delta_{\hat{X}'}\delta_{\hat{X}} A &= \pounds_{\xi'}\pounds_\xi A - \partial\pounds_\xi \Lambda' - (\pounds_\xi \psi' + \psi'\Omega)e \\
&\quad + [\pounds_\xi \partial\Lambda' + \pounds_\xi(\psi' e) + (\hat{X} \leftrightarrow \hat{X}')] \\
\delta_{\hat{X}'}\delta_{\hat{X}} a &= \pounds_{\xi'}\pounds_\xi a - a(\pounds_\xi \Omega' - \Omega\Omega') - [a(\pounds_\xi \Omega' - \Omega\Omega')]^T \\
&\quad - \tilde{e}\nabla(\pounds_\xi \psi' + \psi'\Omega) - [\tilde{e}\nabla(\pounds_\xi \psi' + \psi'\Omega)]^T \\
&\quad + [(\Omega^T \tilde{e}\nabla\psi') + (\Omega^T \tilde{e}\nabla\psi')^T + (\tilde{e}\nabla\psi'\Omega) + (\tilde{e}\nabla\psi'\Omega)^T \\
&\quad + (\Omega^T a \Omega') + \pounds_\xi(\Omega'^T a + (\Omega'^T a)^T + \tilde{e}\nabla\psi' + (\tilde{e}\nabla\psi')^T) \\
&\quad + (\hat{X} \leftrightarrow \hat{X}')],
\end{aligned}$$

from which we, indeed, find

$$\begin{aligned}
[\delta_{\hat{X}'}, \delta_{\hat{X}}] n &= \delta_{[\hat{X}', \hat{X}]} n \\
[\delta_{\hat{X}'}, \delta_{\hat{X}}] e &= \delta_{[\hat{X}', \hat{X}]} e \\
[\delta_{\hat{X}'}, \delta_{\hat{X}}] A &= \delta_{[\hat{X}', \hat{X}]} A \\
[\delta_{\hat{X}'}, \delta_{\hat{X}}] a &= \delta_{[\hat{X}', \hat{X}]} a,
\end{aligned}$$

resulting in the closed algebra

$$[\delta_{\hat{X}'}, \delta_{\hat{X}}] = \delta_{[\hat{X}', \hat{X}]}.$$

In summary, everything we did so far in this section was to show that our Aristotelian symmetry transformations form a Lie algebra and the Wess-Zumino consistency conditions

$$[\delta_{\hat{X}'}, \delta_{\hat{X}}] W = \delta_{[\hat{X}', \hat{X}]} W$$

hold true.

We can now identify the commutator algebra of our Aristotelian symmetry generators. We will follow the approach of [118] and write $\delta_{\hat{X}}$ in the form

$$\delta_{\hat{X}} = i\xi^\mu n_\mu H - i\xi^\mu e^a_\mu P_a + \frac{i}{2}(\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu) M_{ab} - i(\Lambda + \xi^\mu A_\mu) Q^{(0)} - i(\psi^a + \xi^\mu A^a{}_\mu) D_a.$$

Afterwards, we substitute this form into $[\delta_{\hat{X}'}, \delta_{\hat{X}}]$ and find after many cumbersome calculations

$$
\begin{aligned}
[\delta_{\hat{X}'}, \delta_{\hat{X}}] = &\, \delta_{[\hat{X}',\hat{X}]} + \xi^\mu \xi'^\nu \left(e^a_\mu e^b_\nu [P_a, P_b] - (n_\mu e^a_\nu - n_\nu e^a_\mu)[H, P_a] + i\mathcal{C}_{\mu\nu} \right) \\
&+ \frac{1}{2} \left((\omega^{ab} + \xi^\mu \omega^{ab}{}_\mu)\xi'^\nu n_\nu - (\omega'^{ab} + \xi'^\mu \omega^{ab}{}_\mu)\xi^\nu n_\nu \right) [M_{ab}, H] \\
&- \frac{1}{2} \left((\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu)\xi'^\nu e^c_\nu - (\Omega'^{ab} + \xi'^\mu \omega^{ab}{}_\mu)\xi^\nu e^c_\nu \right) ([M_{ab}, P_c] - 2i\delta_{ac}P_b) \\
&+ \frac{1}{4} \left(\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu \right) \left(\Omega'^{cd} + \xi'^\mu \omega^{cd}{}_\mu \right) ([M_{ab}, M_{cd}] - 4i\delta_{ac}M_{bd}) \\
&- \frac{1}{2} \left((\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu)(\psi'^c + \xi'^\nu A^c_\nu) - (\Omega'^{ab} + \xi'^\mu \omega^{ab}{}_\mu)(\psi^c + \xi^\nu A^c_\nu) \right) ([M_{ab}, D_c] - 2i\delta_{ac}D_b) \\
&- (\xi^\mu n_\mu(\Lambda' + \xi'^\nu A_\nu) - \xi'^\mu n_\mu(\Lambda + \xi^\nu A_\nu)) [H, Q^{(0)}] \\
&+ \left(\xi^\mu e^a_\mu (\Lambda' + \xi'^\nu A_\nu) - \xi'^\mu e^a_\mu (\Lambda + \xi^\nu A_\nu) \right) [P_a, Q^{(0)}] \\
&- \frac{1}{2} \left((\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu)(\Lambda' + \xi'^\nu A_\nu) - (\Omega'^{ab} + \xi'^\mu \omega^{ab}{}_\mu)(\Lambda + \xi^\nu A_\nu) \right) [M_{ab}, Q^{(0)}] \\
&+ \left((\psi^a + \xi^\mu A^a_\mu)(\Lambda' + \xi'^\nu A_\nu) - (\psi'^a + \xi'^\mu A^a_\mu)(\Lambda + \xi^\nu A_\nu) \right) [D_a, Q^{(0)}] \\
&- (\xi^\mu n_\mu(\psi'^a + \xi'^\nu A^a_\nu) - \xi'^\mu n_\mu(\psi^a + \xi^\nu A^a_\nu)) ([H, D_a] + \frac{i}{2} e^\rho_a e^{b\sigma} \pounds_v h_{\rho\sigma} D_b) \\
&+ \left(\xi^\mu e^a_\mu(\psi'^b + \xi'^\nu A^b_\nu) - \xi'^\mu e^a_\mu(\psi^b + \xi^\nu A^b_\nu) \right) ([P_a, D_b] - i\delta_{ab}Q^0) \\
&+ (\psi^a + \xi^\mu A^a_\mu)(\psi'^b + \xi'^\nu A^b_\nu)[D_a, D_b],
\end{aligned}
\tag{4.15}
$$

