The Hecke Algebra of a Finite Group with BN-pair

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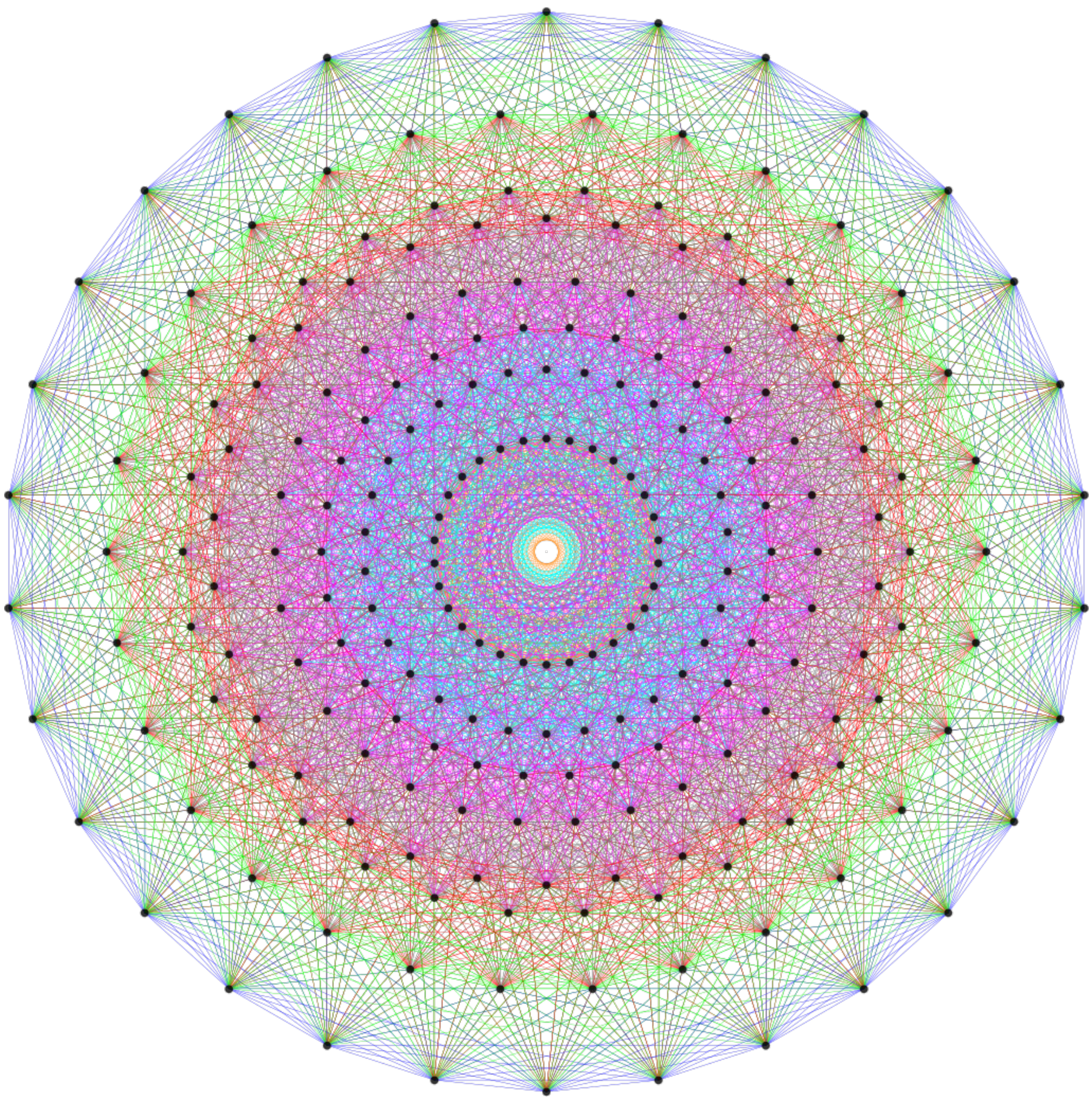

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Contents

1 Introduction

The aim of this dissertation is to study the representation theory of finite Coxeter groups, in the light of their associated Hecke algebras, as well as to highlight the relationship, that exists between their representations of them. A guiding example throughout our study will be the groups $GL_n(q)$ (the general linear group over a finite field of order q, where q is a prime power) and the symmetric group S_n.

The outline of Thesis:

In the 2nd chapter we introduce some basic concepts from the classical representation theory and their correlation with the theory of modules, emphasizing the usage of group algebra of a finite group, as a means of describing the representations of finite groups (i.e in the language of modules).

In the 3rd chapter we refer to the Coxeter groups focusing on their connection to the reflection groups and later on their classification. In particular, we start by introducing all the basic facts in order to define the groups generated by reflections and study their structure, through the concept of roots and fundamental roots. Afterwards, we move to a more combinatorial approach, through the concept of length function of an element, which leads us to very basic theorems such as (cancellation law, exchange condition, Matsumoto Theorem). Also, we introduce a presentation that defines the Coxeter groups through the Coxeter systems, and we state the 1-1 correspondence with the reflection groups. Finally, we introduce the concept of Coxeter graphs, which lead us to the final classification of Coxeter systems.

In 4th chapter we study the concept and the properties of finite groups with BN-pairs, concluding with the fundamental theorem of their connection with Coxeter groups. We also prove that the group $GL_n(q)$ is a group with a BN-pair and associated Weyl group be the symmetric group S_n.

In the 5th chapter we introduce the central concept of Hecke algebras H as endomorphism algebras of induced presentations 1_B^G for groups with BN-pair. Through the study of it's structure we will see an isomorphic interpretation of it, which will lead us to find a basis T_w for H, indexed by the elements of the associated Coxeter group W, which in turn determines for us the proper generators and relations, in order to get a presentation of H. Finally, we are arriving at the central result of this chapter: The 1-1 correspondence between the irreducible representations of Hecke algebras and of irreducible $\mathbb{C}G$-modules appearing as components of an appropriate vector space V.

In the 6th chapter we present a construction of the generic Hecke algebra $\mathcal{H}$, as a special case of the more general concept of generic algebras over an arbitrary commutative ring, by proper choice of the base ring. Through the multiplication rules of the basis elements T_w of the $\mathcal{H}$, we show that they are invertible, while being described through the definition of R-polynomials. Furthermore, we define a specific involution θ on $\mathcal{H}$, which combined with the T-basis and the introduction of Kazhdan-Lusztig polynomials, leads us to the construction of a new basis (C-basis) for $\mathcal{H}$, which has more convenient properties. By further studying of the multiplication formulas between the T-basis and the C-basis, allow us to present an act of the T-basis on C-basis, which lead us to obtain representations for the Hecke algebras.

In the 7th chapter we define the Cells in Coxeter groups, which lead us to the construction of lower-dimension representations of finite Coxeter groups and the associated Hecke algebras, using the theory developed by D.Kazhdan and G.Lusztig. The cells of a particular type, partition the group in a way compatible with the theory that developed in the previous chapters. It turns out that each cell affords a representation of the Hecke algebra, where the dimension of the representation is equal to the number of elements in the cell. By specialising the parameter q in the generic Hecke algebra to 1, it can be shown that we obtain a representation of the Coxeter group. The examples given will be from the symmetric groups for S_3 and S_4, in this case, we obtain all irreducible representations over $\mathbb{C}$ using these methods.

In the 8th chapter we give a description of the Tits Deformation Theorem, focusing more on it's application to the Hecke algebra associated with a finite group with a BN-pair, and a Weyl group W form a Coxeter system (W,S). In particular, it gives us an isomorphism of Hecke algebras with the group algebra of W over $\mathbb{C}$. As corollary, we get a decomposition of the characters for the induced trivial representation 1_B^G in terms of the irreducible characters of the Coxeter group W.

2 Representation Theory

2.1 The basics of Representations

Definition 2.1.1. *Let G be a group and V be K-vector space, where K is a field. Let $\rho : G \longrightarrow GL(V)$ be a homomorphism, where $GL(V)$ be the general linear group on V, such that*

$$\rho(g_1g_2) = \rho(g_1)\rho(g_2), \ \ \forall g_1, g_2 \in G$$

The map ρ is called a representation of G and the vector space V is the representation space.

If further the vector space V is of finite dimension, such as $n = dim_K V$, we can choose a basis of V, and then in the above group homomorphism we can identify the $GL(V)$ with $GL_n(K)$, the group of $n \times n$ invertible matrices over the field K, and the dimension of V is called the degree or dimension of the representation. Also in this situation, the representation ρ is called matrix (or linear) representation.

In this thesis, all the representations that we will see it will be finite representations.

Remark 2.1.1. *We can also think of a representation ρ of G with representation space V, as a pair of V and a map $G \times V \longrightarrow V$, $g \cdot v := \rho(g)(v)$. Then, $G \times V \longrightarrow V$ the axioms that should satisfy are:*

1. *it is a group action (the action is associative and $e \cdot v = v$, for all $v \in V$)*
2. *the map $V \longrightarrow V$, $v \mapsto g \cdot v$ is linear for all $g \in G$.*

So a representation is nothing but a linear action of G on V .

Definition 2.1.2. *A homomorphism of representations $T : (V, \rho) \longrightarrow (V^{'}, \rho^{'})$ is a linear map $T : V \longrightarrow V^{'}$ such that*

$$T o \rho(g) = \rho^{'}(g) o T$$

for all $g \in G$. T is an isomorphism if T is invertible. Two representations are isomorphic if there exists an isomorphism between them.

Definition 2.1.3. *A subrepresentation of a representation V of a group is a subspace $W \leq V$ which is invariant under all the operators ρ_g, for $g \in G$, i.e $\rho(g)(W) \subseteq W$, for all $g \in G$. We will called such a subspace W of V as a G-invariant subspace.*

Definition 2.1.4. *A representation (V, ρ) is irreducible if it is nonzero and there does not exist any proper nonzero subrepresentation of V. So a representation (V, ρ) of G is irreducible if there dose not exists any proper G-invariant subspace W of V.*

2.2 Group Algebra

Let G be a group and R a ring. We will construct an R-module that having the elements of G as a basis, and then by the use of the operations of both the group G and the ring R, we define a ring structure on it. We will donate that construction by RG.

So in this direction, we let RG to be the set of all formal linear combinations of the form

$$\sum_{g \in G} \lambda_g g$$

where $\lambda_g \in R$ and $\lambda_g = 0$ almost everywhere, i.e only finite number of coefficients are from 0 in each of these sums.

Notice that it follows from our definition that given two elements, $a = \sum_{g \in G} a_g g$ and $b = \sum_{g \in G} b_g g \in RG$, we have that $a = b$ if and only if $a_g = b_g$, for every $g \in G$.

Now we define the sum of two elements in RG pointwise, i,e :

$$(\sum_g a_g g) + (\sum_g a_g g) = \sum_g (a_g + b_g)g$$

Also, given two elements $a = \sum_g a_g g$ and $b = \sum_g b_g g \in RG$ we define their product by

$$ab = \sum_{g,h \in G} a_g b_h gh$$

Notice that, the way we have define their product is just defining the product of two elements in the basis by means of their product in G.

Reordering the terms in the formula above, we can write the product ab as

$$ab = \sum_{u \in G} c_u u$$

where $c_u = \sum_{gh=u} a_g b_h$ It is easy to verify that, with the operations above RG becoming a ring, which has unity the element $1 := \sum_g u_g g$, where the coefficients corresponding to the unit element of the group is equal to 1 and $u_g = 0$ for every other element $g \in G$.

We now can also define and an scalar multiplication of elements in RG by elements $\lambda \in R$ as

$$\lambda(\sum_g a_g g) = \sum_g (\lambda a_g) g$$

And again we can verify that RG with the above scalar multiplication rule becoming an R-module. Furthermore, if R is commutative it follows that RG is an algebra over R.

Definition 2.2.1. *The set RG, with the operations defined above, is called the group ring of G over R, and in the case where R is commutative, RG is called group algebra of G over R.*

In the following chapters we will focusing more in the case where $R = \mathbb{C}$, and so to the group algebra of a finite group G over the complex numbers, notated by $\mathbb{C}G$.

Remark 2.2.1. *An alternative way of define the the group ring RG is as the set of all functions $f : G \longrightarrow R$ such that $f(g) \neq 0$ only for finite numbers of elements in G. In this situation, we could realize the formulas of the sum and scalar multiplication, as the usual definitions of sum of two functions and of product of a function by a scalar. Also, the definition of the product, is this case, corresponds to the convolution product of two functions.*

Moreover, we can define an embedding map $i : G \longrightarrow RG$ by assigning to each element $x \in G$ the element $i(x) = \sum_g a_g g$, where $a_x = 1$ and $a_g = 0$ if $g \neq x$. So we could see the group G as a subset of RG, and hence with this identification we can say that G is a basis of RG over R. As an immediate consequence we see that, if R is commutative, the dimension of a free module over R is well defined, and thus if G is finite we get that $dim_R RG = |G|$

For the rest of thesis, as already stated, we will refer to the group algebra $\mathbb{C}G$, except something else said.

2.3 The representation theory in the point view of modules

Proposition 2.3.1. *The representations of a group G over a $\mathbb{C}-$vector space V are into 1-1 correspondence with the $\mathbb{C}G$-modules.*

Definition 2.3.1. *An R-module is called simple (or irreducible) if is non-zero and has no proper, nontrivial submodules.*

So by combined the above definition with interpretation of the representations as $\mathbb{C}G$-modules, we have the proposition.

Proposition 2.3.2. *A representation it will be irreducible if and only if the corresponds $\mathbb{C}G$-module is irreducible.*

Now we present some important facts about the group algebra $\mathbb{C}G$ and the association of it, to the representation theory of finite groups.

Theorem 2.3.1.

- *(Maschke's Theorem): If G is finite then KG is semisimple if and only if K is semisimple and $|G| \in \mathcal{U}(K)$, where K is a field. So for our purposes where $K = \mathbb{C}$ the above holds.*
- *$\mathbb{C}G$ is semisimple.*

- *Form Wedderburn-Artin Theorem we obtain that*

$$\mathbb{C}G = \prod_{i=1}^{r} M_{n_i}(\mathbb{C})$$

where

  - *r is the number of non-isomorphic simple $\mathbb{C}G$-modules, i.e equivalent r is presenting the number of the irreducible representations of G over $\mathbb{C}$, which it can been showed that is equal to the number of the different conjugacy classes of G.*
  - *$n_i = dim_{\mathbb{C}G}V_i$, with V_i be simple $\mathbb{C}G$-modules, i.e equivalent n_i is the degree of the representations that corresponds to the V_i $\mathbb{C}-$vector spaces.*
  - $|G| = \sum_{i=1}^{r} {n_i}^2$

2.4 Character Theory

Another important concept that appears in the representation theory for finite representations is the character theory.

Definition 2.4.1. *The character of a matrix representation $\rho : G \longrightarrow GL_n(V)$, where V is a finite dimensional $\mathbb{C}-$vector space, is the function $\chi_\rho : G \longrightarrow \mathbb{C}$ such that*

$$\chi_\rho(g) = trace\rho(g)$$

Its common in notation to write χ_V for the character of the representation (V, ρ).

Proposition 2.4.1. *Isomorphic representations have the same character. For the converse, if G is finite is also true.i.e*

$$(V, \rho) \cong (W, \sigma) \Leftrightarrow \chi_V = \chi_W$$

In the case that G is finite by Maschke's theorem, we have that every finite representation is completely reducible, i.e every finite representation of a finite group G is the direct sum of irreducible subrepresentations, where with direct sum of representations we mean that given two representations (V_1, ρ_1) and (V_2, ρ_2) is the representation $(V_1 \oplus V_2, \rho := \rho_1 \oplus \rho_2)$, where $\rho_1 \oplus \rho_2(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$.

Now we introduce the idea of a character table:

Definition 2.4.2. *The character table of G is the table whose the (i,j)-entry is $\chi_{V_i}(g_j)$.*
Where $\{(V_1, \rho_1), \cdots, (V_m, \rho_m)\}$ be a complete set of irreducible representations of a finite group G and the number m equals to the number of conjugacy classes of G. Also label $C_1, \cdots, C_m$ these conjugacy classes and let $g_i \in C_i$ be a representative of conjugacy class C_i.

Note that the definition does not depend on the choice of V_i up to isomorphism nor on the choice of representatives g_j.

2.5 Representations of Algebras

Similar definitions we could write for the representations of algebras. So

Definition 2.5.1. *A representation of an algebra A (also called a left A-module) is a vector space V together with a homomorphism of algebras $\rho : A \longrightarrow End(V)$*

Definition 2.5.2. *A subrepresentation of a representation (V, ρ) of an algebra A is a subspace $W \leq V$ such that $\rho(a)(W) \subseteq W$, for all $a \in A$.*

Definition 2.5.3. *A representation (V, ρ) of an algebra A is irreducible is non zero and does not exists non proper, non trivial subrepresentations.*

3 Coxeter Groups

3.1 Finite Reflection Groups

Definition 3.1.1 (Euclidean space). *Let V be a finite $\mathbb{R}-$vector space, with dim $V = n < \infty$ and assume that V is equipped with a symmetric, positive define bilinear form $(\ ,\ ) : V \times V \longrightarrow \mathbb{R}$ such that $(x, y) \in \mathbb{R}$, for $x, y \in V$. We call $(V, (\ ,\ ))$ Euclidean space and we refer to the bilinear form $(\ ,\ )$ as an inner product on V. Obviously, the form is non-degenerate.*

Definition 3.1.2 (The group of Orthogonal transformations). *The group of orthogonal transformations on is defined by*

$$O(V) = \{s \in End_{\mathbb{R}}V \ :\ (sx, sy) = (x, y),\ for\ all\, x, y \in V\}$$

Definition 3.1.3 (Orthogonal Complement). *For a subspace U of V, we call orthogonal complement of U the set*

$$U^{\perp} = \{y \in V :\ (y, x) = 0,\ for\ all\, x \in U\}$$

Remark 3.1.1.

Let $a \in V$, with $a \neq 0$, then the space V decompose it as $V =< a > \oplus < a >^{\perp}$, where $< a >$ is the vector space generated by a and the orthogonal complement of the space $< a >$ denoted by $H_a := < a >^{\perp}$. The H_a is called the orthogonal hyperplane to the vector a, and dim $H_a = n - 1$

Definition 3.1.4 (Reflection). *A reflection is a linear map $s : V \longrightarrow V$ such that $s \in O(V)$, $s \neq 1$, and s fixes every vector in some hyperplane in V. In particular, for every $a \in V$, with $a \neq 0$, we define as a reflection in the direction of the vector a, with respect to the hyperplane H_a, the linear map $s_a : V \longrightarrow V$, such that*

$$s_a(a) = -a,$$
$$s_a(x) = x,\ \forall x \in H_a$$

Proposition 3.1.1 (Properties of Reflections). *Let $a \in V$, with $a \neq 0$, and H_a be the orthogonal hyperplane to the vector a, as previous. Then we have that:*

(i) There exists a unique reflection $s \in O(V)$ that leaves fixed the elements of H_a. Then $s^2 = 1$ and s can be calculated by the formula

$$s_a(v) = v - \frac{2(a, v)}{(a, a)}\ \ for\ all\, v \in V$$

(ii) $a = kb$, where $k \in \mathbb{R}$ if and only if $s_a = s_b$. In particular, $s_a = s_{-a}$

(iii) $a = kb$, where $k \in \mathbb{R}$ if and only if $H_a = H_b$

(iv) The minimal polynomial $m_a(x)$ of the reflection s_a is $m_a(x) = (x - 1)(x + 1)$. Also s_a is diagonizable in $\mathbb{R}$, with $1, -1$ be the eigenvalues of s_a and the eigenspaces be $V_{s_a}(-1) =< a >$ and $V_{s_a}(1) = H_a$. The space $V =< a > \oplus H_a$ has a basis as to which one the matrix of the reflection s_a is the diagonal matrix $diag(-1, 1, \cdots, 1)$

(v) From (i) we have that $s_a \in O(V)$ and for every $g \in O(V)$ we get

$$g\ o\ s_a\ o\ g^{-1} = s_{g(a)}$$

In particular, let $x, y \in V$ and $g \in O(V)$. If $y = g(x)$ then $s_y = s_{g(x)} = gs_xg^{-1}$

(vi) The reflections s_a maintain the inner product as a orthogonal transformations, i.e

$$(s_a(x), s_a(y)) = (x, y)\ \ \forall x, y \in V$$

(vii) $s_as_b = s_bs_a \Longleftrightarrow (a, b) = 0$, for $a, b \neq 0$ and a, b are linearly independent.

Definition 3.1.5 (Reflection Group). *A subgroup G of the orthogonal group $O(V)$, that generated by reflections, will be called the group generated by reflections.*

3.2 Roots Systems

Definition 3.2.1 (Root System). *A set of vectors, Δ, it we be called a root system if $\Delta \subseteq V$, where V is Euclidean space, and satisfying the following conditions:*

(i) *Δ consists of non-zero elements, and the vectors in Δ are spanned the space V.*

(ii) *(Reduced Condition) If $a \in \Delta$, then $-a \in \Delta$. Furthermore, if $a \in \Delta$ and $k \in \mathbb{R}$ such that $ka \in \Delta$, then $k = \pm 1$.*

(iii) *For each $a \in \Delta$, we have that $s_a(\Delta) = \Delta$, where s_a is the reflection fixing the hyperplane H_a.*
A crystallographic root system is a root system Δ satisfying the additional condition:

(iv) *(Crystallographic Condition) For all pairs of roots $a, b \in \Delta$, we have that $2\frac{(a,b)}{(b,b)} \in \mathbb{Z}$.*

Remark 3.2.1. *The group $W = W(\Delta)$ that is generated by reflections s_a, such that $a \in \Delta$ is called the group generated by reflections associated with the root system Δ. Thus*

$$W = W(\Delta) =< s_a \ : \ a \in \Delta >$$

Moreover, the only reflections that are in the group $W(\Delta)$, are the reflections that they come from the root system Δ.

Proposition 3.2.1. *For every group generated by reflections associated with some root system Δ, i.e $W = W(\Delta)$, we have that W is isomorphic to a subgroup of the permutation group of Δ. Thus there exist a subgroup $H \leq S_\Delta$ such that*

$$W = W(\Delta) \cong H \leq S_\Delta$$

Corollary 3.2.1. *If the root system, Δ is finite, then the group generated by reflection associated to this root system, $W(\Delta)$, is also finite group.*

From now on the root systems that we will be concentrated it will be finite, so and the reflection group associated to the root system it will be finite, from the previous corollary.

Proposition 3.2.2. *If $\Delta, \Delta^{'}$ are two different root systems that are associated with the same reflection group W, i.e*

$$W =< s_a \ : \ a \in \Delta >=< s_b \ : \ b \in \Delta^{'} >$$

then we have that:

- *For each $b \in \Delta^{'}$ there exist $k \in \mathbb{R}$: $b = ka$, for some $a \in \Delta$.*
- *For each $a \in \Delta^{'}$ there exist $\lambda \in \mathbb{R}$: $a = \lambda b$, for some $b \in \Delta$.*

Furthermore, if let $V_\Delta, V_{\Delta^{'}}$ be the subspaces of that are generated by the roots of $\Delta, \Delta^{'}$, respectively.

Proposition 3.2.3. *Let W be a finite reflection group, i.e $W =< s_{a_1}, \cdots, s_{a_\ell} >$, where s_{a_i} are reflections. Then by letting Δ be the set of unit vectors orthogonal to the hyperplanes fixed by reflections in W.*
i.e

$$\Delta = \{x \in V \ : \ s_x \text{ be a reflection in } W \text{ and } ||x|| = 1\}$$

Then Δ is a root system, where the reflection group W is associated with it. Thus

$$W = W(\Delta) =< s_a \ : \ a \in \Delta >$$

Proposition 3.2.4. *For every subgroup G of the orthogonal group $O(V)$ there exist a group $G^{'}$ such that*

$$G \cong G^{'} \text{ and also } V_o(G^{'}) = \{0\}$$

where $V_o(G) = Fix_V(G) = \bigcap_{g \in G} V_g = \bigcap_{g \in G}\{x \in V \ : \ gx = x\}$

Definition 3.2.2 (Effective group). *A group G that is satisfying the relation $V_o(G) = \{0\}$ it will be called effective group.*

Corollary 3.2.2. *Since every reflection group $W = W(\Delta)$ is a subgroup of the $O(V)$ we have that there exist a group $W^{'}$ such that:*

$$W \cong W^{'} \text{ with } V_o(W^{'}) = 0$$

In particular, by letting V_Δ be the space that generated by the root system Δ which is associated with the reflection group W, i.e $V_\Delta =< \Delta >$, then

$$V_o(W) = {V_\Delta}^{\perp} \text{ and } V_o(W)^{\perp} = V_\Delta$$

Proposition 3.2.5. *Let $W = W(\Delta) =< s_{r_1}, \cdots, s_{r_n} \; : \; r_i \in \Delta >$, Δ be a root system and $V_\Delta =< r_1, \cdots, r_n >$. Then*

$$\begin{aligned} W \; is \; effective &\iff V_o(W) = \{0\} \\ &\iff {V_\Delta}^{\perp} = \{0\} \\ &\iff V_\Delta = V \\ &\iff \text{ the root system } \Delta \text{ include a basis of the space } V \end{aligned}$$

Remark 3.2.2. *In general a reflection group associated with a root system, i.e $W = W(\Delta)$, is not an effective group. But if we restrict ourselves to the space that is generated by the roots of Δ, hence $W|_{V_\Delta}$, the action of W leaves invariant the root system Δ, and thus the same is true for V_Δ. So by the above proposition we get a isomorphic group $W^{'}$ to W, that is effective. So w.l.o.g we can let $V = V_\Delta$.*

3.3 Positive and Fundamental Roots

Now let Δ be a root system associated with the reflection group $W = W(\Delta)$, where the group W generated by the set $\{s_a \; : \; a \in \Delta\}$. As we seen already we can let $V = V_\Delta$.

For every $a \in \Delta$ we recall that we can define the hyperplanes H_a, as

$$H_a = \{x \in V \; : \; (a, x) = 0\}$$

Then we can define the semi-spaces

$$H_a{}^{+} = \{x \in V \; : \; (a, x) > 0\} \text{ and } H_a{}^{-} = \{x \in V \; : \; (a, x) < 0\}$$

Theorem 3.3.1. *$V \setminus \bigcup_{a \in \Delta} H_a \neq \emptyset$.*

The proof of the above Theorem is based to the following Lemma:

Lemma 3.3.1. *Let V be a vector space over a infinity field E. Then V it can't be written as finite union of proper subspaces of it. Thus if $V = X_1 \cup \cdots x_n$ with $x_i \gneqq V$, then at least one of the subspaces X_i is the whole space V. i.e for some i_0 we have that $V = X_{i_0}$*

From the above Theorem we get that there exist a vector u such that $(a, u) \neq 0$, for all $a \in \Delta$. So for every $x \in V \setminus \bigcup_{a \in \Delta} H_a$ and for every $a \in \Delta$ we have that either $(x, a) > 0$ or $(x, a) < 0$. Equivalently, $x \in H_a{}^{+}$ or $H_a{}^{-}$.

Now we can define a equivalence relation on the set $V \setminus \bigcup_{a \in \Delta} H_a$, which will lead us to the definition of the Weyl chambers, which in their turn, will be help us define the positive and negative roots.

Definition 3.3.1 (Weyl Chambers). *We define a equivalence relation on $V \setminus \bigcup_{a \in \Delta} H_a$, $\sim$, by : If $x, x^{'} \in V \setminus \bigcup_{a \in \Delta} H_a$, then*

$$x \sim x^{'} \iff \forall \, a \in \Delta, \; (x, a), \; (x^{'}, a) \text{ have the same sign.}$$

i.e $x \sim x^{'}$ if and only if for every $a \in \Delta$ they 're exist in the same semi-space $H_a{}^{+}$ or $H_a{}^{-}$.

The equivalence class of an element $xV \setminus \bigcup_{a \in \Delta} H_a$ it will be called the Weyl Chamber and it will be the set

$$[x] = \{y \in V \; : \; (x, a), \; (y, a) \text{ have the same sign } \forall a \in \Delta\}$$

Proposition 3.3.1. *Every Weyl chamber give us a partition of the root system Δ. In particular, if let C be a Weyl chamber then we can define the sets*

$$\Delta^+ = \{a \in \Delta \; : \; (a, x) > 0, \; \forall x \in C\}$$
$$\Delta^- = \{a \in \Delta \; : \; (a, x) < 0, \; \forall x \in C\}$$

Notice that, the partition Δ into Δ^+ and Δ^- depends in our choice of the Weyl chamber C.

Definition 3.3.2 (Positive Roots)**.** *We define the positive roots to be the elements of Δ^+ and similarly the negative roots to be the elements of Δ^-. So we have a partition of Δ into positive roots Δ^+ and negative roots Δ^-.*

Let Δ^+ be such a positive root system, that occurs from a Weyl Chamber C. We have the following definition.

Definition 3.3.3 (Fundamental Root System)**.** *A fundamental (or simple) root system Σ we will call a subset of Δ^+, such that satisfying the following properties:*

(i) Every positive root in Δ can be expressed as linear combination of elements of Σ, with non-negative real coefficients. i.e $\forall a \in \Delta^+$ exist $\lambda_t \in \mathbb{R}: \; \lambda_t \geq 0$ such that $a = \sum_{\lambda_t \geq 0} \lambda_t a_t$, where $a_t \in \Sigma$.

(ii) The elements of Σ are linear independently.

i.e the Σ constitutes a basis of the space V_Δ so because we have restrict to the V_Δ we obtain that Σ is a basis of V.

Theorem 3.3.2. *Every positive root system Δ^+ contains a unique fundamental root system Σ.*

Now a natural question that is created from this construction is: If we give us a fundamental root system, can we found a Weyl chamber from whom they come the positive roots.

Proposition 3.3.2. *Let $\Sigma = \{a_1, \cdots, a_n\}$ be a fundamental root system. We define the set $C_0 = \{v \in V : (v, a_i) > 0, \; \forall i = 1, \cdots, n\}$ then:*

(i) $C_0 \neq \emptyset$.

(ii) C_0 is a Weyl chamber and will be called the fundamental Weyl chamber.

(iii) C_0 determines the specific fundamental root system Σ.

Theorem 3.3.3. *There exist a 1-1 correspondence from the set of all the Weyl chamber to the set of all the fundamentals root systems.*

$$\{\textit{Weyl Chambers}\} \longleftrightarrow \{\textit{fundamentals root systems}\}$$

Thus we obtain that:

(i) For each Weyl chamber C we get a unique fundamental root system.

(ii) For each fundamental root system we construct the fundamental Weyl chamber C_0.

Let Δ be a finite root system which is associated with the group of reflections $W = W(\Delta)$. We fixed a fundamental root system Σ that is comes from fundamental Weyl chamber C_0. Then, from there, we determined a partition to the root system Δ into $\Delta = \Delta^+ \sqcup \Delta^-$. The Δ^+ is a positive root system with $\Sigma = \{a_1, \cdots, a_n\}$

Definition 3.3.4. *The roots $a_i \in \Sigma$ are called fundamental roots (or simple roots). The reflections $\{s_{a_i} : a_i \in \Sigma\}$ are called fundamental reflections and we denoted by $s_{a_i} := s_i$, for each i.*

Proposition 3.3.3. *Let Σ be a fundamental root system and Δ^+ the positive root system containing Σ. Then every fundamental reflection change the sign only of two roots, in particular of the roots a_i and $-a_i$.*

Thus each fundamental reflection s_i permutes every positive roots expect the root a_i.
i.e

$$s_i(\Delta^+ \backslash \{a_i\}) = \Delta^+ \backslash \{a_i\}$$

Theorem 3.3.4. *Let $\Sigma = \{a_1, \cdots, a_n\}$ be a fundamental system in Δ. Then the reflection group associated with the root system Δ, i.e $W = W(\Delta)$, is generated by the fundamental reflections s_i. Moreover, every root r is in the W-orbit of some fundamental root. i.e for every $r \in \Delta^+$ exist $w \in W$ s.t $w = s_{a_i} \cdots s_{a_n}$, where s_{a_i} are fundamental reflections, such that $w(r) \in \Sigma$.*

3.4 The length function

Now from the above theorem we get that, for every $w \in W$ can be written as $w = s_{a_{i_1}} \cdots s_{a_{i_k}}$, where it is possible to have a repetition of some roots a_{i_ρ} in the expression of w. The fundamentals roots s_{a_i} of an expression of some $w \in W$ called words.