where we defined the auxiliary quantity

$$
\mathcal{C}_{\mu\nu} \equiv -F^n_{\mu\nu} H + 2T^a{}_{\mu\nu} P_a - \frac{1}{2} R^{ab}{}_{\mu\nu} M_{ab} + F^a{}_{\mu\nu} D_a.
$$

We will call $\mathcal{C}_{\mu\nu}$ the curvature operator. In deriving the commutator algebra we will need the following identities

$$
\begin{aligned}
\delta_{\hat{X}'}(\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu) &= \pounds_{\xi'}(\Omega^{ab} + \xi^\mu \omega^{ab}{}_\mu) + 2\Omega'^{[a}{}_c(\Omega^{b]c} + \xi^\mu \omega^{b]c}{}_\mu) \\
\delta_{\hat{X}'}(\Lambda + \xi^\mu A_\mu) &= \pounds_{\xi'}(\Lambda + \xi^\mu A_\mu) + \xi^\mu e^a_\mu \psi'_a \\
\delta_{\hat{X}'}(\psi^a + \xi^\mu A^a_\mu) &= \pounds_{\xi'}(\psi^a + \xi^\mu A^a_\mu) - \Omega'^a{}_b(\psi^b + \xi^\mu A^b_\mu) \\
&\quad + (\Omega^a{}_b + \xi^\mu \omega^a{}_{b\mu})\psi'^b + \frac{1}{2}\xi^\mu n_\mu \psi'^b e^{a\rho} e^\sigma_b \pounds_v h_{\rho\sigma}.
\end{aligned}
$$

The above calculation 4.15 must satisfy the Lie algebra condition 4.7, giving us the Aristotelian symmetry algebra

$$\boxed{\begin{aligned}
[M_{ab}, M_{cd}] &= i(\delta_{ac}M_{bd} - \delta_{bc}M_{ad} - \delta_{ad}M_{bc} + \delta_{bd}M_{ac}) && (4.16a)\\
[M_{ab}, P_c] &= i(\delta_{ac}P_b - \delta_{bc}P_a) && (4.16b)\\
[M_{ab}, D_c] &= i(\delta_{ac}D_b - \delta_{bc}D_a) && (4.16c)\\
[P_a, D_b] &= i\delta_{ab}Q^{(0)} && (4.16d)\\
[P_a, P_b] &= -ie^{\mu}_{a}e^{\nu}_{b}\mathcal{C}_{\mu\nu} && (4.16e)\\
[H, P_a] &= iv^{\mu}e^{\nu}_{a}\mathcal{C}_{\mu\nu} && (4.16f)\\
[H, D_a] &= -\frac{i}{2}e^{\rho}_{a}e^{b\sigma}\pounds_v h_{\rho\sigma}D_b && (4.16g)
\end{aligned}}$$

with the rest vanishing. We can observe now that equations 4.16a to 4.16d are identical to the familiar 2.13 we derived in flat spacetime in Chapter 2. The new components to our symmetry algebra are equations 4.16e to 4.16g whose form exposes the exotic characteristics of Aristotelian geometry, present especially in the curvature operator $\mathcal{C}_{\mu\nu}$.

4.3 Quantum Anomalies

4.3.1 Wess-Zumino conditions

After having carefully studied all the fundamental features of Aristotelian geometry and its symmetries, we are finally ready to tackle the important matter of its quantum anomalies. Let us look at the generating functional

$$W = W[n_\mu, e^a_\mu, A_\mu, A^a{}_\mu]$$

with background source gauge fields n_μ, e^a_μ, A_μ and $A^a{}_\mu$. It can be shown that under our Aristotelian transformations $\hat{X} = (\xi^\mu, \Omega^a{}_b, \Lambda, \psi_a)$, the generating functional changes infinitesimally as

$$\delta_{\hat{X}} W = \int d^{d+1}x\sqrt{\gamma}\,\left(\xi^\mu \mathcal{D}_\mu + \Omega^a{}_b \mathcal{R}^b{}_a + \Lambda\mathcal{U} + \psi_a \mathcal{S}^a\right), \qquad (4.17)$$

where $\mathcal{D}_\mu$, $\mathcal{R}^b{}_a$, $\mathcal{U}$ and $\mathcal{S}^a$ are local functional of the background fields, meaning they all have the form

$$\mathcal{G} = \mathcal{G}(x)[n_\mu, e^a_\mu, A_\mu, A^a{}_\mu].$$

Here the symbols $\mathcal{D}$, $\mathcal{R}$, $\mathcal{U}$ and $\mathcal{S}$ were chosen because they represent diffeomorphism, spatial $SO(d)$ rotation, $U(1)$ and dipole shift transformations, respectively. As we have already explained, if W is invariant under our Aristotelian transformations, i.e. $\delta_{\hat{X}} W = 0$, then our Aristotelian theory is symmetric under these Aristotelian transformations and it has no quantum anomalies, i.e. $\mathcal{D}_\mu = \mathcal{R}^b{}_a = \mathcal{U} = \mathcal{S}^a = 0$. On the other hand, if

$$\delta_{\hat{X}} W \neq 0$$

our Aristotelian theory has quantum anomalies, so some of $\mathcal{D}_\mu$, $\mathcal{R}^b{}_a$, $\mathcal{U}$ or $\mathcal{S}^a$ do not vanish. We will try to find these Aristotelian anomalies using as our main tool the *Wess-Zumino conditions*

$$[\delta_{\hat{X}'}, \delta_{\hat{X}}]W = \delta_{[\hat{X}',\hat{X}]}W.$$

Again, for more details on quantum anomalies see the relevant literature [56, 111, 112, 113, 114, 115, 116, 117].

First consider

$$\delta_{\hat{X}} W = \int d^{d+1}x\sqrt{\gamma}\,\left(\xi^\mu \mathcal{D}_\mu + \Omega^a{}_b \mathcal{R}^b{}_a + \Lambda\mathcal{U} + \psi_a \mathcal{S}^a\right).$$

Note that $\mathcal{D}_\mu$, $\mathcal{R}^a{}_b$, $\mathcal{U}$ and $\mathcal{S}^a$ are not necessarily tensor quantities. Acting with $\delta_{\hat{X}'}$ on the above equation we get

$$\begin{aligned}\delta_{\hat{X}'}\delta_{\hat{X}} W = &\int d^{d+1}x\,\delta_{\hat{X}'}\sqrt{\gamma}\,\left(\xi^\mu \mathcal{D}_\mu + \Omega^a{}_b \mathcal{R}^b{}_a + \Lambda\mathcal{U} + \psi_a \mathcal{S}^a\right)\\ &+\int d^{d+1}x\sqrt{\gamma}\,\left(\xi^\mu\,\delta_{\hat{X}'}\mathcal{D}_\mu + \Omega^a{}_b\,\delta_{\hat{X}'}\mathcal{R}^b{}_a + \Lambda\,\delta_{\hat{X}'}\mathcal{U} + \psi_a\,\delta_{\hat{X}'}\mathcal{S}^a\right)\\ &+\int d^{d+1}x\sqrt{\gamma}\,\left(\delta_{\hat{X}'}\xi^\mu\,\mathcal{D}_\mu + \delta_{\hat{X}'}\Omega^a{}_b\,\mathcal{R}^b{}_a + \delta_{\hat{X}'}\Lambda\mathcal{U} + \delta_{\hat{X}'}\psi_a\,\mathcal{S}^a\right).\end{aligned}$$

But we also have

$$\begin{aligned}\delta_{[\hat{X}',\hat{X}]}W &= \int d^{d+1}x\sqrt{\gamma}\,\left(\xi^\mu_{[\hat{X}',\hat{X}]}\,\mathcal{D}_\mu + \Omega^a{}_{b[\hat{X}',\hat{X}]}\,\mathcal{R}^b{}_a + \Lambda_{[\hat{X}',\hat{X}]}\mathcal{U} + \psi_{a[\hat{X}',\hat{X}]}\,\mathcal{S}^a\right)\\ &= \int d^{d+1}x\sqrt{\gamma}\,\left(\delta_{\hat{X}'}\xi^\mu\,\mathcal{D}_\mu + \delta_{\hat{X}'}\Omega^a{}_b\,\mathcal{R}^b{}_a + \delta_{\hat{X}'}\Lambda\mathcal{U} + \delta_{\hat{X}'}\psi_a\,\mathcal{S}^a\right).\end{aligned}$$

Thus, if we define the quantity

$$\begin{aligned}F_{\hat{X}'\hat{X}} \equiv &\int d^{d+1}x\,\delta_{\hat{X}'}\sqrt{\gamma}\,\left(\xi^\mu \mathcal{D}_\mu + \Omega^a{}_b \mathcal{R}^b{}_a + \Lambda\mathcal{U} + \psi_a \mathcal{S}^a\right)\\ &+\int d^{d+1}x\sqrt{\gamma}\,\left(\xi^\mu\,\delta_{\hat{X}'}\mathcal{D}_\mu + \Omega^a{}_b\,\delta_{\hat{X}'}\mathcal{R}^b{}_a + \Lambda\,\delta_{\hat{X}'}\mathcal{U} + \psi_a\,\delta_{\hat{X}'}\mathcal{S}^a\right),\end{aligned}$$

we get

$$\delta_{\hat{X}'}\delta_{\hat{X}} W = F_{\hat{X}'\hat{X}} + \delta_{[\hat{X}',\hat{X}]}W.$$

Using this the Wess-Zumino conditions take the form

$$\boxed{F_{\hat{X}'\hat{X}} - F_{\hat{X}\hat{X}'} = -\delta_{[\hat{X}',\hat{X}]}W.} \tag{4.18}$$

For the calculations going forward we will need to know $\delta_{\hat{X}}\sqrt{\gamma}$ explicitly. From 3.1 we have

$$\frac{\partial\gamma}{\partial\gamma_{\mu\nu}} = \gamma\,\gamma^{\mu\nu}$$

and we can see that this gives us

$$\delta_{\hat{X}}\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\,\gamma^{\mu\nu}\delta_{\hat{X}}\gamma_{\mu\nu}.$$