Definition 3.4.1 (The length)**.** *Let Δ be a root system, with a given set Δ^+ of positive roots and a fundamental system Σ. We define, for every $w \in W$, it's length $\ell(w)$ to be the minimal number of factors needed to express the element $w \in W$, as a product of fundamental reflections. An expression $w = s_{i_{a_1}} \cdots s_{i_{a_k}}$ is called reduced if the number of the fundamental reflections appears in the expression is the minimal possible, and thus $\ell(w) = k$.*

Remark 3.4.1.

- *Let $w \in W$ we called that this element has a length $\ell(w) = r$, if we could expressed as $w = s_1 \cdots s_r$, but it can't be written as a product of smaller number of fundamental reflections. Also we made the admission that $\ell(1) = 0$.*
- *If $s_i \in S$ then $\ell(s_i) = 1$, where $S = \{s_1, \cdots, s_n\}$ the set of all fundamental reflections.*

Proposition 3.4.1 (Properties of the length)**.** *(i) $\ell(ww^{'}) \leq \ell(w) + \ell(w^{'})$ and $|\ell(w) - \ell(w^{'})| \leq \ell(ww^{'})$, for each $w, w^{'} \in W$.*

(ii) $\ell(w) = \ell(w^{-1})$, for each $w \in W$.

(iii) If $w = s_1 \cdots s_q$ be a reduced expression of w with $\ell(w) = q$, then every subexpression of w is also reduced.

Theorem 3.4.1. *Let $\Sigma = \{a_1, \cdots, a_n\} \subseteq \Delta^+$ be a fundamental root system and $S = \{s_1, \cdots s_n\}$ the set of all the fundamental reflections. Then for each $w \in W$ and $s \in S$ we have that $\ell(sw) = \ell(w) + 1$ or $\ell(sw) = \ell(w) - 1$. Similarly, $\ell(ws) = \ell(w) + 1$ or $\ell(ws) = \ell(w) - 1$.*

Corollary 3.4.1.

- *If $\ell(sw) = \ell(w)+1$ then there not exist a reduced expression of w that it begins with s.*
- *If $\ell(sw) = \ell(w) - 1$ then there exist a reduced expression of w that it begins with s.*
- *If $\ell(ws) = \ell(w) + 1$ then there not exist a reduced expression of w that it ends with s.*
- *If $\ell(ws) = \ell(w) - 1$ then there exist a reduced expression of w that it ends with s.*

3.4.1 A Geometric Interpretation of length function

Definition 3.4.2. *For each $w \in W$, we define the subset of Δ to be the set*

$$N(w) = \Delta^+ \cap w^{-1}(\Delta^-)$$

i.e

$$N(w) = \{\theta \in \Delta^+ : \theta = w^{-1}(a), \text{ where } a \in \Delta^-\} = \{\theta \in \Delta^+ : w(\theta) \in \Delta^-\}$$

Finally we define

$$n(w) := |N(w)|$$

Proposition 3.4.2. *If $s_i \in S$, i.e s_i be a fundamental reflection occurs from a fundamental root $a_i \in \Sigma$. Then for every $w \in W$ we have that:*

- $n(ws_i) = \begin{cases} n(w) + 1 & , \text{ if } w(a_i) > 0 \\ n(w) - 1 & , \text{ if } w(a_i) < 0 \end{cases}$
- $n(s_iw) = \begin{cases} n(w) + 1 & , \text{ if } w^{-1}(a_i) > 0 \\ n(w) - 1 & , \text{ if } w^{-1}(a_i) < 0 \end{cases}$

It's clear that its also true and the converse statements of the above.

Proposition 3.4.3. $n(w) = n(w^{-1})$ *for every $w \in W$.*

3.5 The Cancellation law and the Exchange Condition

Theorem 3.5.1 (Cancellation Law). *Let $w = s_1 \cdots s_k$, where $s_i \in S$, not necessarily reduced. If $n(w) < k$ then there exist integers i, j, with $1 \leq i < j \leq j$, such that $w = s_1 s_2 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_k$.*

Corollary 3.5.1. *For each $w \in W$ we have that $n(w) = \ell(w)$.*

Proposition 3.5.1. *(i) If $w \in W$ is such that $w(\Delta^+) = \Delta^+$, then $w = 1$.*

(ii) There exists a unique element $w_o \in W$ of maximal length. This element has the properties:

- $\ell(w_o) \geq \ell(w), \ \forall \, w \in W$
- $\ell(w_o) = |\Delta^+|$
- $w_o(\Delta^+) = \Delta^-$
- ${w_o}^2 = 1$

Theorem 3.5.2 (Exchange Condition). *1. If $w = s_1 \cdots s_k$, not necessarily reduced and $s \in S$, with $\ell(ws) < \ell(w)$. Then there exists a unique integer i, with $1 \leq i \leq k$ such that $ws = s_1 \cdots \hat{s_i} \cdots s_k$*

2. Similarly we can formulate the exchange condition for the case that $\ell(sw) < \ell(w)$. Then there exists a unique integer i, with $1 \leq i \leq k$ such that $sw = s_1 \cdots \hat{s_i} \cdots s_k$

We can also make the following alternative forms of the exchange condition according to Matsumoto.

Proposition 3.5.2. *1. Suppose that $w = s_1 \cdots s_m$ be a reduced expression of w, i.e $\ell(w) = m$, and also $\ell(s_1 s_2 \cdots s_{m+1}) < m + 1$ where each $s_i \in S$. Then there exist an integer j, with $1 \leq j \leq m$, such that*

$$s_1 s_2 \cdots s_j = s_2 s_3 \cdots s_{j+1}$$

2. We can go even further, and obtain two integers i,j, with $1 \leq i < j \leq m$ such that

$$s_{i+1} s_{i+2} \cdots s_{j-1} s_j = s_i s_{i+1} \cdots s_{j-1}$$

Theorem 3.5.3 (Strong Exchange Condition). *1. If $w = s_1 \cdots s_k$, not necessarily reduced and s_a a reflection with $a \in \Delta^+$, with $\ell(ws_a) < \ell(w)$. Then there exists a unique integer i, with $1 \leq i \leq k$ such that $ws_a = s_1 \cdots \hat{s_i} \cdots s_k$*

2. Similarly we can formulate the exchange condition for the case that $\ell(sw) < \ell(w)$. Then there exists a unique integer i, with $1 \leq i \leq k$ such that $s_a w = s_1 \cdots \hat{s_i} \cdots s_k$

3.6 Coxeter Systems and Coxeter groups

Definition 3.6.1 (Coxeter System). *Let W be a finite group and $S = \{s_1, \cdots, s_n\}$ be a set of involutory generators of W. Then if the group W has a presentation*

$$W =< s_1, \cdots, s_n : (s_i s_j)^{m_{ij}} = 1 \, for \; all \, i, j >$$

where the m_{ij} are positive integers such that

$$m_{ii} = 1, \; m_{ij} > 1 \quad if \, i \neq j, \; and \quad m_{ij} = m_{ji} \quad for \; all \;\; i, j$$

the pair (W,S) called it a finite Coxeter system.

Definition 3.6.2 (Coxeter Group). *A group W it will be called Coxeter group if there exists a proper Coxeter system, thus a pair (W,S) such that the group W has a presentation as described in the previous definition. The cardinality of the set S called the rank of the Coxeter system, i.e $rank(W, S) = |S|$ and the defining relations in the presentation called Coxeter relations.*

Remark 3.6.1. *1. The same group W could admits different Coxeter systems, so it might have different Coxeter presentations.*

2. Additionally to the (1) we could have even that the same group W not only admits different Coxeter system, but also of different ranks.

Theorem 3.6.1 (Matsumoto). *Let $W = W(\Delta)$ be a finite reflection group associated to a $S = \{s_1, \cdots, s_n\}$ be the induced set of fundamental reflections, and $(M, \cdot)$ be a monoid with identity e. Let $f : S \longrightarrow M$ be a well-defined map such that*

$$f(s)f(t)f(s)f(t)\cdots = f(t)f(s)f(t)f(s)\cdots \quad if \ stst\cdots = tsts\cdots$$

Then there exist a unique extension of f, $\tilde{f} : W \longrightarrow M$ such that for each $w = s_1 \cdots s_q$ arbitrary reduced expression of w, the following holds:

$$\tilde{f}(w) = f(s_1)\cdots f(s_q)$$

Theorem 3.6.2. *Let $W = W(\Delta)$ be a finite reflection group associated with a root system Δ, and let $\Sigma = \{a_1, \cdots, a_n\}$ be fundamental root system, with $\Sigma \subseteq \Delta$. Also let $S = \{s_1, \cdots, s_n\}$ be the induced set of fundamental reflections of the set Σ. Let the order of the elements $s_i s_j$ be $m_{ij} = o(s_i s_j)$. i.e*

$$(s_i s_j)^{m_{ij}} = 1, \ \ for \ all \ i, j \ \ and \ \ m_{ij} \geq 2, \ \forall i \neq j, \ m_{ii} = 1$$

Then (W,S) is a Coxeter system and hence the group $W = W(\Delta)$ has a presentation

$$W = W(\Delta) =< s_i \in S : {s_i}^2 = 1, (s_i s_j)^{m_{ij}} = 1 >$$

So W is a Coxeter group that occurs from the Coxeter system (W,S) and the above presentation for the group W is unique in terms of isomorphism, i.e is independent of the choice of the fundamental root system.

In other words, if we choose another fundamental root system $\Sigma^{'}$, with $\Sigma \neq \Sigma^{'}$, and $S, S^{'}$ be the set of the fundamental reflections, respectively from the fundamental root systems $\Sigma, \Sigma^{'}$. Then from these two fundamental root systems arise the isomorphic presentations for the group W.

Our goal now is to prove the converse of the above theorem, i.e we would like to claim that every finite Coxeter group can be identified with a reflection group that acting on a proper Euclidean space. In particular, let W be a Coxeter group, with associated Coxeter system (W, S), we shall show that there exist (up to isomorphism) a finite reflection group, that is also effective, such that W is isomorphic to this reflection group and define a action of W on proper Euclidean space V.

Theorem 3.6.3. *Let (W, S) be a finite Coxeter system, with generators $S = \{s_1, \cdots, s_n\}$. Let V be a finite dimensional $\mathbb{R}-$vector space, with $dim_{\mathbb{R}} V = n$, and let $\{a_{s_i} : \ i = 1, \cdots, n\}$ be a basis of V. Define a bilinear form $B : V \times V \longrightarrow \mathbb{R}$ by*

$$B(a_{s_i}, a_{s_j}) = -cos\frac{\pi}{m_{ij}}, \ \ 1 \leq i, j \leq n,$$

where m_{ij} is the order of the element $s_i s_j$ in W. For each $s_i \in S$, we define a linear transformation $\rho_{s_i} : V \longrightarrow V$ by $\rho_{s_i}(a_{s_j}) = a_{s_j} - 2B(a_{s_j}, a_{s_i})e_i, \ \ 1 \leq j \leq n$. Then we have the following statements, that lead us to the claim we mentioned earlier :

1. *First for the linear transformation ρ_{s_i}, for each $s_i \in S$ we have that:*
   (a) *${\rho_{s_i}}^2 = 1_V$, where $s_i \in S$.*
   (b) *$\rho_{s_i}(a_{s_i}) = -a_s$ and $\rho_{s_i}(v) = v$, for every $v \in V$ such that $(v, a_{s_i}) = 0$, where $s_i \in S$ and a_{s_i} be the corresponding basis element of V.*
   (c) *We can decompose the space V into $V =< a_{s_i}, a_{s_j} > \oplus (H_{a_{s_i}} \cap H_{a_{s_j}})$ and the maps ρ_{s_i}, ρ_{s_j} leaves the space $< a_{s_i}, a_{s_j} >$ invariant*
   (d) *The order of the $\rho_{s_i}\rho_{s_j}$ is m_{ij}.*
   (e) *$B(\rho_s(v), \rho_s(u)) = B(u, v)$, for every $u, v \in V$.*
   (f) *For each $s_i \in S$ we have $\rho_{s_i} \in GL(V)$.*

2. *From the linear transformation we can define a map $\rho : S \longrightarrow End(V)$, that can be extended to a faithful representation $\rho : W \longrightarrow GL(V)$. Moreover, the representation ρ define an action of the group W to V.*

3. *The bilinear form B is symmetric, positive defined, and as an extension to (c), is invariant with respect to ρ_w, for all $w \in W$. Hence the space (V,B) is an Euclidean space, with inner product the bilinear form B.*

4. *The group $\rho(W)$ is a finite reflection group and the set $\{\rho_{s_1}, \cdots, \rho_{s_n}\}$ can be identified with a set of fundamental reflections in $\rho(W)$. In particular, we can call, for each $s_i \in S$, the map ρ_{s_i} a reflection of V according to the direction of the element a_{s_i} into the hyperplane $H_{a_{s_i}} := \{v \in V : B(v, a_{s_i}) = 0\}$.*

Corollary 3.6.1. *Following the notation from the above theorem we get that the groups W and $\rho(W)$ is isomorphic, and since the $\rho(W)$ is a reflection group we conclude that every Coxeter group is corresponding (up to isomorphism) to a reflection group.*

Corollary 3.6.2. *Now by the above two theorems we can established a 1-1 correspondence such that :*

$$\left\{ \begin{array}{c} \textit{Isomorphic classes of} \\ \textit{finite reflection groups} \end{array} \right\} \underset{onto}{\overset{1\text{-}1}{\longleftrightarrow}} \left\{ \begin{array}{c} \textit{Isomorphic classes of} \\ \textit{finite Coxeter groups} \end{array} \right\}$$

3.7 Classification of Coxeter Systems

Definition 3.7.1 (Coxeter Graph)**.** *Let (W,S) be a finite Coxeter system. We define as a Coxeter graph associated to the (W,S), the graph that consists of vertices, edges, and positive integers as labels to the edges, that satisfying the following properties:*

- *For the set of the vertices : We identify as vertices the elements of the set S.*
- *For the set of edges: We join two vertices $s, s^{'}$ by an edge if and only if the elements $s, s^{'}$ of S doesn't commute, i.e iff the order $m_{ss'} \neq 2$. In this case we label the edge by the positive integer $m_{ss'}$, where $m_{ss'}$ be the order of the $ss^{'}$, with $m_{ss'} \geq 3$. Note that it is common to omit the label in the case where the order $m_{ss'} = 3$. Also, recall that $s, s^{'}$ commutes if and only if $m_{ss'} = 2$, so in this case the elements $s, s^{'}$ doesn't joined by an edge.*

Theorem 3.7.1. *There is 1-1 and onto correspondence from the class of finite Coxeter system to the Coxeter graphs, i.e :*

$$\{finite\ Coxeter\ systems\} \underset{onto}{\overset{1-1}{\longleftrightarrow}} \{Coxeter\ graphs\}$$

Definition 3.7.2 (Reducible Coxeter system)**.** *A Coxeter system (W,S) is called reducible if we can written the set S as a disjoint union of subsets of W, $S_1, S_2 \subseteq W$, such that (W_1, S_1) and (W_2, S_2) be Coxeter systems and $W = W_1 \times W_2$. Differently, we say that the Coxeter system (W,S) is irreducible.*

Proposition 3.7.1. *The Coxeter system is irreducible if and only if the corresponding Coxeter graph is connected.*

Definition 3.7.3. *Let (W,S) is a Coxeter system the irreducible representation ρ, as defined in the Theorem 3.0.11 is called reflection representation of W.*

Proposition 3.7.2. *If (W,S) is an irreducible Coxeter system then the reflection representation is irreducible.*

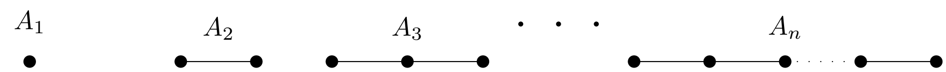
Theorem 3.7.2. *Let (W,S) be a finite Coxeter system that is reducible, and let Γ be the induced Coxeter graph from this system. Also, let $S_1, \cdots, S_k$ be the subsets of S associated with the connected components Γ_i of the Coxeter graph $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$. Then for every $i \in \{1, \cdots, k\}$ (W_i, S_i) is a finite Coxeter system where $W_i =< S_i >$ and the Coxeter group W decomposed into the direct product*

$$W = W_1 \times \cdots \times W_k$$

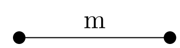
So is clear that in order to classification the finite Coxeter systems it's enough to classified the induced Coxeter graphs that come from them, and moreover by combining the theorems the problem of classification comes down to classification of the connected Coxeter graphs.