Now, given that $\gamma_{\mu\nu} = n_\mu n_\nu + h_{\mu\nu}$, and we know that

$$\delta_{\hat{X}}n_\mu = \pounds_\xi n_\mu$$

and

$$\delta_{\hat{X}}h_{\mu\nu} = \pounds_\xi h_{\mu\nu}$$

we get

$$\delta_{\hat{X}}\gamma_{\mu\nu} = \pounds_\xi\gamma_{\mu\nu},$$

so

$$\delta_{\hat{X}}\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\,\gamma^{\mu\nu}\pounds_\xi\gamma_{\mu\nu}.$$

It is instructive to see why $\delta_{\hat{X}}h_{\mu\nu} = \pounds_\xi h_{\mu\nu}$ in the first order formulation, the steps being laid out in detail below

$$\begin{aligned}
\delta_{\hat{X}}h_{\mu\nu} &= \delta_{\hat{X}}\left(\delta_{ab}e^a_\mu e^b_\nu\right) = \delta_{ab}\,\delta_{\hat{X}}\,e^a_\mu e^b_\nu + \delta_{ab}\,e^a_\mu\,\delta_{\hat{X}}e^b_\nu \\
&= \delta_{ab}\left(\pounds_\xi e^a_\mu - \Omega^a{}_c e^c_\mu\right)e^b_\nu + \delta_{ab}e^a_\mu\left(\pounds_\xi e^b_\nu - \Omega^b{}_d e^d_\nu\right) \\
&= \delta_{ab}\,\pounds_\xi e^a_\mu\,e^b_\nu + \delta_{ab}e^a_\mu\pounds_\xi e^b_\nu - \delta_{ab}\Omega^a{}_c e^c_\mu e^b_\nu - \delta_{ab}e^a_\mu\Omega^b{}_d e^d_\nu \\
&= \pounds_\xi\left(\delta_{ab}e^a_\mu e^b_\nu\right) - \Omega_{bc}e^c_\mu e^b_\nu - e^a_\mu\Omega_{ad}e^d_\nu \\
&= \pounds_\xi h_{\mu\nu} - 2\Omega_{(ab)}e^a_\mu e^b_\nu \\
&= \pounds_\xi h_{\mu\nu},\ \text{since } \Omega_{(ab)} = 0.
\end{aligned}$$

In fact, since

$$\delta_\Omega n_\mu = \delta_\Omega h_{\mu\nu} = \delta_\Omega A_\mu = \delta_\Omega A^\mu{}_\nu = 0,$$

if we had a local functional of the form

$$\mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = \mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots}(x)[n_\mu, h_{\mu\nu}, A_\mu, A^\mu{}_\nu],$$

it would obey

$$\delta_\Omega\mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = 0.$$

Similarly, since

$$\delta_\psi n_\mu = \delta_\psi e^a_\mu = 0,$$

if we had

$$\mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = \mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots}(x)[n_\mu, e^a_\mu],$$

then

$$\delta_\psi \mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = 0,$$

while since

$$\delta_\Lambda n_\mu = \delta_\Lambda e^a_\mu = \delta_\Lambda F_{\mu\nu} = \delta_\Lambda A^a{}_\mu = 0,$$

a local functional

$$\mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = \mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots}(x)[n_\mu, e^a_\mu, F_{\mu\nu}, A^a{}_\mu]$$

would satisfy

$$\delta_\Lambda \mathcal{G}^{\kappa\cdots}{}_{\lambda\cdots} = 0.$$

From all the above the operator

$$\delta_{\hat{X}} = \delta_\xi + \delta_\Omega + \delta_\Lambda + \delta_\psi$$

will give us

$$\delta_{\hat{X}}\gamma_{\mu\nu} = \delta_\xi \gamma_{\mu\nu} = \pounds_\xi \gamma_{\mu\nu}.$$

We are now ready to break apart the Wess-Zumino conditions to equivalent partial conditions that $\mathcal{D}_\mu, \mathcal{R}^a{}_b, \mathcal{U}$ and $\mathcal{S}^a$ should satisfy. This is done by taking transformations $\hat{X}'$ and $\hat{X}$ such that only one of its gauge parameters in non-zero. For example, we will start with $\hat{X}' = (0, 0, \Lambda', 0)$ and $\hat{X} = (0, 0, \Lambda, 0)$. This gives us

$$[\hat{X}', \hat{X}] = 0, \quad \delta_{[\hat{X}', \hat{X}]} W = 0, \quad \delta_{\hat{X}'}\gamma_{\mu\nu} = \delta_{\hat{X}}\gamma_{\mu\nu} = 0,$$
$$F_{\hat{X}'\hat{X}} = \int d^{d+1}x\sqrt{\gamma}\,\Lambda\,\delta_{\Lambda'}\mathcal{U}, \quad F_{\hat{X}\hat{X}'} = \int d^{d+1}x\sqrt{\gamma}\,\Lambda'\,\delta_\Lambda \mathcal{U}$$

and putting these in 4.18 we find

$$\int d^{d+1}x\sqrt{\gamma}\,(\Lambda\,\delta_{\Lambda'}\mathcal{U} - \Lambda'\,\delta_\Lambda\mathcal{U}) = 0.$$