A list of the Coxeter graphs of systems associated with the irreducible:

(i) Type A_n :

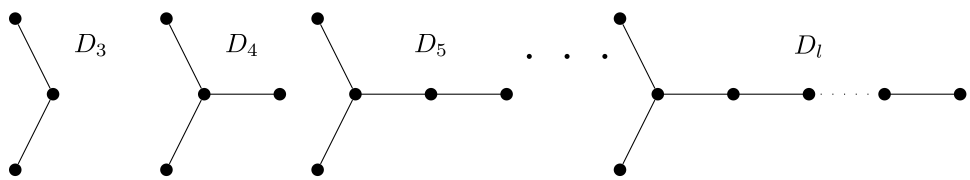


(ii) Type $I_2(m),\ m \geq 4,\ m \neq 6.$

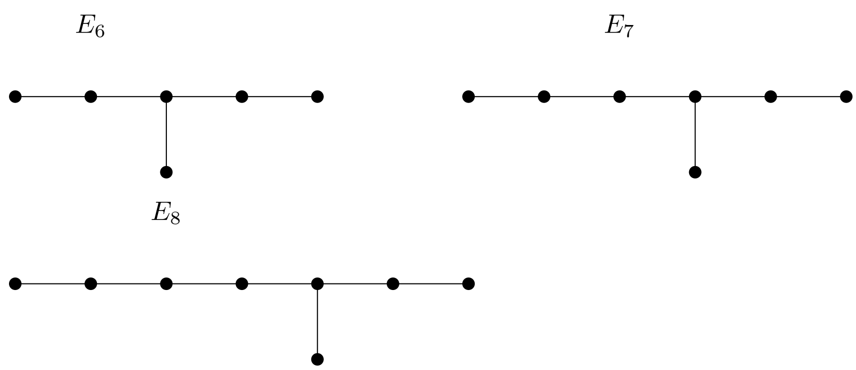


(iii) Type B_l :

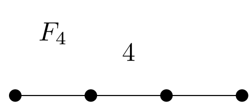
(iv) Type D_l :



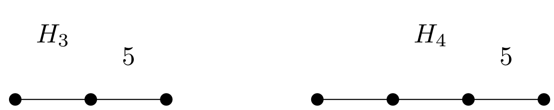
(v) 3 τύπου E_n :



(vi) 1 Type F_n :



(vii) 2 Type H_n :



4 Groups with BN-pair

4.1 Definition and Consequences

Definition 4.1.1 (Groups with BN-pair)**.** *Let G be a finite group and $B, N \leq G$. We say these subgroups form a BN-pair of the group G or that the group G is a group with BN-pair, if the following axioms satisfied:*

1. *$G =< B, N >$ (i.e the group G is generated by the elements of B, N)*
2. *Let $H := B \cap N$ and $H \lhd N$. So we can form the quotient group*

$$\frac{N}{B \cap N} := W$$

   Let $\pi : N \longrightarrow W$ be the natural homomorphism.

3. *$W = {}^{N}/_{H}$ generated by a subset $S = \{s_1, ..., s_n\}$ of involution's, i.e ${s_i}^2 = 1\ \forall s_i \in S$.*

$$W = \ < s_i \ \mid {s_i}^2 = 1 >$$

4. *For each $w \in W$ choose a coset representative $\dot{w}H \in {}^{N}/_{H}$ then :*
   (i) $\dot{s}_i B \dot{w} \subseteq B\dot{w}B \cup B\dot{s}_i wB$
   (ii) $\dot{s}_i B \dot{s}_i \neq B$

Remark 4.1.1.

- *The group $W = {}^{N}/_{H}$ will be called the Weyl group associated with the BN-pair.*
- *The generators S of W will be called distinguished generators of W, and the cardinality $|S|$ is called the rank of the BN-pair. (Both S and the rank are uniquely determined.)*
- *The subgroup $B \leq G$ is called Borel subgroup of G.*

Remark 4.1.2.

1. *The relations (4i),(4ii) in the definition, are equivalent to the relations via the natural homomorphism $\pi : N \longrightarrow W$,*
   (a) $n_i B n \subseteq BnB \cup Bn_i nB$
   (b) $n_i B n_i \neq B$

   where $n_i, n \in N$ s.t $\pi(n_i) = s_i$ and $\pi(n) = w$. (i.e n_i, n are representatives of the elements s_i, w respectively)

2. *The relation (b) can be written in the equivalent forms*

$$n_i B {n_i}^{-1} \neq B \quad or \quad {n_i}^{-1} B n_i \neq B$$

   where $n_i \in N$ s.t $\pi(n_i) = s_i$, for $s_i \in S$.

Proof. Indeed, ${s_i}^2 = 1$ in W, $\forall i = 1, ..., n$, thus $\pi({n_i}^2) = \pi(n_i)^2 = {s_i}^2 = 1$. So $\pi({n_i}^2) = 1$ in W. Hence, ${n_i}^2 \in H \Longleftrightarrow {n_i}^2 H = H \Longleftrightarrow n_i H = {n_i}^{-1} H$, so exists $h \in H : n_i = {n_i}^{-1} h$. Therefore,

$$n_i B n_i \neq B \Longleftrightarrow n_i B {n_i}^{-1} h \neq B$$

But, since $H \lhd N$, ${n_i}^{-1} h = h' {n_i}^{-1}$, for some $h' \in H$. Thus

$$n_i B {n_i}^{-1} h \neq B \Longleftrightarrow n_i B h' {n_i}^{-1} \neq B$$

From $H = B \cap N$, follow that $Bh' = B$. So

$$n_i B h' {n_i}^{-1} \neq B \Longleftrightarrow n_i B {n_i}^{-1} \neq B$$

Finally, from all the above

$$n_iBn_i \neq B \Longleftrightarrow n_iBn_i^{-1} \neq B$$

In the same way, we can show the equivalent

$$n_iBn_i \neq B \Longleftrightarrow n_i^{-1}Bn_i \neq B$$

□

3. *Another equivalent form for the relation (b) is*

$$n_iBn_i \neq B \Longleftrightarrow n_iBn_i \nsubseteq B$$

Proof. Indeed, ($\Longleftarrow$) is obvious.

($\Longrightarrow$) Let $n_iBn_i \neq B$ and let $n_iBn_i \subseteq B$. Then $n_iBn_i^{-1} \subseteq B$ which is equivalent to $B \subseteq n_i^{-1}Bn_i = n_iBn_i$. Thus, $B = n_iBn_i$, which is false due to assumption $n_iBn_i \neq B$ □

4. *The relation (a) has the equivalent form :*

$$n_iBn \subseteq Bn_inB \cup BnB \Longleftrightarrow Bn_inB \cdot BnB \subseteq Bn_inB \cup BnB \Longleftrightarrow BnB \cdot Bn_iB \subseteq Bnn_iB \cup BnB$$

Proof. Let $C(w) := BwB$ the double cosets BwB.

(a) $C(1) = B \cdot 1 \cdot B = B$

(b) $C(ww^{'}) \subseteq C(w) \cdot C(w^{'})$, for all $w, w^{'} \in W$ (i.e $Bww^{'}B \subseteq BwB \cdot Bw^{'}B$.)

Indeed, if $x \in C(ww^{'}) \Longrightarrow \exists\, b, b^{'} \in B \ : \ x = bww^{'}b^{'}$. Thus,

$$x = bwb^{-1}bw^{'}b^{'} \in BwB \cdot Bw^{'}B = C(w)C(w^{'})$$

(c) $C(w^{-1}) = C(w)^{-1}$. (i.e $Bw^{-1}B = (BwB)^{-1}$)

Indeed, if $x \in Bw^{-1}B \Longrightarrow x = b_1w^{-1}b_2 = (b_2{}^{-1}wb_1{}^{-1})^{-1} \in (BwB)^{-1}$.
Conversely, if $x \in (BwB)^{-1} \Longrightarrow x = (b_1wb_2)^{-1} = b_2{}^{-1}w^{-1}b_1{}^{-1} \in Bw^{-1}B$. Thus, $C(w^{-1}) = C(w)^{-1}$.

Now, by multiplying left and right with B the relation

$$n_iBn \subseteq Bn_inB \cup BnB$$

we have equivalent that

$$Bn_iBnB \subseteq B(Bn_inB \cup BnB)B \subseteq B(Bn_inB)B \cup B(BnB)B = Bn_inB \cup BnB = C(n_in) \cup C(n)$$

So,

$$C(n_i) \cdot C(n) = Bn_iB \cdot BnB = Bn_iBnB \subseteq C(n_in) \cup C(n)$$

Now, by taking inverses in the same relation as previous we have :

$$(n_iBn)^{-1} \subseteq (Bn_inB \cup BnB)^{-1} \Longleftrightarrow n^{-1}Bn_i^{-1} \subseteq Bn^{-1}n_i^{-1}B \cup Bn^{-1}B$$

but $n_i^2 \in H = B \cap N \Longrightarrow n_i^2 \in B \Longrightarrow Bn_i^{-1} = Bn_i$ *and* $n_iB = n_i^{-1}B$.

Therefore,

$$n^{-1}Bn_i \subseteq Bn^{-1}n_iB \cup Bn^{-1}B$$

The last relation is true for every $n \in N$, so by replacing $n \in N$ by n^{-1} we obtain :

$$nBn_i \subseteq Bnn_iB \cup BnB$$

Now, again by multiplying left and right the above relation with B we have :

$$BnB \cdot Bn_iB \subseteq (Bnn_iB) \cup (BnB)$$

Finally, we have the equivalent forms for the relation $n_iBn \subseteq Bn_inB \cup BnB$

i. $Bn_iB \cdot BnB \subseteq Bn_inB \cup BnB$

ii. $nBn_i \subseteq Bnn_iB \cup BnB$

iii. $BnB \cdot Bn_iB \subseteq Bnn_iB \cup BnB$

□

5. *For every $s_i \in S$ applies that $s_i \neq 1$*

   Proof. If $s_i = 1$, for some $s_i \in S$ then $\exists\, n_i \in N \;:\; \pi(n_i) = s_i = 1 \Longrightarrow n_i \in H = B \cap N$. Thus, $n_iBn_i = B$ which is a contradiction due to $n_iBn_i \neq B$. □

6. *Lastly, also since W has a set of involutory generators S, we can define the length $\ell(w)$ of elements $w \in W$, as well as the concept of reduced expression for elements of W.*

4.2 Bruhat Decomposition

Theorem 4.2.1 (Bruhat Decomposition)**.** *Let G be a finite group with a BN-pair. Then*

$$G = BNB = \bigsqcup_{n \in N} BnB$$

In particular, the map $w \mapsto BwB$ gives a bijection $W \longleftrightarrow B\backslash G/B$

Proof. Since G is a group with BN-pair we have that $G = < B, N >$. So G is the minimum group that containing both the subgroups B and N. Furthermore, $B, N \subseteq BNB$. Thus we just have to show that BNB is subgroup of G, i.e we show that BNB is closed under multiplication and inversion.

- $(BNB)^{-1} = B^{-1}N^{-1}B^{-1} = BNB$, i.e is closed under inversions.
- $BNB \cdot BNB = BNBNB$ (since $B \leq G$). We have to show that, $BNB \cdot BNB \subseteq BNB$ or equivalent that $BNBNB \subseteq BNB$.

  Indeed, let $nBn' \subseteq NBN$, with $n, n' \in N$. Now, consider the reduced expression $s_{i_1}...s_{i_k} = w = \pi(n)$, with $s_{i_a} \in S$. (needn't consider powers since ${s_{i_a}}^2 = 1$). Choose $n_{i_1}...n_{i_k} \in N$ with $\pi(n_{i_a}) = s_{i_a}$ s.t $n_{i_1}...n_{i_k} = n$. Notice that $n_{i_k}Bn' \subseteq Bn_{i_k}n'B \cup Bn'B \subseteq BNB$ (from axiom (4i)). So,

  $$n_{i_k}Bn' \subseteq BNB$$

  Then, again from axiom (4i) we have that

  $$n_{i_{k-1}}n_{i_k}Bn' \subseteq n_{i_{k-1}}BNB \subseteq BNB \cdot B = BNB$$

  Now, continuing inductively, we take that

  $$n_{i_1}...n_{i_k}Bn' \subseteq BNB$$

  So

  $$nBn' \subseteq BNB$$

  and therefore $NBN \subseteq BNB$.

  Thus,

  $$BNB \cdot BNB \subseteq BNBNB \subseteq B \cdot BNB \cdot B \subseteq BNB$$

  (i.e we have show that BNB is closed under multiplication.)

  Therefore, from all the above, we take that BNB is subgroup of G and that completes the proof.

$\square$

Corollary 4.2.1. *Every double coset of B in G contains an element of N. Hence any double coset can be written as BnB for $n \in N$.*

Proof. For every $g \in G$ by the Bruhat Decomposition Theorem we have that

$$BgB = Bbnb^{'}B = BnB$$

for some $b, b^{'} \in B$ and $n \in N$. $\square$

Theorem 4.2.2. *Let $n, n^{'} \in N$. Then,*

$$BnB = Bn^{'}B \Longleftrightarrow \pi(n) = \pi(n^{'})$$

Proof. If $\pi(n) = \pi(n^{'}) \implies nH = n^{'}H \implies n^{'} \in nH$. So, $n^{'} = nh$, for some $h \in H$.Then $Bn^{'}B = BnhB = BnB$ (since $h \in H = B \cap N \subseteq B$).

Conversely, suppose $BnB = Bn^{'}B$. Let $\pi(n) = w$ and $\pi(n^{'}) = w^{'}$. We want to show that $w = w^{'}$. Now, every element of W is a product of elements of S. Let $\ell(w)$ be the shortest length of any such expression. W.l.o.g suppose that $\ell(w) \leq \ell(w^{'})$ and we apply induction on length $\ell(w)$:

- Suppose $\ell(w) = 0$. Then $w = 1$. So $n \in H = B \cap N$ and thus $BnB = B$. Moreover, from hypothesis $Bn^{'}B = BnB$. Hence, $Bn^{'}B = B \Longrightarrow n^{'} \in B \cap N = H \Longrightarrow \pi(n^{'}) = 1$. i.e $w^{'} = 1$.
- Suppose now that $\ell(w) > 0$. Then $w = s_i w^{''}$ where $\ell(w^{''}) = \ell(w) - 1$. Choose, $n_i \in N$ and $n^{''} \in N$ s.t $\pi(n_i) = s_i$ and $\pi(n^{''}) = w^{''}$. Then,

$$\pi(n_i n^{''}) = s_i w^{''} = w = \pi(n) \xrightarrow{\textit{from the 1st part of thm}} Bn_i n^{''}B = BnB$$

So $Bn^{'}B = BnB = Bn_i n^{''}B$, and thus

$$n_i n^{''}B \subseteq Bn^{'}B \Rightarrow n^{''}B \subseteq {n_i}^{-1}Bn^{'}B$$

But ${n_i}^2 \in B \Rightarrow {n_i}^{-1}B = n_i B$, since $\pi(n_i) = s_i$ and ${s_i}^2 = 1$, so

$$n^{''}B \subseteq n_i B n^{'}B$$

but by axiom (4i) $n_i B n^{'} \subseteq Bn_i n^{'}B \cup Bn^{'}B$, so

$$n^{''}B \subseteq (Bn_i n^{'}B \cup Bn^{'}B)B = Bn_i n^{'}B \cup Bn^{'}B$$

Hence,

$$Bn^{''}B = Bn_i n^{'}B \text{ or } Bn^{'}B$$

But by the property of double cosets it must be equal to one or the other or its intersection will be $= \varnothing$
Recall $\ell(w^{''}) = \ell(w) - 1$ so we can apply induction and thus we have that

$$\pi(n^{''}) = \pi(n_i n^{'}) \text{ or } \pi(n^{'})$$

$$\Longleftrightarrow w^{''} = s_i w^{'} \text{ or } w^{'}$$

But, $\ell(w^{''}) < \ell(w) \leq \ell(w^{'})$, so $w^{''} \neq w^{'}$.
So we must have

$$w^{''} = s_i w^{'}$$

Then,

$$w = s_i w^{''} = {s_i}^2 w^{'} = w^{'}$$

since ${s_i}^2 = 1$

Thus, $\pi(n) = \pi(n^{'})$.

$\square$

Corollary 4.2.2. *The number of double cosets of B in G is equal to $|W|$. In particular, when $G = GL_n(k)$, $B =$ triangular subgroup , the number of double cosets is $n!$.*

Proposition 4.2.1. *Let $w \in W$, $s_i \in S$ and $\pi(n_i) = s_i$, $\pi(n) = w$, where $n_i, n \in N$. Then*

1. *If $\ell(s_i w) \geq \ell(w)$ then $n_i B n \subseteq B n_i n B$*
2. *If $\ell(s_i w) \leq \ell(w)$ then $n_i B n \cap B n B \neq \varnothing$*

Remark 4.2.1. *Note that they must always intersect since $n_i n \in n_i B n \cap B n_i n B$.*

Proof. For (1):

We apply induction on $\ell(w)$

- If $\ell(w) = 0$: then $w = 1$ and so $n \in B \cap N$. Hence, $n_i B n = n_i B$ and $B n_i n B = B n_i B$, but $n_i B \subseteq B n_i B$ so $n_i B n \subseteq B n_i n B$.
- Now suppose $\ell(w) > 0$. Then, exists $w' \in W$ such that $w = w' s_j$, with $s_j \in S$ and $\ell(w') = \ell(w) - 1$.

  Suppose the result is false for contradiction, i.e $n_i B n \nsubseteq B n_i n B$. Then, by the axiom (4i) $n_i B n \subseteq B n_i n B \cup B n B$, we take that

  $$n_i B n \cap B n B \neq \varnothing \ \ (\star)$$

  Choose $n' \in N$ with $\pi(n') = w'$ and $n_j \in N \ : \ \pi(n_j) = s_j$. Then, $n = n' n_j$, so

  $$n_i B n' n_j \cap B n B \neq \varnothing$$

  from $(\star)$
  but, $\pi(n' n_j) = w = \pi(n) \xRightarrow{theorem} B n' n_j B = B n B$.
  So,

  $$n_i B n' n_j \cap B n B \neq \varnothing \Rightarrow n_i B n' \cap B n B {n_j}^{-1} \neq \varnothing$$

  Now, since ${n_j}^2 \in B$ we have that

  $$n_i B n' \cap B n B n_j \neq \varnothing$$

  We have $\ell(s_i w') \geq \ell(w')$ (because otherwise we would have $\ell(s_i w') < \ell(w') = \ell(w) - 1$, but from $s_i w = s_i w' s_j \Rightarrow \ell(s_i w) \leq \ell(s_i w') + 1 < \ell(w) - 1 + 1 = \ell(w)$, which is contradiction due to hypothesis $\ell(s_i w) \geq \ell(w)$.)

  So, $\ell(s_i w') \geq \ell(w')$ and from the inductive hypothesis we have that

  $$n_i B n' \subseteq B n_i n' B$$

  Now, since $n_i B n' \cap B n B n_j \neq \varnothing$ we obtain that

  $$B n_i n' B \cap B n B n_j \neq \varnothing \tag{1}$$

  But from axiom (4i) $n B n_j \subseteq B n n_j B \cup B n B \Rightarrow$ we get

  $$B n B n_j \subseteq B n n_j B \cup B n B \tag{2}$$

  From the previous relations,(1) and (2), we have that there exists a common element in both $B n_i n' B$, $B n n_j B \cup B n B$, so as double coset we take that

  $$B n_i n' B = B n n_j B \text{ or } B n B$$

Hence, by Theorem $\Rightarrow s_i w^{'} = ws_j$ or w.
If $s_i w^{'} = ws_j \Rightarrow s_i w^{'} s_j = w \Rightarrow s_i w = w \Rightarrow s_i = 1$, contradicting the construction of S.
So, $s_i w^{'} = w \Rightarrow w^{'} = s_i w$, since ${s_i}^2 = 1$. Thus, from $\ell(w^{'}) < \ell(w)$ we obtain that

$$\ell(s_i w) < \ell(w)$$

contradicting our hypothesis.

So, we conclude that if

$$\ell(s_i w) \geq \ell(w) \Rightarrow n_i B n \subseteq B n_i n B$$

For (2) :

We have that, $n_i B n_i \subseteq B {n_i}^2 B \cup B n_i B$ by axiom (4i) for $n = n_i$.
Also, since ${n_i}^2 \in B$ we have that $B {n_i}^2 B \cup B n_i B = B \cup B n_i B$.
On the other side by axiom (4ii) $n_i B n_i = n_i B {n_i}^{-1} \neq B$.

So, from all the above, we obtain that

$$n_i B n_i \cap B n_i B \neq \varnothing$$

$$\Rightarrow n_i B \cap B n_i B {n_i}^{-1} \neq \varnothing$$

$$\Rightarrow n_i B \cap B n_i B n_i \neq \varnothing$$

$$\Rightarrow n_i B n \cap B n_i B n_i n \neq \varnothing$$

Now, $\ell(s_i s_i w) = \ell(w) \geq \ell(s_i w)$ so $s_i w$ satisfies the conditions of (1), of this proposition, and hence

$$n_i B n_i n \subseteq B {n_i}^2 n B = BnB$$

So,

$$B n_i B n_i n \subseteq BnB$$

Thus,

$$n_i B n \cap BnB \neq \varnothing$$

i.e we have shown that if

$$\ell(s_i w) \leq \ell(w) \Longrightarrow n_i B n \cap BnB \neq \varnothing$$

□

Corollary 4.2.3. $\ell(s_i w) \neq \ell(w)$. *Specifically,* $\ell(s_i w) = \ell(w) \pm 1$.

Proposition 4.2.2. *Suppose* $w \in W,\ s_i, s_j \in S$ *satisfy*

- $\ell(s_i w) = \ell(w) + 1$,
- $\ell(w s_j) = \ell(w) + 1$,
- $\ell(s_i w s_j) = \ell(w)$.

Then

$$s_i w = w s_j \ \text{ and so } \ s_i w s_j = w$$

Proof. Let $w^{'} = ws_j$ and $n, n^{'}, n_i, n_j \in N$ with $\pi(n^{'}) = w^{'}, \pi(n) = w, \pi(n_i) = s_i, \pi(n_j) = s_j$.
Since

$$\ell(s_i w) = \ell(w) + 1 \xRightarrow{prop\ 4.2.1} n_i B n \subseteq B n_i n B \tag{3}$$

and from

$$\ell(s_i w^{'}) = \ell(w^{'}) - 1 \xRightarrow{prop\ 4.2.1} n_i B n^{'} \cap B n^{'} B \neq \varnothing$$

So

$$n_i B n^{'} n_j \cap B n^{'} B n_j \neq \varnothing$$

and hence we get

$$n_i B n \cap B n^{'} B n_j \neq \varnothing \tag{4}$$

From (3), (4) we obtain that

$$B n_i n B \cap B n^{'} B n_j \neq \varnothing$$

Now, from axiom (4i)

$$n^{'} B n_j \subseteq B n^{'} n_j B \cap B n^{'} B$$

So by the property of double cosets,

$$B n_i n B = B n^{'} n_j B \text{ or } B n^{'} B$$

since it intersects and their union is non-trivially and they're all double cosets.

So from Theorem above,

$$s_i w = w^{'} s_j \text{ or } w^{'}$$

But if $s_i w = w^{'} s_j \Rightarrow s_i w = w s_j s_j = w {s_j}^2 = w$ then $s_i = 1$, which is contradiction by the construction of S.

So $s_i w = w^{'} \Rightarrow s_i w = w s_j \Longleftrightarrow s_i w s_j = w$ □

4.3 The Fundamental Theorem

Theorem 4.3.1. *For every finite group G with BN-pair, with Weyl group* $W = \frac{N}{B \cap N}$ *and the S be the set of distinguished generators of W, we have that W is a Coxeter group and S is a set of Coxeter generators. In particular (W,S) is Coxeter System.*

Proof. Let $s_i, s_j \in S$ with $s_i \neq s_j$. Let m_{ij} be the order of $s_i s_j$ if this is finite.
We have $(s_i s_j)^{m_{ij}} = 1$. We need to show that it's a set of defining relations.

So we must prove that, if $G^{\star}$ is any group generated by elements g_i in 1-1 correspondence with elements $s_i \in S$ s.t ${g_i}^2 = 1$, $(g_i g_j)^{m_{ij}} = 1$, then there exist a homomorphism $\theta : W \longrightarrow G^{\star}$ s.t $\theta(s_i) = g_i$.

This will show that W is the universal group with these generators and relations.

- **STEP 1 :** We first show that if $w \in W$ has two reduced expressions

  $$w = s_{i_1} ... s_{i_k} = s_{j_1} ... s_{j_k}$$

  then

  $$g_{i_1} ... g_{i_k} = g_{j_1} ... g_{j_k}$$

  We use induction on $\ell(w)$.

  - If $\ell(w) = 0$ then $w = 1$ so we are ok.
  - Suppose this is true for every w with $\ell(w) < k$, but that exist $w \in W$ with $\ell(w) = k$ for which it fails. Then

    $$w = s_{j_1} ... s_{j_k} = s_{j_{2k}} ... s_{j_{k+1}}$$

    but

    $$g_{j_1} ... g_{j_k} \neq g_{j_{2k}} ... g_{j_{k+1}}$$

    Note that $s_{j_2} ... s_{j_k}$ is reduced since $s_{j_1} ... s_{j_k}$ is reduced.

Claim 1. *We shall show that $s_{j_2}...s_{j_k}s_{j_{k+1}}$ is also reduced.*

Proof. Indeed, suppose that isn't. Then $s_{j_2}...s_{j_k}s_{j_{k+1}} = s_{a_1}...s_{a_{k-2}}$ reduced. So, $s_{j_2}...s_{j_k} = s_{a_1}...s_{a_{k-2}}s_{j_{k+1}}$ since ${s_{j_{k+1}}}^2 = 1$

Both of these expressions are reduced of length $k-1$ (since $s_{j_2}...s_{j_k}$ is reduced, they're the same length and they're equal)

So by our choice of k (minimality) we must have that

$$g_{j_2}...g_{j_k} = g_{a_1}...g_{a_{k-2}}g_{j_{k+1}} \quad (\star)$$

Also

$$s_{j_{2k}}...s_{j_{k+2}} = s_{j_1}...s_{j_k}s_{j_{k+1}} = s_{j_1}s_{a_1}...s_{a_{k-2}}$$

from the fact that $s_{j_2}...s_{j_k} = s_{a_1}...s_{a_{k-2}}s_{j_{k+1}}$ and ${s_{j_{k+1}}}^2 = 1$

Both sides are reduced of length $k-1$, so again by minimality of k

$$g_{j_{2k}}...g_{j_{k+2}} = g_{j_1}g_{a_1}...g_{a_{k-2}} \quad (\star\star)$$

So eliminating the ${g_a}'s$ from the relations $(\star), (\star\star)$ we get

$$g_{j_2}...g_{j_k}g_{j_{k+1}} = g_{j_1}g_{j_{2k}}...g_{j_{k+2}} \Longleftrightarrow g_{j_1}...g_{j_k} = g_{j_{2k}}...g_{j_{k+1}}$$

which is contradiction.

So, we have shown that $s_{j_2}...s_{j_k}s_{j_{k+1}}$ is reduced.

□

Claim 2. *We shall show that $s_{j_1}...s_{j_k} = s_{j_2}...s_{j_{k+1}}$ but $g_{j_1}...g_{j_k} \neq g_{j_2}...g_{j_{k+1}}$.*

Proof. Now, we have that

$$s_{j_2}...s_{j_k} \text{ is reduced of length k-1}$$
$$s_{j_1}s_{j_2}...s_{j_k} \text{ is reduced of length k}$$
$$s_{j_2}...s_{j_k}s_{j_{k+1}} \text{ is reduced of length k}$$
$$s_{j_1}s_{j_2}...s_{j_k}s_{j_{k+1}} \text{ is not reduced } \ell(s_{j_1}...s_{j_{k+1}}) = k-1$$

(since $s_{j_1}...s_{j_k} = s_{j_{2k}}...s_{j_{k+1}} \Rightarrow s_{j_1}...s_{j_{k+1}} = s_{j_{2k}}...s_{j_{k+2}}$)

So by previous proposition we get

$$s_{j_1}...s_{j_k} = s_{j_2}...s_{j_{k+1}} \quad (\star\star\star)$$

We shall show that

$$g_{j_1}...g_{j_k} \neq g_{j_2}...g_{j_{k+1}}$$

Suppose that is possible $g_{j_1}...g_{j_k} = g_{j_2}...g_{j_{k+1}}$.

Now, $s_{j_{2k}}...s_{j_{k+2}} = s_{j_1}...s_{j_k}s_{j_{k+1}} = s_{j_2}...s_{j_k}$ by $(\star\star\star)$ and ${s_{j_{k+1}}}^2 = 1$
Hence, $g_{j_{2k}}...g_{j_{k+2}} = g_{j_2}...g_{j_k}$ by induction.

So

$$g_{j_1}...g_{j_k} = g_{j_2}...g_{j_{k+1}} = g_{j_{2k}}...g_{j_{k+2}}g_{j_{k+1}}$$

which is contradiction, by assumption and above.

Thus

$$g_{j_1}...g_{j_k} = g_{j_{2k}}...g_{j_{k+1}}$$

So

$$s_{j_1}...s_{j_k} = s_{j_2}...s_{j_{k+1}} \text{ but } g_{j_1}...g_{j_k} \neq g_{j_2}...g_{j_{k+1}}$$

□

Hence, having started with

$$s_{j_1}...s_{j_k} = s_{j_{2k}}...s_{j_{k+1}} \text{ but } g_{j_1}...g_{j_k} \neq g_{j_{2k}}...g_{j_{k+1}}$$

we have derived that

$$s_{j_1}...s_{j_k} = s_{j_2}...s_{j_{k+1}} \text{ but } g_{j_1}...g_{j_k} \neq g_{j_2}...g_{j_{k+1}}$$

We repeat this process, considering

$$s_{j_2}...s_{j_{k+1}} = s_{j_1}...s_{j_k}$$

to get

$$s_{j_2}...s_{j_{k+1}} = s_{j_3}...s_{j_{k+1}}s_{j_k} \text{ but } g_{j_2}...g_{j_{k+1}} \neq g_{j_3}...g_{j_{k+1}}g_{j_k}$$

Repeat again, also swapping l.h.s and r.h.s again, gives

$$s_{j_3}...s_{j_{k+1}}s_{j_k} = s_{j_4}...s_{j_{k+1}}s_{j_k}s_{j_{k+1}} \text{ but } g_{j_3}...g_{j_{k+1}}g_{j_k} \neq g_{j_4}...g_{j_{k+1}}g_{j_k}g_{j_{k+1}}$$

By continuing in this way we end up with

$$s_{j_k}s_{j_{k+1}}s_{j_k}s_{j_{k+1}}... = s_{j_{k+1}}s_{j_k}s_{j_{k+1}}s_{j_k}$$

but

$$g_{j_k}g_{j_{k+1}}g_{j_k}g_{j_{k+1}}... \neq g_{j_{k+1}}g_{j_k}g_{j_{k+1}}g_{j_k}...$$

Hence, since the ${s_a}'s$ have order 2 this says that

$$(s_{j_k}s_{j_{k+1}})^k = 1 \text{ and } (g_{j_k}g_{j_{k+1}})^k \neq 1$$

So k is a multiple of the order of $s_{j_k}s_{j_{k+1}}$, i.e of $m_{j_k j_{k+1}}$.

Thus, $(g_{j_k}g_{j_{k+1}})^k = 1$, which is contradiction by our construction of $G^\star$.