As another example, we take $\hat{X}' = (\xi'^\mu, 0, 0, 0)$ and $\hat{X} = (0, 0, \Lambda, 0)$ and find

$$[\hat{X}', \hat{X}] = (0, 0, \Lambda_{[\hat{X}', \hat{X}]}, 0), \quad \Lambda_{[\hat{X}', \hat{X}]} = \pounds_{\xi'}\Lambda, \quad \delta_{[\hat{X}', \hat{X}]} W = \int d^{d+1}x\sqrt{\gamma}\,\pounds_{\xi'}\Lambda\,\mathcal{U},$$
$$\delta_{\hat{X}'}\gamma_{\mu\nu} = \pounds_{\xi'}\gamma_{\mu\nu}, \quad \delta_{\hat{X}}\gamma_{\mu\nu} = 0,$$
$$F_{\hat{X}'\hat{X}} = \int d^{d+1}x\,\left(\delta_{\hat{X}'}\sqrt{\gamma}\,\Lambda\,\mathcal{U} + \sqrt{\gamma}\Lambda\,\delta_{\hat{X}'}\mathcal{U}\right), \quad F_{\hat{X}\hat{X}'} = \int d^{d+1}x\sqrt{\gamma}\,\xi'^\mu\,\delta_\Lambda\mathcal{D}_\mu,$$

which give us the partial condition

$$\delta_{\xi'}\left(\int d^{d+1}x\sqrt{\gamma}\,\Lambda\,\mathcal{U}\right) - \int d^{d+1}x\sqrt{\gamma}\,\xi'^\mu\,\delta_\Lambda\mathcal{D}_\mu = 0. \tag{4.19}$$

Following similar steps to the above, we should get a total of $\binom{4}{2}+4=10$ partial conditions. After calculations, these partial conditions turn out to be

$$\int d^{d+1}x\sqrt{\gamma}\,\left(\Lambda\,\delta_{\Lambda'}\mathcal{U}-\Lambda'\,\delta_{\Lambda}\mathcal{U}\right)=0 \tag{4.20a}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\left(\Lambda\,\delta_{\Omega'}\mathcal{U}-\Omega'^{a}{}_{b}\,\delta_{\Lambda}\mathcal{R}^{b}{}_{a}\right)=0 \tag{4.20b}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\left(\Lambda\,\delta_{\psi'}\mathcal{U}-\psi'_{a}\,\delta_{\Lambda}\mathcal{S}^{a}\right)=0 \tag{4.20c}$$

$$\delta_{\xi'}\left(\int d^{d+1}x\sqrt{\gamma}\,\Lambda\,\mathcal{U}\right)-\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\Lambda}\mathcal{D}_{\mu}=0 \tag{4.20d}$$

$$\begin{aligned}\int d^{d+1}x\sqrt{\gamma}\,\Big[&(\Omega^{a}{}_{b}\,\delta_{\Omega'}\mathcal{R}^{b}{}_{a}-\Omega'^{a}{}_{b}\,\delta_{\Omega}\mathcal{R}^{b}{}_{a})+\\ &+(\Omega^{a}{}_{b}\Omega'^{b}{}_{c}-\Omega'^{a}{}_{b}\Omega^{b}{}_{c})\mathcal{R}^{c}{}_{a}\Big]=0\end{aligned} \tag{4.20e}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\psi'_{a}\left(\Omega^{a}{}_{b}\mathcal{S}^{b}+\delta_{\Omega}\mathcal{S}^{a}-\Omega^{a}{}_{b}\,\delta_{\psi'}\mathcal{R}^{b}{}_{a}\right)=0 \tag{4.20f}$$

$$\delta_{\xi'}\left(\int d^{d+1}x\sqrt{\gamma}\,\Omega^{a}{}_{b}\mathcal{R}^{b}{}_{a}\right)-\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\Omega}\mathcal{D}_{\mu}=0 \tag{4.20g}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\left(\psi_{a}\,\delta_{\psi'}\mathcal{S}^{a}-\psi'_{a}\,\delta_{\psi}\mathcal{S}^{a}\right)=0 \tag{4.20h}$$

$$\delta_{\xi'}\left(\int d^{d+1}x\sqrt{\gamma}\,\psi_{a}\mathcal{S}^{a}\right)-\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\psi}\mathcal{D}_{\mu}=0 \tag{4.20i}$$

$$\begin{aligned}\delta_{\xi'}\left(\int d^{d+1}x\sqrt{\gamma}\,\xi^{\mu}\mathcal{D}_{\mu}\right)-\delta_{\xi}\left(\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\mathcal{D}_{\mu}\right)+\\ +\int d^{d+1}x\sqrt{\gamma}\,\pounds_{\xi}\xi'^{\mu}\,\mathcal{D}_{\mu}=0.\end{aligned} \tag{4.20j}$$

4.3.2 Candidate Anomalies

In this section we will try to find solutions to the partial conditions 4.20. In doing that, we will *assume* that $\mathcal{D}_{\mu}$, $\mathcal{R}^{a}{}_{b}$, $\mathcal{U}$ and $\mathcal{S}^{a}$ are tensor quantities. Specifically, we will take $\mathcal{D}_{\mu}$ to be an 1-form, $\mathcal{R}^{a}{}_{b}$ a spatial $(1,1)$ tensor, $\mathcal{U}$ a scalar and $\mathcal{S}^{a}$ a spatial vector. Of course, under these conditions any gravitational anomalies are excluded from our current study. We must also note that if we have the integral of a scalar $\mathcal{F}=\mathcal{F}(x)[n_{\mu},h_{\mu\nu},A_{\mu},A^{\mu}{}_{\nu}]$ in the form

$$\int d^{d+1}x\sqrt{\gamma}\,\mathcal{F},$$

we know that due to covariance it will be diffeomorphism invariant, so

$$\delta_{\xi}\left(\int d^{d+1}x\sqrt{\gamma}\,\mathcal{F}\right)=0\quad\forall\xi^{\mu}.$$

Using this property, conditions like 4.19 are simplified to

$$\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\Lambda}\mathcal{D}_{\mu}=0.$$

From the preceding discussion, the partial conditions 4.20 are reduced to

$$\int d^{d+1}x\sqrt{\gamma}\,(\Lambda\,\delta_{\Lambda'}\mathcal{U}-\Lambda'\,\delta_{\Lambda}\mathcal{U})=0 \tag{4.21a}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\left(\Lambda\delta_{\Omega'}\mathcal{U}-\Omega'^{a}{}_{b}\delta_{\Lambda}\mathcal{R}^{b}{}_{a}\right)=0 \tag{4.21b}$$

$$\int d^{d+1}x\sqrt{\gamma}\,(\Lambda\,\delta_{\psi'}\mathcal{U}-\psi'_{a}\,\delta_{\Lambda}\mathcal{S}^{a})=0 \tag{4.21c}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\Lambda}\mathcal{D}_{\mu}=0 \tag{4.21d}$$

$$\begin{aligned}\int d^{d+1}x\sqrt{\gamma}\,\big[&(\Omega^{a}{}_{b}\,\delta_{\Omega'}\mathcal{R}^{b}{}_{a}-\Omega'^{a}{}_{b}\,\delta_{\Omega}\mathcal{R}^{b}{}_{a})+\\ &+(\Omega^{a}{}_{b}\Omega'^{b}{}_{c}-\Omega'^{a}{}_{b}\Omega^{b}{}_{c})\mathcal{R}^{c}{}_{a}\big]=0\end{aligned} \tag{4.21e}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\psi'_{a}\,\left(\Omega^{a}{}_{b}\mathcal{S}^{b}+\delta_{\Omega}\mathcal{S}^{a}-\Omega^{a}{}_{b}\,\delta_{\psi'}\mathcal{R}^{b}{}_{a}\right)=0 \tag{4.21f}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\Omega}\mathcal{D}_{\mu}=0 \tag{4.21g}$$

$$\int d^{d+1}x\sqrt{\gamma}\,(\psi_{a}\,\delta_{\psi'}\mathcal{S}^{a}-\psi'_{a}\,\delta_{\psi}\mathcal{S}^{a})=0 \tag{4.21h}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\xi'^{\mu}\,\delta_{\psi}\mathcal{D}_{\mu}=0 \tag{4.21i}$$

$$\int d^{d+1}x\sqrt{\gamma}\,\pounds_{\xi}\xi'^{\mu}\,\mathcal{D}_{\mu}=0. \tag{4.21j}$$

Let us try to simplify and draw conclusions from the above partial conditions 4.21. Firstly, we can easily see that equations 4.21d, 4.21g, 4.21i lead to

$$\delta_{\Lambda}\mathcal{D}_{\mu}=\delta_{\Omega}\mathcal{D}_{\mu}=\delta_{\psi}\mathcal{D}_{\mu}=0,$$

which are satisfied for $\mathcal{D}_{\mu}$ of the form

$$\mathcal{D}_{\mu}=\mathcal{D}_{\mu}(x)[n_{\kappa},h_{\kappa\lambda}].$$

Putting $\xi^{\mu}=\delta^{\mu}_{\kappa}\delta(x-x_0)$ in 4.21j, we get

$$\sqrt{\gamma}\,\partial_{\nu}\xi'^{\mu}\mathcal{D}_{\mu}+\partial_{\mu}\left(\sqrt{\gamma}\,\xi'^{\mu}\mathcal{D}_{\nu}\right)=0\quad\forall\xi'^{\mu}.$$

Setting $\xi'^{\mu}=\delta^{\mu}_{\kappa}$ in some chart of our manifold, we find

$$\partial_{\mu}\left(\sqrt{\gamma}\,\mathcal{D}_{\nu}\right)=0$$

or, inside this randomly chosen coordinate chart,

$$\mathcal{D}_\mu = \frac{1}{\sqrt{\gamma}} c_\mu$$

for some constants c_μ. We know that $\sqrt{\gamma}$ is a tensor density of weight 1, i.e.

$$\sqrt{\gamma'} = J^{-1}\sqrt{\gamma}$$

with $J \equiv \det\left(\frac{\partial x'}{\partial x}\right)$ the Jacobian. This means that in another chart we will have

$$\mathcal{D}'_\mu = \frac{1}{\sqrt{\gamma'}} c'_\mu = J\frac{1}{\sqrt{\gamma}} c'_\mu$$

with c'_μ constants. If a $\mathcal{D}$ is a non-zero 1-form, then there exists some non-zero c_μ in the original chart and we have

$$\mathcal{D}'_\mu = J\frac{1}{\sqrt{\gamma}} c'_\mu = J\,\frac{c'_\mu}{c_\mu}\,\frac{1}{\sqrt{\gamma}} c_\mu = J\,\frac{c'_\mu}{c_\mu}\,\mathcal{D}_\mu.$$

It is clear that for $\mathcal{D}$ to be an 1-form, transforming as

$$\mathcal{D}'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\mathcal{D}_\nu,$$

all the constants c_μ must vanish. From these arguments we see that

$$\mathcal{D}_\mu = 0.$$

Now we will look at equations 4.21b, 4.21e, 4.21f and analyze the action of δ_Ω. To start, notice that

$$\delta_\Omega n_\mu = \delta_\Omega h_{\mu\nu} = \delta_\Omega A_\mu = \delta_\Omega a_{\mu\nu} = \delta_\Omega A^\mu{}_\nu = 0,$$

while

$$\delta_\Omega e^a_\mu = -\Omega^a{}_b e^b_\mu \qquad \delta_\Omega e^\mu_a = \Omega^b{}_a e^\mu_b$$
$$\delta_\Omega a_{ab} = \Omega^c{}_a a_{cb} + \Omega^c{}_b a_{ac} \qquad \delta_\Omega A^a{}_\mu = -\Omega^a{}_b A^b{}_\mu$$

and we even have

$$\begin{aligned}\delta_{\Omega'}\xi^\mu &= \delta_{\Omega'}\Lambda = 0\\ \delta_{\Omega'}\psi_a &= \Omega'^b{}_a\psi_b\\ \delta_{\Omega'}\Omega^a{}_b &= -\Omega'^a{}_c\Omega^c{}_b + \Omega'^c{}_b\Omega^a{}_c.\end{aligned}$$