So, we have shown that any two reduced expressions for $w \in W$ give equal expressions in $G^\star$.

- **STEP 2 :** Know we define the map $\theta : W \longrightarrow G^\star$ in the following way:

Take a reduced expression for $w \in W$. Define $\theta(w)$ to be the corresponding product in $G^\star$. Then, from the Step 1 we obtain that the map θ is well-defined.

Now, we have to show that θ is a homomorphism.

Claim 3. *In this direction, it's sufficient to show that*

$$\theta(s_i w) = \theta(s_i)\theta(w)$$

$\forall s_i \in S,\ w \in W$, *since all the elements of W are products of elements of S.*

Proof. If $\ell(s_i w) = \ell(w) + 1$ then the result is obvious.

Now suppose that $\ell(s_i w) = \ell(w) - 1$. Then put $w^{'} = s_i w$ and so we get

$$w = s_i w^{'}$$

with $\ell(s_i w^{'}) = \ell(w) + 1$, since ${s_i}^2 = 1$

So

$$\theta(s_i w^{'}) = \theta(s_i)\theta(w^{'})$$
$$\Rightarrow \theta(w) = \theta(s_i)\theta(w^{'})$$

But $\theta(s_i) = g_i$ and has order 2 so $\theta(s_i)\theta(w) = \theta(w^{'})$

i.e

$$\theta(s_i w) = \theta(s_i)\theta(w)$$

in this case also. □

So we have shown that the map θ as defined above is homomorphism.

It's also clear from the definitions of the homomorphism θ that is 1-1 and onto, is isomorphism.

This shows that W is isomorphic to the abstract group with generators and relations as given.

So W is a Coxeter group.

□

4.4 An Example of a BN-pair Group

Let K be any field and $G = GL_n(K)$ be the group of all invertible $n \times n$ matrices with entries over K. (Note that for an arbitrary field K the group G may be infinite)

Let:

$B =$ group of upper triangular matrices $= \{(a_{ij}) \in G : a_{ij} = 0, \text{ if } i < j\}$
$N =$ group of monomial matrices in G $= \{(a_{ij}) :$ exactly one non-zero element for every row and column$\}$
$H = B \cap N =$ group of diagonal matrices in G

Theorem 4.4.1. *The subgroups B and N defined above form a BN-pair in $G = GL_n(K)$ of rank n, whose Weyl group W is isomorphic to the symmetric group S_n.*

Proof.

Claim 4. $H = B \cap N =$ *group of diagonal matrices in G*

Proof. Let $h \in H$. Then $h = (h_{ij})_{n \times n}$ is an $n \times n$ matrix such that is upper triangular and simultaneously every row and every column has exactly one non-zero element. Thus,

$$h_{ii} \neq 0 \text{ and } h_{ij} = 0 \ \ \forall i < j$$

Now

$$h_{ij} = 0 \ \ \forall i > j$$

since if exists $i > j$ s.t $h_{ij} \neq 0$ then either the $j-$column or the $i-$row would have a second non-zero element beside the element h_{ii}, which is contradiction due to the definition of matrices in the subgroup N and $h \in N$.

So

$$H = B \cap N = \text{ group of diagonal matrices in G}$$

□

- Let S_n be the symmetric group of degree n. We will construct a epimomorphism $\pi : N \longrightarrow S_n$

1. First we defined the homomorphism $\pi : N \longrightarrow S_n$ in the following way:

   Let $A \in N$ a monomial matrix

   e.g A will be of the form

$$\begin{pmatrix} \star & 0 & 0 & \cdots & 0 \\ 0 & 0 & \star & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & \star & 0 & \cdots & 0 \end{pmatrix}$$

   i.e a $n \times n$ matrix with exactly one non-zero element for every row and column.

   Now suppose the non-zero entry in $i-$row appear in j_i-column. i.e

$$\begin{pmatrix} 0 & \cdots & \underset{\substack{\uparrow \\ j_i - \text{column}}}{\star} & \end{pmatrix} 0$$

   Then we construct the permutation σ_A in the following way :

We map the index of every row to the corresponding index of the column that has the non-zero element.

Thus we have:

$$\sigma_A = \begin{Bmatrix} 1 & \mapsto & j_1 \\ 2 & \mapsto & j_2 \\ & \vdots & \\ n & \mapsto & j_n \end{Bmatrix}$$

i.e : we have the permutation

$$\sigma_A = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

Claim 5. *Indeed, $\sigma_A \in S_n$*

Proof. – $\underline{\boldsymbol{\sigma}_{\ A}\ \textbf{is 1-1}}$:

If $i \neq i^{'}$ (i.e we have choose two different rows from the matrix A) then we obtain that in the j_i and $j_{i'}$−column we have a non-zero element of the matrix. Thus by the property of the monomial matrices we have that it isn't possible to have two non-zero elements in the same column. So $j_i \neq j_{i'}$. Hence σ_A is 1-1.

– $\underline{\boldsymbol{\sigma}\ \textbf{is onto}}$:

Let $k \in [n]$. We will show that exists $i \in [n]: \ \sigma_A(i) = k$

From the definition of the matrix $A \in N$ we obtain that exits a row s.t in the $k-$column has a non-zero element. Let k_i-row be the row with the non-zero element. Then from the definition of the σ_A we have that:

$$\sigma_A(k_i) = k$$

Thus σ_A is onto.

So $\sigma_A \in S_n$ □

Now, we are ready to define the map $\pi : N \longrightarrow S_n$:

We map the matrix A to the permutation σ_A by

$$\begin{pmatrix} 0 & \cdots & \underset{\substack{\uparrow \\ j_i - \text{column}}}{\star} & \end{pmatrix} 0 \longmapsto \ \sigma_A = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix} \in S_n$$

2. **The map $\pi : N \longrightarrow S_n$ is well defined :**

Suppose $A = A^{'} \in N$. From the way they have been defined, the same non-zero elements must correspond to the same rows for the same columns. So

$$\sigma_A = \sigma_{A'}$$

since for every $i \in [n]$ we will have that the same column j_i will correspond to the $i-$row and then

$$\sigma_A(i) = j_i = \sigma_{A'}(i)$$

Let the unity monomial matrix $I_N \in N = diag(1, \cdots, 1)$. Then

$$\begin{array}{ccc} \text{non-zero element of 1st-row} & \rightsquigarrow & \text{non-zero element of 1st-column} \\ & \vdots & \\ \text{non-zero element of nth-row} & \rightsquigarrow & \text{non-zero element of nth-column} \end{array}$$

So

$$\pi(I_N) = \sigma_N = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix} = id_N$$

i.e the map π is sending the unity matrix of the subgroup N to the identity permutation of the S_n.

3. **The map π is indeed homomorphism and moreover epimorphism :**

   In this direction, we will present an equivalent way of defining the above map π:

   Notice that the subgroup N permutes the lines Ke_i, i.e the subspaces $< e_i >$, where $\{e_1, \cdots e_n\}$ be the canonical basis of the vector space K^n, and so we obtain an act of N on $< e_i >$. Therefore, from this action it defined an epimorphism $\rho : N \longrightarrow S_n$ such that maps every i-row of a matrix $A \in N$ into the permutation σ_A, where the σ_A is such that $i \mapsto j_i$, with j_i-column be the column of matrix A with the non-zero element. Now it's obvious from the way the maps π and ρ have been defined, that are the same map, and hence we get that the map π is an epimorphism.

4. $Ker\pi = H$

   Let $A \in Ker\pi$.

   Then

   $$\pi(A) = id_{S_n}$$

   i.e $id_{S_n}(i) = i$ but on the other hand

   $$\pi(A) = \sigma_A = \begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

   where j_i-column is the column of the matrix A that has the non-zero element for $i-$row.

   So

   $$j_i = \sigma_A(i) = id_{S_n}(i) = i$$

   i.e the matrix A is diagonal.

   Obviously, vice versa if A is a diagonal matrix then

   $$\pi(A) = \sigma_A = id_{S_n}$$

   Hence we obtain that

   $$Ker\pi = H = \text{diagonal matrices of G}$$

5. Now from the **1st Isomorphism Theorem** for groups we obtain that

   $$\frac{N}{Ker\pi} \cong im\pi = S_n$$

   Since $Ker\pi = H$ by (4) we have that $H \lhd N$ and we conclude that

   $$N/H = W \cong S_n$$

   So

   $$W \cong S_n$$

- Now we will show that the group G satisfies the axioms in the definition of the group with BN-pair.

1. **For Axiom 1:**

   Consider left multiplication by elements of B

   $$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & t \end{pmatrix} = \begin{pmatrix} ax+cz & ay+ct \\ bz & bt \end{pmatrix}$$

   So left multiplication by an element of B transforms any row into a multiple of itself and a linear combination of later rows.

   On the other hand consider right multiplication by elements of B

   $$\begin{pmatrix} x & y \\ z & t \end{pmatrix} \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} = \begin{pmatrix} ax & cx+by \\ az & cz+bt \end{pmatrix}$$

   So right multiplication by an element of B transforms each column into a multiple of itself and a linear combination of earlier columns.

   Hence the left and right multiplication by elements of B is corresponding to row and column transformations.

   Thus any element of G can be transformed by left multiplication by $b \in B$ into a matrix s.t the first non-zero positions in the rows $1, .., n$ are in different columns. So any matrix of the G after left multiplication by an element of B it will be of the form :

   $$\begin{pmatrix} \star & \cdots & & & & \\ \vdots & & & & & \\ 0 & \star & \cdots & & & \\ \vdots & & & & & \\ 0 & \cdots & 0 & \star & \cdots & \\ \vdots & & & & & \\ 0 & \cdots & \star & \cdots & & \end{pmatrix}$$

   Now by right multiplication by $b^{'} \in B$ we can make the above matrix monomial. So finally it will be of the form :

   $$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & \star & \cdots \\ \vdots & \vdots & \vdots & \\ \star & 0 & 0 & \cdots \\ 0 & \star & 0 & \cdots \end{pmatrix}$$

   Thus for every $g \in G$ and appropriate choose of elements in B, we obtain that exists $b, b^{'} \in B$ $s.t$ $bgb^{'} \in N$, i.e $bgb^{'} = n$, for some $n \in N$. So

   $$g = b^{-1}n(b^{'})^{-1} \in BNB \quad \text{(Bruhat Decomposition)}$$

   So

   $$G = < B, N >$$

i.e we have show that the group G satisfy the axiom 1.

2. **For Axiom 3 :**

   $W \cong S_n$ we have that
   $$W = < s_i \ : \ {s_i}^2 = 1 >$$
   where $s_i = (i \; i+1)$, for $i \in [n]$.

   So there exists a set of generators $S = \{s_1, \cdots, s_{n-1}\}$ for W with ${s_i}^2 = 1$ (s_i is transportation's)

   Thus S clearly satisfy the axiom 3.

3. **For Axiom (4i) and (4ii):**

   Let $n_i \in N$, where n_i be a monomial matrix that has unit in the diagonal except the (i,i) and (i+1,i+1) positions where we have zero's, and zero anywhere else except the positions (i,i+1) and (i+1,i) where we have unity. Also let $\pi(n_i) = s_i$

   Let U be the subgroup of upper-triangular matrices with unit in the diagonal i.e
   $$U = \{(a_{ij}) \ : \ a_{ij} = 0 \text{ if } i > j, \ a_{ii} = 1\}$$
   and $U_i \leq U$ be the subgroup of the upper-triangular matrices with unit in the diagonal and 0 in the $i+1-$position above the diagonal i.e
   $$U_i = \{(a_{ij}) \in U \ : \ a_{i \; i+1} = 0\}$$
   Let e_{ij} be the elementary matrix which has in every position zero except the (i,j)-position that has unit. For $i \neq j$ define $X_{ij} = \{I + \lambda e_{ij} \ : \ \forall \lambda \in k\}$ so X_{ij} it will be consists of matrices that have
   - zero under the diagonal
   - unit in the diagonal
   - λ in the $(i,j)-$position

   We will write $X_i = X_{i \; i+1}$, $X_{-i} = X_{i+1 \; i}$ these are subgroups of G.

   Now, we have that $U \lhd B$ and so if $b \in B, u \in U \implies b^{-1}ub \in B$. It's obvious from the way multiplication works between matrices that $B = UH = HU$ since
   $$\begin{pmatrix} 1 & \star & \cdots & \star \\ 0 & 1 & \cdots & \star \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} diag(\star, \cdots, \star) = \begin{pmatrix} \star & \star & \cdots & \star \\ 0 & \star & \cdots & \star \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \star \end{pmatrix} \in B$$
   Similarly if we multiple by an element of H an element of U we obtain a matrix in B. Also obviously a upper-triangular matrix in B can decomposed it, as a product of a matrix in U and a matrix in H or vice versa.

   Furthermore $U \cap H = \{I\}$ since a matrix in $U \cap H$ is simultaneously a diagonal matrix (because is in H) with unit in the diagonal (because is in U).

   Also clearly $U = X_iU_i = U_iX_i$ and $U_i \cap X_i = \{I\}$

   Now by simple calculations we obtain that $n_iX_i{n_i}^{-1} = X_{-i}$, e.g in the case of 2×2:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

Also $n_i U_i {n_i}^{-1} = U_i$

Thus if $n_i B {n_i}^{-1} = B$ we would have that

$$X_{-i} = n_i X_i {n_i}^{-1} \subseteq n_i B {n_i}^{-1} \subseteq B$$

which is contradiction since X_{-i} is lower-triangular and B is the subgroup of upper-triangular matrices of G, hence must $X_{-i} \nsubseteq B$.

So

$$n_i B {n_i}^{-1} \neq B$$

thus the axiom (4ii) holds.

Now for axiom (4i):

$n_i B n \overset{B=HU}{=\!=} n_i H U n = n_i H {n_i}^{-1} n_i U n \overset{H \lhd N}{=\!=} H n_i U n \overset{U=U_i X_i}{=\!=} H n_i U_i {n_i}^{-1} n_i X_i n \overset{n_i U_i {n_i}^{-1} = U_i}{=\!=} H U_i n_i X_i n \overset{HU_i \subseteq B}{\subseteq} B n_i X_i n$

Consider $\pi(n) = \sigma$ and $\pi(n_i) = \sigma_i = (i\ i+1) \in S_n$

Then

$$n^{-1} X_{ij} n = X_{\sigma(i)\sigma(j)}$$

so

$$n_i X_i n = n_i n n^{-1} X_i n = n_i n X_{\sigma(i)\sigma(i+1)}$$

We have two cases to consider:

(a) **If** $\sigma(i) < \sigma(i+1)$:

Then $X_{\sigma(i)\sigma(i+1)} \subseteq B$ since $X_{\sigma(i)\sigma(i+1)}$ is upper-triangular. So

$$n_i X_i n = n_i n X_{\sigma(i)\sigma(i+1)} \subseteq n_i n B$$

and hence

$$n_i B n \subseteq B n_i X_i n \subseteq B n_i n B$$

so the axiom 4ii) holds for this case.

(b) **If** $\sigma(i) > \sigma(i+1)$:

Let $n^{'} = n_i n$ and $\pi(n^{'})\sigma^{'}$. So $\sigma^{'} = \pi(n^{'}) = \pi(n_i)\pi(n) = s_i \sigma$ since $\pi(n_i) = s_i$. Thus $\sigma^{'}(i) < \sigma^{'}(i+1)$ since $s_i = (i\ i+1)$

Then

$$n_i X_i n = n_i X_i {n_i}^{-1} n_i n = X_{-i} n^{'}$$

$$\begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & a & b & & & \\ & & c & d & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} \quad \text{as} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for short, where a occurs in the (i,i) position.

We notice that multiplication of the large matrices of this form corresponds to multiplying the ones as if they were normal 2×2 matrices and vice versa.