From these relations we can see that $\mathcal{R}^a{}_b$ should be of the form

$$\mathcal{R}^a{}_b = e^a_\mu e^\nu_b \mathcal{R}^\mu{}_\nu \quad \text{with} \quad \mathcal{R}^\mu{}_\nu = \mathcal{R}^\mu{}_\nu(x)[n_\kappa, h_{\kappa\lambda}, A_\kappa, A^\kappa{}_\lambda].$$

Similarly, we should have

$$\mathcal{S}^a = e^a_\mu \mathcal{S}^\mu \quad \text{with} \quad \mathcal{S}^\mu = \mathcal{S}^\mu(x)[n_\kappa, h_{\kappa\lambda}, A_\kappa, A^\kappa{}_\lambda]$$

and

$$\mathcal{U} = \mathcal{U}(x)[n_\kappa, h_{\kappa\lambda}, A_\kappa, A^\kappa{}_\lambda].$$

Under these assumptions we will have

$$\delta_\Omega \mathcal{U} = 0, \quad \delta_\Omega \mathcal{R}^a{}_b = -\Omega^a{}_c \mathcal{R}^c{}_b + \Omega^c{}_b \mathcal{R}^a{}_c \quad \text{and} \quad \delta_\Omega \mathcal{S}^a = \Omega^a{}_b \mathcal{S}^b,$$

which turn equations 4.21b, 4.21e, 4.21f into

$$\int d^{d+1}x\sqrt{\gamma}\,\Omega^a{}_b \delta_{\Lambda'} \mathcal{R}^b{}_a = 0$$

$$\int d^{d+1}x\sqrt{\gamma}\,(\Omega^a{}_b \Omega'^b{}_c - \Omega'^a{}_b \Omega^b{}_c)\mathcal{R}^c{}_a = 0$$

$$\int d^{d+1}x\sqrt{\gamma}\,\Omega^a{}_b\, \delta_{\psi'} \mathcal{R}^b{}_a = 0.$$

If we set $\Omega_{ab} = \delta^k_{[a}\delta^l_{b]}\delta(x - x_0)$ in the above relations we get

$$\begin{aligned} \delta_\Lambda \mathcal{R}_{[ab]} &= 0 \\ \left(\Omega'^b{}_c \mathcal{R}^{ca} - \Omega'^a{}_c \mathcal{R}^{cb}\right) - \left(\Omega'^{ca} R^b{}_c - \Omega'^{cb} \mathcal{R}^a{}_c\right) &= 0 \\ \delta_\psi \mathcal{R}_{[ab]} &= 0, \end{aligned} \tag{4.22}$$

respectively. Putting $\Omega'_{ab} = \delta^k_{[a}\delta^l_{b]}$ in 4.22 we find

$$\delta_{kb}\mathcal{R}_{[la]} - \delta_{lb}\mathcal{R}_{[ka]} - \delta_{ka}\mathcal{R}_{[lb]} + \delta_{la}\mathcal{R}_{[kb]} = 0.$$

Then setting in the above $k = b$ and summing on b we obtain the condition

$$\mathcal{R}_{[ab]} = 0.$$

However, $\mathcal{R}_{ab}$ is already antisymmetric from the relation 4.17, so we deduce that

$$\mathcal{R}_{ab} = 0.$$

Finally, after assuming that $\mathcal{D}_\mu$, $\mathcal{R}^a{}_b$, $\mathcal{U}$ and $\mathcal{S}^a$ are tensor quantities, our analysis shows that the partial conditions 4.21 will take the simplified form

$$\boxed{\begin{aligned} \int d^{d+1}x\sqrt{\gamma}\,(\Lambda\, \delta_{\Lambda'}\mathcal{U} - \Lambda'\, \delta_\Lambda \mathcal{U}) &= 0 && \text{(4.23a)} \\ \int d^{d+1}x\sqrt{\gamma}\,(\Lambda\, \delta_{\psi'}\mathcal{U} - \psi'_a\, \delta_\Lambda \mathcal{S}^a) &= 0 && \text{(4.23b)} \\ \int d^{d+1}x\sqrt{\gamma}\,(\psi_a\, \delta_{\psi'}\mathcal{S}^a - \psi'_a\, \delta_\psi \mathcal{S}^a) &= 0 && \text{(4.23c)} \\ \mathcal{R}^a{}_b &= 0 && \text{(4.23d)} \\ \mathcal{D}_\mu &= 0. && \text{(4.23e)} \end{aligned}}$$

We can easily identify a class of solutions to the above equations. Firstly, to satisfy 4.23c it is easiest to choose $\delta_\psi \mathcal{S}^a = 0$, meaning

$$\mathcal{S}^a = e^a_\mu \mathcal{S}^\mu \quad \text{with} \quad \mathcal{S}^\mu = \mathcal{S}^\mu(x)[n_\kappa, h_{\kappa\lambda}].$$