If $\lambda \neq 0$ then $\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{pmatrix}$

So

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \in UHn_iU \subseteq Bn_iB$$

Now when $\lambda = 0$ we get the unit matrix $I_n \in B$

Thus in general we obtain that

$$X_{-i} \subseteq B \cup Bn_iB \Rightarrow X_{-i}n^{'} \subseteq Bn^{'} \cup Bn_iBn^{'}$$

because $\sigma^{'}(i) < \sigma^{'}(i+1)$ so by the previous case we have that

$$n_iBn^{'} \subseteq Bn_in^{'}B$$

and hence

$$X_{-i}n^{'} \subseteq Bn^{'} \cup B \cdot Bn_in^{'}B = Bn^{'} \cup Bn_in^{'}B$$

Also $n_iX_in = X_{-i}n^{'}$ from above so we get that

$$n_iX_in \subseteq Bn^{'} \cup Bn_in^{'}B = Bn_in \cup Bn_in_inB$$

Now ${n_i}^2 \in B$ so

$$n_iX_in \subseteq Bn_in \cup BnB$$

So

$$n_iBn \subseteq Bn_iX_in \subseteq B(Bn_in \cup BnB) \subseteq Bn_inB \cup BnB$$

Hence

$$n_iBn \subseteq Bn_inB \cup BnB$$

and so the axiom (4ii) holds.

□

5 The Hecke Algebra

5.1 Double Cosets

Definition 5.1.1. *Let G be a group, and let H and K be subgroups. We define on group G the equivalence relation :*

For $x, y \in G$:

$$x \sim y \iff \text{ if } \ \exists\, h \in H,\ k \in K :\ y = hxk$$

For each $x \in G$ the equivalence classes of x under this equivalence relation is called the $(H,K)-$ double coset of x and is the set

$$HxK = \{hxk \ : \ h \in H, k \in K\}$$

The set of all (H,K)-double cosets is denoted $H\backslash G/K$.

If $K = H$ the $(H,H)-$double coset of G due to subgroup H, is the sets of the equivalence classes HxH, for each $x \in G$ and the x is called representative os the double coset.

The set of all $(H,H)-$double cosets is denoted by $H\backslash G/H$.

[Double Cosets]

Remark 5.1.1. *Equivalently the (H, K)-double cosets of G may be described as orbits for the product group $H \times K$ acting on G by*

$$(h,k)\cdot x = hxh^{-1}, \quad for\, h \in H, k \in K, x \in G$$

Proposition 5.1.1. ***Some Properties of Double Cosets :***

1. *Two double cosets HxK and HyK are either disjoint or identical.*
2. *G is the disjoint union of its double cosets. i.e $G = \bigsqcup_{x\in G} HxK$*
3. *There is a $1-1$ correspondence between the two double coset spaces $H\backslash G/K$ and $K\backslash G/H$ given by identifying HxK with $Kx^{-1}H$.*
4. *A double coset HxK is a union of right cosets of H and left cosets of K.*

Proposition 5.1.2. *Let HxH a double coset for $x \in G$.*

Then the HxH double coset is written as a disjoint union of right cosets. Specifically,

$$HxH = \bigsqcup_{s_i} Hxs_i$$

where $\{s_i\}$ is a complete system of right representatives of the subgroup $K := H \cap x^{-1}Hx$ in H, (i.e $H = Ks_1 \sqcup Ks_2 \sqcup \cdots$)

Proposition 5.1.3. *It follows that the number of right cosets of H contained in HxK is the index $[K : K \cap x^{-1}Hx]$ and the number of left cosets of K contained in HxK is the index $[H : H \cap xKx^{-1}]$. Therefore*

$$|HxK| = [H : H \cap xKx^{-1}]|K| = |H|[K : K \cap x^{-1}Hx]$$

If $|G| < \infty$ then

$$|HxK| = \frac{|H||K|}{|H \cap xKx^{-1}|} = \frac{|H||K|}{|K \cap x^{-1}Hx|}$$

Particularly, if $H = K$ and G is finite then

$$|HxH| = |H| \cdot [H : H \cap x^{-1}Hx] = |H|^2 \cdot |H \cap x^{-1}Hx|$$

5.2 The Structure of Hecke Algebra

We now assume that G is a finite group, with BN-pair (e.g $G = GL_n(F_q)$, F_q is a field with q elements, where $q = p^e$, with p be a prime)

Let $\mathbb{C}$G be the group algebra of G. Thus $\mathbb{C}$G is the set of elements of the form $\sum_g \lambda_g g$, where $\lambda_i \in \mathbb{C}$, $g_i \in G$. We recall also that, sometimes is convenient to identify the group algebra $\mathbb{C}$G with the set of $\mathbb{C}-$functions on G. The element $\sum_g \lambda_g g \in \mathbb{C}G$ corresponding to the function $f : G \longrightarrow \mathbb{C}$ defined by $f(g) = \lambda_g,\ \ g \in G,\ \ \lambda_g \in \mathbb{C}$.

Now the group algebra $\mathbb{C}G$ is a vector space over $\mathbb{C}$ and also $\mathbb{C}G$ is a ring, because the elements of G admit multiplication $g_i g_j \in G$, extended linearly.

A $\mathbb{C}-$algebra is a vector space over $\mathbb{C}$ which is also a ring.

We can regard $\mathbb{C}G$ as a left G-module since for every $g \in G$, $v \in \mathbb{C}G$ we get $gv \in \mathbb{C}G$ and $(gg^{'})v = g(g^{'}v)$, for $g, g^{'} \in G$, $v \in \mathbb{C}G$.

Let $e = \frac{1}{|B|}\sum_{b\in B} b \in \mathbb{C}B$ is a idempotent in $\mathbb{C}B$, since $e^2 = e$ ($e^2 = \frac{1}{|B|^2}\sum_{b,b'\in B} bb^{'} = \frac{1}{|B|^2}|B|\sum_{b\in B} b = e$). Also note that for every $b \in B$ is true that $be = e$. Thus the space $\mathbb{C}e$ is a $\mathbb{C}B-$module, and in particular is a 1-dimensional left $\mathbb{C}B$-module, that giving the trivial representation of B (since $\mathbb{C}$e is B-invariant). i.e defined the representation $1_B : B \longrightarrow GL_1(\mathbb{C})$, with $b \mapsto 1$.

Now we could make the following definition:
Let $V = \mathbb{C}Ge \leq \mathbb{C}G$ be the left $\mathbb{C}G$-module generated by e. We will call V the induced module of $\mathbb{C}e$. So $\mathbb{C}e$ gives the trivial representation 1_B of B and $V = \mathbb{C}Ge$ gives the induced representation ${1^G}_B$ of G.

5.2.1 The Definition of Hecke algebra H

Proposition 5.2.1. *dim V = [G : B]*

Proof. Let $g_1, ..., g_r$ be a set of left coset representatives of B in G. So

$$G = g_1B \sqcup g_2B \sqcup ... \sqcup g_rB$$

We notice that,

$$re = \sum_{g\in G} \lambda_g ge = \sum_{g\in G} \lambda_g(ge)$$

for $r = \sum_{g\in G} \lambda_g g \in \mathbb{C}G$

So the elements ge, with $g \in G$ will spanned the $\mathbb{C}$G-module V.i.e

$$V =< ge \mid g \in G >$$

as a $\mathbb{C}$-vector space.

Now, since g is in some left coset of B in G we have that $g = g_ib$ where $b \in B$ for some $i \in [r]$. Then

$$ge = g_ibe = g_ie$$

and so the elements g_ie spanned V. Thus

$$V =< g_1e, \cdots, g_re >$$

We will show that the elements g_ie are linearly independent.

Suppose

$$\sum_i \lambda_i g_i e = 0 \Rightarrow \frac{1}{|B|}\sum_i \lambda_i (\sum_{b\in B} g_i b) = 0$$

But $\sum_{b\in B} g_i b$ is the sum of all the elements of cosets g_iB and two such different sums are disjoint between them i.e the addends are in different cosets. Hence, as $i \in [r]$ we get the linear combination of all the elements of the group G, which are linear independent since they are form a basis for the $\mathbb{C}$G as $\mathbb{C}$-vector space. So $\lambda_i = 0$ and thus the set $\{g_1e, ..., g_re\}$ are linearly independent. Furthermore, the set $\{g_ie \ : \ i \in [r]\}$ form a basis for the $\mathbb{C}$-vector space V. Thus

$$dim_{\mathbb{C}} V = [G : B]$$

$\square$

Remark 5.2.1. *The basis $\{g_1e, ..., g_re\}$ we construct in previous proposition, for the $\mathbb{C}$-vector space V, is independent of the choice of the representatives of the left cosets.*

Now a $\mathbb{C}$-linear map $\theta : V \longrightarrow V$ is called a $\mathbb{C}G-$endomorphism if

$$\theta(gv) = g(\theta v), \quad \forall g \in G \text{ and } v \in V$$

Let θ_1, θ_2 be $\mathbb{C}G$-endomorphisms, then so are $\theta_1 + \theta_2, \theta_1\theta_2$(defined by $(\theta_1\theta_2)v = \theta_1(\theta_2 v)$), and $\lambda\theta_1$, for $\lambda \in \mathbb{C}$. Thus the $\mathbb{C}G$-endomorphisms form the structure of an $\mathbb{C}-$algebra. i.e The space $End_{\mathbb{C}G}V$ is an $\mathbb{C}-$algebra.

We are ready to define the Hecke algebra of a finite group G with BN-pair.

Definition 5.2.1 (Hecke Algebra)**.** *Let G be a finite group with BN-pair and $e = \frac{1}{|B|}\sum_{b\in B} B$ be a idempotent element of $\mathbb{C}B$ such that the left ideal $\mathbb{C}Ge$ affords the induced representation 1_B^G, We define as Hecke algebra the space of the $\mathbb{C}G$-endomorphisms of the $\mathbb{C}G$-module V generated by e, where $V = \mathbb{C}Ge$. i.e the Hecke algebra of G with respect to B is the space $End_{\mathbb{C}G}V$ and we will denote by $H = H(G, B, 1_B)$.*

Now let $A = e\mathbb{C}Ge = \{ere : r \in \mathbb{C}G\} \leq \mathbb{C}G$. A is a subalgebra of $\mathbb{C}G$ since it's closed under addition, multiplication $(e(r_1e \cdot er_2)e \in e\mathbb{C}Ge)$ and scalar multiplication.

Proposition 5.2.2. *$dimA = |B\backslash G/B|$*

Proof. Let $x_1, x_2, ..., x_s$ be a set of double coset representatives of B in G.

$$G = \bigsqcup_i Bx_iB$$

Now $e\mathbb{C}Ge$ is spanned by the elements ege, with $g \in G$. But $g = bx_ib'$, for $b, b' \in B$. So

$$ege = ebx_ib'e = ex_ie$$

since $eb = b'e = e$.

So $A = e\mathbb{C}Ge$ is spanned by the elements ex_ie, i.e

$$e\mathbb{C}Ge =< ex_ie \mid i \in [s] >$$

We show that these elements are linearly independent. Suppose

$$\sum_i \lambda_i e x_i e = 0 \Rightarrow \frac{1}{|B|^2}\sum_{b\in B}\sum_{b'\in B}\sum_i \lambda_i b x_i b' = 0$$

Now

$$bx_ib' = \bar{b}x_i\bar{b}' \iff {x_i}^{-1}\bar{b}^{-1}bx_i = \bar{b}'{b'}^{-1} \Rightarrow \bar{b}' \in (B \cap {x_i}^{-1}Bx_i)b'$$

Conversely for each $\bar{b}' \in (B \cap {x_i}^{-1}Bx_i)b'$ there's a unique $\bar{b}$ s.t

$${x_i}^{-1}\bar{b}^{-1}bx_i = \bar{b}'{b'}^{-1}$$

So $g = bx_ib'$ occurs $|B \cap {x_i}^{-1}Bx_i|-$times in the sum. The coefficients of this element in the given sum is

$$\frac{1}{|B|^2}|B \cap {x_i}^{-1}Bx_i|\lambda_i$$

which are equal to zero by the same argument as in the previous proposition. So $\lambda_i = 0$ and thus the $\{ex_1e, ..., ex_se\}$ are a basis for A. Hence

$$dimA = |B \backslash G/B|$$

$\square$

Remark 5.2.2. *The basis we construct in previous proposition, for the subalegra A of $\mathbb{C}G$, is independent of the choice of the representatives of the double cosets.*

5.2.2 An Isomorphic expression for the Hecke algebra

Proposition 5.2.3. *1. If $v \in V$, $a \in A$ then $va \in V$*

2. The map $\rho_a : V \longrightarrow V$ s.t $v \mapsto va$ lies in $H = End_{\mathbb{C}G}V$

3. The map $(a \longmapsto \rho_a)$ is a bijective map $A \longrightarrow H$

4. The bijection $a \longmapsto \rho_a$ is an anti-isomorphism. (i.e $\rho_{ab} = \rho_b\rho_a$)

Proof. 1. $V = \mathbb{C}Ge$, $A = e\mathbb{C}Ge$ so $VA = \mathbb{C}Ge \cdot e\mathbb{C}Ge = (\mathbb{C}Gee\mathbb{C}Ge) \subseteq \mathbb{C}Ge = V$

2. ρ_a is clearly linear. Now if $g \in G$ then $\rho_a(gv) = (gv)a = g(va) = g\rho_a(v)$

   (acting by g on left and a on right. Hence actions commuting.)

3. Suppose $\rho_a(v) = \rho_b(v)$ for every $v \in V$. So

$$\rho_a(e) = \rho_b(e) \Rightarrow ea = eb$$

   but $ea = a,\ eb = b$ since $a, b \in e\mathbb{C}Ge$.

   So $a = b$.

   Thus the map $a \longmapsto \rho_a$ is injective.

   Now suppose $\theta \in H$, $\theta : V \longrightarrow V$. Let $\theta(e) = a \in \mathbb{C}Ge$

   We have that

$$a \in e\mathbb{C}Ge$$

   since $e\theta(e) = \theta(e \cdot e) = \theta(e) \Rightarrow ea = a$. So $a = ea \in e\mathbb{C}Ge$

   We show that $\theta = \rho_a$. Let $v \in \mathbb{C}Ge$, so $ve = v$. Thus

$$\theta(v) = \theta(ve) = v\theta(e) = va$$

   Hence $\theta = \rho_a$.

4. $\rho_{ab}(v) = v(ab) = (va)b = \rho_b(va) = \rho_b\rho_a(v)$. So $\rho_{ab} = \rho_b\rho_a$.

$\square$

Corollary 5.2.1. $dimH = |B\backslash G/B| = |W|$

Proof. Consider the map $i : G \longrightarrow G$ with $i(g) = g^{-1}$ then

$$i(g_1g_2) = (g_1g_2)^{-1} = {g_2}^{-1}{g_1}^{-1} = i(g_2)i(g_1)$$

So i is an anti-automorphism of G.

Extend linear this map to $\mathbb{C}G$ to get a map $i : \mathbb{C}G \longrightarrow \mathbb{C}G$ with $i(\sum \lambda_i g_i) = \sum \lambda_i {g_i}^{-1}$.

Then i is an anti-automorphism of $\mathbb{C}G$.

Let $x \in \mathbb{C}G$ then $i(exe) = i(e)i(x)i(e)$. Now since $e = \frac{1}{|B|}\sum_{b\in B} b$ we get that

$$i(e) = \frac{1}{|B|}\sum_{b\in B} b^{-1} = e$$

So

$$i(exe) = ei(x)e \in e\mathbb{C}Ge$$

So i gives an anti-automorphism of $e\mathbb{C}Ge$, $i : A \longrightarrow A$.

Combining the above map i, with an anti-isomorphism $A \longrightarrow H$, such as in the previous proposition, we obtain an isomorphism $A \longrightarrow H$. So the Hecke algebra $H = H(G, B, 1_B)$ is isomorphic to the subalgebra $e\mathbb{C}Ge$ of $\mathbb{C}G$, □

5.2.3 A basis for the Hecke algebra H

Proposition 5.2.4. *Let $w \in W$, $s_i \in S$ with $\ell(s_iw) = \ell(w)+1$. Let $n_i, n \in N$ have $\pi(n) = w$, $\pi(n_i) = s_i$. Then*

$$en_ie \cdot ene = en_ine$$

Proof.

$$en_ie \cdot ene = en_iene = \frac{1}{|B|}\sum_{b\in B} en_ibne$$

But $n_iBn \subseteq Bn_inB$

So

$$n_ibn \in Bn_inB \Rightarrow n_ibn = b_1n_inb_2$$

for some $b_1, b_2 \in B$.

Now

$$en_ibne = eb_1n_inb_2e = en_ine$$

since $be = e = eb$ for each $b \in B$.