This will automatically satisfy $\delta_\Lambda S^a = 0$, which means that conditions 4.23a and 4.23b become

$$\int d^{d+1}x\sqrt{\gamma}\,(\Lambda\,\delta_{\Lambda'}\mathcal{U} - \Lambda'\,\delta_\Lambda \mathcal{U}) = 0 \tag{4.24a}$$

$$\delta_\psi \mathcal{U} = 0. \tag{4.24b}$$

But to satisfy 4.24b $\mathcal{U}$ must be of the form

$$\mathcal{U} = \mathcal{U}(x)[n_\kappa, h_{\kappa\lambda}],$$

which automatically satisfies 4.24a. To summarize, our solutions are of the form

$$\boxed{\begin{aligned} &\mathcal{U} = \mathcal{U}(x)[n_\kappa, h_{\kappa\lambda}], && (4.25a)\\ &\mathcal{S}^a = e^a_\mu \mathcal{S}^\mu \quad \text{with} \quad \mathcal{S}^\mu = \mathcal{S}^\mu(x)[n_\kappa, h_{\kappa\lambda}], && (4.25b)\\ &\mathcal{R}^a{}_b = 0, && (4.25c)\\ &\mathcal{D}_\mu = 0. && (4.25d)\end{aligned}}$$

We must mention that, as there are no dipole symmetric quantities that depend only on our background fields, the only way for $\mathcal{U}$ and $\mathcal{S}^a$ to be dipole shift invariant is to be independent of A_μ and $A^\mu{}_\nu$, the background sources of the dipole shift transformation. The solutions

$$\mathcal{U} = \mathcal{U}(x)[n_\kappa, h_{\kappa\lambda}]$$

and

$$\mathcal{S}^a = e^a_\mu \mathcal{S}^\mu \quad \text{with} \quad \mathcal{S}^\mu = \mathcal{S}^\mu(x)[n_\kappa, h_{\kappa\lambda}]$$

constitute essentially a purely geometric class of solutions that depend only on the fundamental mathematical structure of Aristotelian geometry, its clock form n_μ and its spatial metric $h_{\mu\nu}$. Such solutions emerge due to the peculiar characteristics of Aristotelian geometry, like its Aristotelian connection not being metric compatible and torsion-free. We saw a similar behavior in the Aristotelian algebra 4.16 with the emergence of the new commutation relations 4.16e to 4.16g.

We should emphasize that, geometric solutions for $\mathcal{U}$ and $\mathcal{S}^a$ can be chosen independently from each other. This means that we can have $\mathcal{U} \neq 0$ and $\mathcal{S}^a = 0$, $\mathcal{U} = 0$ and $\mathcal{S}^a \neq 0$, or, $\mathcal{U} \neq 0$ and $\mathcal{S}^a \neq 0$. Some examples for non-vanishing $\mathcal{U}$ and $\mathcal{S}^a$ are[3]

$$\begin{aligned} \mathcal{U} &= F^n_{\mu\nu}\, h^{\mu\kappa} h^{\nu\lambda}\, F^n_{\kappa\lambda} \\ \mathcal{U} &= h_{\mu\nu}\, \pounds_v h^{\mu\nu} \\ \mathcal{U} &= n_\mu n_\nu\, \pounds_v h^{\mu\nu} \\ \mathcal{U} &= \nabla_\mu v^\mu \end{aligned}$$

[3]The relation $\pounds_v n_\mu = T^\kappa{}_{\kappa\mu}$ restricts the number of dissimilar geometric solutions.

and

$$\mathcal{S}^a = e^{a\mu}\,\pounds_v n_\mu = -e^a_\mu\,\pounds_v h^{\mu\nu} n_\nu$$
$$\mathcal{S}^a = e^{a\mu} F^n_{\mu\nu} h^{\nu\kappa}\,\pounds_v n_\kappa.$$

At even or odd spacetime dimensions $d+1$ we can use the Levi-Civita tensor $\varepsilon_{\mu_1\cdots\mu_{d+1}}$ to find even more choices for $\mathcal{U}$ and $\mathcal{S}^a$. For information on the Levi-Civita tensor see [89, 104, 93]. At even spacetime dimensions $d+1 = 2k$ we can define geometric quantities like

$$\mathcal{U} = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}} F^n_{\mu_1\mu_2}\cdots F^n_{\mu_{2k-1}\mu_{2k}}$$
$$\mathcal{U} = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}} \pounds_v F^n_{\mu_1\mu_2}\cdots \pounds_v F^n_{\mu_{2k-1}\mu_{2k}}$$

and

$$\mathcal{S}^a = e^a_{\mu_1}\varepsilon^{\mu_1\mu_2\mu_3\mu_4\cdots\mu_{2k-1}\mu_{2k}} n_{\mu_2} F^n_{\mu_3\mu_4}\cdots F^n_{\mu_{2k-1}\mu_{2k}}$$
$$\mathcal{S}^a = e^a_{\mu_1}\varepsilon^{\mu_1\mu_2\mu_3\mu_4\cdots\mu_{2k-1}\mu_{2k}} n_{\mu_2} \pounds_v F^n_{\mu_3\mu_4}\cdots \pounds_v F^n_{\mu_{2k-1}\mu_{2k}}.$$

At odd spacetime dimensions $d+1 = 2k+1$ we can find

$$\mathcal{U} = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\mu_{2k+1}} F^n_{\mu_1\mu_2}\cdots F^n_{\mu_{2k-1}\mu_{2k}} n_{\mu_{2k+1}}$$
$$\mathcal{U} = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\mu_{2k+1}} \pounds_v F^n_{\mu_1\mu_2}\cdots \pounds_v F^n_{\mu_{2k-1}\mu_{2k}} n_{\mu_{2k+1}}$$

and

$$\mathcal{S}^a = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\mu_{2k+1}} F^n_{\mu_1\mu_2}\cdots F^n_{\mu_{2k-1}\mu_{2k}} e^a_{\mu_{2k+1}}$$
$$\mathcal{S}^a = \varepsilon^{\mu_1\mu_2\cdots\mu_{2k-1}\mu_{2k}\mu_{2k+1}} \pounds_v F^n_{\mu_1\mu_2}\cdots \pounds_v F^n_{\mu_{2k-1}\mu_{2k}} e^a_{\mu_{2k+1}}.$$

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