So

$$en_ie \cdot ene = \frac{1}{|B|}\sum_{b\in B} en_ine = \frac{1}{|B|}|B|en_ine = en_ine$$

□

Proposition 5.2.5. *Let $q_i = [B : n_iB{n_i}^{-1} \cap B]$. Then*

$$en_ie \cdot en_ie = \frac{1}{q_i}e + \frac{q_i - 1}{q_i}en_ie$$

Proof. By axiom 4 in the definition of the BN-group we have that $n_iBn_i \subseteq B \cup Bn_iB$. So

$$|n_iBn_i| = |n_iBn_i| + |n_iBn_i \cap Bn_iB|$$

Now $n_iBn_i = n_iBn_i{}^{-1}$ and $|n_iBn_i{}^{-1}| = |B|$

$$|n_iBn_i{}^{-1} \cap B| = \frac{1}{q_i} \cdot |B|$$

by the definition of q_i.

So

$$|n_iBn_i{}^{-1} \cap Bn_iB| = |n_iBn_i{}^{-1}| - \frac{1}{q_i}|B| = |B| - \frac{1}{q_i}|B| = (1 - \frac{1}{q_i})|B|$$

Thus

$$en_ie \cdot en_ie = en_ien_ie = \frac{1}{|B|}\sum_{b\in B} e(n_ibn_i)e$$

Then

$$\begin{aligned}\frac{1}{|B|}\sum_{b\in B} e(n_ibn_i)e &= \frac{1}{|B|}[(\#\text{ times})\ n_ibn_i \in B) \cdot e + (\#\text{ times } n_ibn_i \in Bn_iB) \cdot en_ie] \\ &= \frac{1}{|B|}(\frac{1}{q_i}|B| \cdot e + \frac{q_i - 1}{q_i}|B| \cdot en_ie) \\ &= \frac{1}{q_i}e + \frac{q_i - 1}{q_i}en_ie\end{aligned} \tag{5}$$

□

Example 5.2.1. *Let $G = GL_n(q)$ where $q = p^a$ for some prime p.*

Recall $B = UH = X_iU_iH$ and $n_iHn_i{}^{-1} = H$, $n_iU_in_i{}^{-1} = U_i$, $n_iX_in_i{}^{-1} = X_{-i}$. So

$$n_iBn_i{}^{-1} = n_iX_iU_iHn_i{}^{-1} = X_{-i}U_iH$$

Also

$$B \cap n_iBn_i{}^{-1} = X_iU_iH \cap X_{-i}U_iH = (X_i \cap X_{-i})U_iH = U_iH$$

since $X_i \cap X_{-i} = I_n$.

So

$$|B \cap n_iBn_i{}^{-1}| = |U_i||H| = \frac{1}{q}|U||H|$$

since in U the entry $(i,\ i+1)$ could be chosen in q different ways.

Then

$$|B \cap n_iBn_i{}^{-1}| = \frac{1}{q}|B|$$

and so

$$[B : B \cap n_iBn_i{}^{-1}] = q$$

So in $GL_n(q)$ we obtain that all $q_i = q$.

We shall consider only the case when all $q_i = q$. So

$$en_ie \cdot en_ie = \frac{1}{q}e + \frac{q-1}{q}en_ie$$

Let us choose, for each $w \in W$, an $n \in N$ with $\pi(n) = w$. Now we have that $ebne = ene$, since $b \in B \cap N$ because $\pi(bn) = w$ and hence ene is independent of the choice of n. So take such an element ene for each $w \in W$. These elements are linearly independent in $\mathbb{C}G$, from the fact that the double cosets of G with respect to B are precisely those of the form BnB as n ranges over a set containing one preimage under π of each element of W. Also, from the corollary 5.2.1 we have that the number of such elements is $|W| = dim\ e\mathbb{C}Ge$. So from the above we obtain that these elements form a basis for the $A = e\mathbb{C}Ge$.

So now we choose an isomorphism $\phi : e\mathbb{C}Ge \longrightarrow H$, with $q^{\ell(w)}ene \mapsto \phi(q^{\ell(w)}ene) := T_w$, where $\pi(n) = w$. Thus, through the isomorphism ϕ we get a basis T_w, $w \in W$ for the Hecke algebra H.

We now want to find out how these basis elements multiply together. The following Theorem gives the results we want.

Theorem 5.2.1. *1. Suppose $\ell(s_i w) = \ell(w) + 1$. Then $T_{s_i}T_w = T_{s_i w}$*

2. Suppose $\ell(s_i w) = \ell(w) - 1$. Then $T_{s_i}T_w = qT_{s_i w} + (q-1)T_w$

Proof. 1. The element $T_{s_i}T_w$ corresponds to

$$qen_ie \cdot q^{\ell(w)}ene = q^{\ell(w)+1}en_ie \cdot ene = q^{\ell(w)+1}en_ine$$

So

$$T_{s_i}T_w = T_{s_i w}$$

by the previous corresponds.

2. Let $w = s_iw^{'}$. Then $\ell(s_iw^{'}) = \ell(w^{'}) + 1$. Now $T_{s_i}T_{s_i}$ corresponds to

$$\begin{aligned} q \cdot en_ie \cdot q \cdot en_ie &= q^2(\frac{1}{q}e + \frac{q-1}{q} \cdot en_ie) \\ &= qe + (q-1) \cdot qen_ie \end{aligned} \tag{6}$$

Thus

$$T_{s_i}T_{s_i} = qT_1 + (q-1)T_{s_i}$$

but $T_w = T_{s_i}T_{w'}$ by (1) of the theorem.

So

$$\begin{aligned} T_{s_i}T_{s_i}T_{w'} &= qT_1T_{w'} + (q-1)T_{s_i}T_{w'} \\ T_{s_i}T_w &= qT_{w'} + (q-1)T_w \\ T_{s_i}T_w &= qT_{s_iw} + (q-1)T_w \end{aligned} \tag{7}$$

□

These are known as the Iwahori relations.

These results suffice to define the multiplication between any two basis elements (since the T_{s_i} generate H as an algebra), and thus the entire algebra structure is determined in terms of this basis.

Remark 5.2.3. *Also notice that, if $w = s_{i_1}...s_{i_k}$ reduced then $T_w = T_{s_{i_1}}...T_{s_{i_k}}$. So to calculate $T_wT_{w'}$ we express $w^{'}$ as a reduced word in s_j's and then we apply the theorem repeatedly.*

Corollary 5.2.2. *If $\ell(ww^{'}) = \ell(w) + \ell(w^{'})$ then $T_{ww'} = T_wT_{w'}$.*

So the product of any pair $T_wT_{w'}$ can be deduced from the Iwahori relations.

The Hecke algebra H is determined up to isomorphism by the Weyl group W and the parameter q.

5.2.4 A presentation for Hecke algebra

Corollary 5.2.3. *The Hecke algebra H is generated by elements $\{T_1, \cdots, T_{s_n}\}$, where $S = \{s_1, \cdots, s_n\}$. These generators satisfy the quadratic relations*

$$T_{s_i}{}^2 = qT_1 + (q+1)T_{s_i}, \;\; 1 \leq i \leq n$$

and the homogeneous relations

$$(T_{s_i}T_{s_j})^{m_{ij}} = (T_{s_j}T_{s_i})^{m_{ij}} \quad \text{if } m_{ij} = \text{ even}$$

$$(T_{s_i}T_{s_j})^{m_{ij}}T_{s_i} = (T_{s_j}T_{s_i})^{m_{ij}}T_{s_j} \quad \text{if } m_{ij} = \text{ odd}$$

where m_{ij} is the order of the elements s_is_j in W.

Theorem 5.2.2. *The generators and relations given in the above corollary, define a presentation of the Hecke algebra H.*

Since the proof of the above Theorem is similarly to the corresponding proof for the presentation of the Generic Hecke algebra, which we will introduce it in the next chapter, we will skip it for now. Notice that, in particular the quadratic relation $T_{s_i}{}^2 = qT_1 + (q - 1T_{s_i}), \;\; 1 \leq i \leq n$ can also be expressed in the form

$$(T_{s_i} - qT_1)(T_{s_i} + T_1) = 0, \;\; 1 \leq i \leq n$$

since

$$T_{s_i}{}^2 = qT_1 + (q_1)T_{s_i} \Longrightarrow T_{s_i}{}^2 - (q-1)T_{s_i} - qT_1 = 0 \Longrightarrow (T_{s_i} - qT_1)(T_{s_i} + T_1) = 0$$

Remark 5.2.4. *By the above observation we can see that in order to construct representations of the Hecke algebra H, it's enough to define a homomorphism such that for each generator T_{s_i} of H, we mapped it to a matrix A_i, with the following properties:*

- *The matrix A_i has eigenvalues either q or -1*
- *Also every such matrix A_i satisfy the homogeneous relations.*

Now, let demonstrate some simple examples of constructions of representations for the Hecke algebra H.

Example 5.2.2. *The simplest construction of representations for the Hecke algebra H, are the 1-dimensional representations of H. So*

(i) We define the homomorphism $ind : H \longrightarrow \mathbb{C}$, where $indT_{s_i} \mapsto q, \;\; 1 \leq i \leq n$. The representation that occur from this homomorphism we will call it the index representation.

(ii) Another 1-dimensional representation is the sign representation, that is defined by the homomorphism $sgn : H \longrightarrow \mathbb{C}$, such that $sgn\ T_{s_i} = -1, \;\; 1 \leq i \leq n$. Also, notice that, it can been shown that the sign representation corresponds to Steinberg representation.

5.3 The Semisimplicity of the Hecke Algebra

Now will show that the Hecke algebra H is semisimple algebra.

Lemma 5.3.1. *Let R be a semisimple $\mathbb{C}-$algebra with unit. Then any irreducible left R-module M is isomorphic to a left ideal L of R.*

Proof. Let $m \in M$, with $m \neq 0$.

Consider the map $\theta : R \longrightarrow M$ s.t $r \longmapsto rm$, for each $r \in R$.

θ is a homomorphism of left R-modules. Also $\theta \neq 0,\ \theta(1) = m \neq 0$.

So $\theta(R)$ is a non-zero submodule of M. Now since M is irreducible we get that $\theta(R) = M$.

Let $K = ker\theta$, K is a left R-submodule of R.

Since R is semisimple we have that $R = K \oplus L$, for some left R-submodule L. Thus

$$M = \theta(R) \cong \frac{R}{K} \cong L$$

□

Proposition 5.3.1. *If R is a semisimple $\mathbb{C}-$algebra with 1 and $e \in R$ is idempotent, then eRe is also a semisimple $\mathbb{C}-$algebra.*

Proof. Let N be submodule of eRe, i.e $eReN \subseteq N$. We must find a complement $N^{'} \subseteq eRe$ s.t $eReN^{'} \subseteq N^{'}$ and $eRe = N \oplus N^{'}$.

Let

$$M = RN =< rn \mid n \in N, r \in R >$$

Now since $Ne = N = eN$ we have that

$$M = RN = RNe \subseteq Re$$

By the fact that R is semisimple there is an R-module $M^{'}$ s.t $Re = M \oplus M^{'}$.

Claim 6. $N = eM$

Proof. Indeed,

$$eM = eRN = eReN \subseteq N$$

Conversely,

$$N = eN \subseteq eRN = eM$$

□

Let $N^{'} = eM^{'}$. Then $eRe = eM + eM^{'} = N + N^{'}$.

Now $N \subseteq M, N^{'} \subseteq M^{'} \Rightarrow N \cap N^{'} \subseteq M \cap M^{'} = 0$

So $eRe = N \oplus N^{'}$. Furthermore,

$$eReN^{'} = eReeM^{'} \subseteq eM^{'} = N^{'}$$

i.e $N^{'}$ is a complimentary left ideal. □

Corollary 5.3.1. *Take $R = \mathbb{C}G,\ e = \frac{1}{|B|}\sum_{b \in B} b$ and $eRe = H$. So H is semisimple.*

Lemma 5.3.2. *Let R be a semisimple $\mathbb{C}-$algebra with 1. Then any left ideal of R has the form Re for some idempotent e.*

Proof. Let L be a non-zero left ideal of R, and let $L^{'}$ be a complimentary ideal, thus

$$R = L \oplus L^{'}$$

So we have

$$1 = e + e^{'}$$

where $e \in L, e^{'} \in L^{'}$ Then

$$1 = e + e^{'} \Rightarrow e = e^2 + ee^{'} \Rightarrow e = e^2,\ ee^{'} = 0$$

by the uniqueness of direct sum decomposition.

Now $e \in L$ so $Re \subseteq L$

$$\ell \in L \Rightarrow \ell = \ell e + \ell e^{'}$$

where $\ell e \in L,\ \ell e^{'} \in L^{'}$. Now

$$\ell - \ell e = \ell e^{'} \in L \cap L^{'} = 0$$

So

$$\ell e = \ell,\ \ell e^{'} = 0 \Rightarrow L = Le \subseteq Re \subseteq L \Rightarrow L = Re$$

□

Definition 5.3.1 (Primitive element)**.** *An idempotent* $e \in R$ *is called primitive if* e *cannot be expressed as* $e = e_1 + e_2$*, where* $e_1, e_2 \neq 0,\ {e_1}^2 = e_1, {e_2}^2 = e_2,\ e_1e_2 = 0 = e_2e_1$

Lemma 5.3.3. *Re is irreducible $\Longleftrightarrow$ e is primitive.*

Proof. ($\Leftarrow$) Suppose Re is not irreducible. Then

$$Re = M_1 \oplus M_2$$

where $M_1, M_2 \neq 0$ left R-submodules.

So $e = e_1 + e_2$ for $e_i \in M_i$ also $e_i \in Re$, thus $e_ie = e_i$

Now

$$e_1 = e_1e = {e_1}^2 + e_1e_2 \Rightarrow {e_1}^2 = e_1,\ e_1e_2 = 0$$

and

$$e_2 = e_2e = e_2e_1 + {e_2}^2 \Rightarrow {e_2}^2 = e_2,\ e_2e_1 = 0$$

Finally, $e_1 \neq 0$, otherwise $e = e_2,\ Re \subseteq M_2,\ M_1 = 0$

Similarly $e_2 \neq 0$. But e is primitive and so we have a contradiction.

($\Rightarrow$) Suppose e isn't primitive. Then

$$e = e_1 + e_2$$

with ${e_i}^2 = e_i,\ e_ie_j = 0\ (i \neq j),\ e_i \neq 0$.

So

$$Re = Re_1 + Re_2$$

We need to show that $Re_1 \cap Re_2 = 0$:

Indeed, we take $x \in Re_1 \cap Re_2 \Rightarrow x = xe_1 = xe_2 = xe_1e_2 = 0$

So

$$Re = Re_1 \oplus Re_2$$

but Re is irreducible and thus we obtain a contradiction. □

5.4 The Correspondence Theorem

Theorem 5.4.1. *Let $e \in \mathbb{C}G$ be idempotent and $V = \mathbb{C}Ge$. Then there is a 1-1 correspondence between irreducible $\mathbb{C}G-$modules occurring as components of V (up to isomorphism) and irreducible modules for $e\mathbb{C}Ge$ (up to isomorphism). The dimension of an irreducible module for $e\mathbb{C}Ge$ is equal to the multiplicity of the corresponding $\mathbb{C}G-$module as a component of V.*

Proof.

- **STEP 1:**

Let M be an irreducible $\mathbb{C}G$-module.

Claim 7. *M appears as a component of $V = \mathbb{C}Ge \Longleftrightarrow Hom_{\mathbb{C}G}(\mathbb{C}Ge, M) \neq 0$*

Proof. First, suppose that M is a component of V. Then $V = M \oplus M^{'}$, thus $v = m + m^{'}$.

The projection $p_M : V \longrightarrow M$ where $v \mapsto m$ is a non-zero $\mathbb{C}G-$ homomorphism.

Conversely, if $\theta \in Hom_{\mathbb{C}G}(\mathbb{C}Ge, M)$ with $\theta \neq 0$. Then

$$V = K \oplus L$$

where $K = ker\theta$. So

$$L \cong \frac{V}{K} \cong im\theta = M$$

since $\theta \neq 0$ and M is irreducible.

Hence we get that V is a component of M. □

- **STEP 2 :**

Claim 8. *There is a bijection $Hom_{\mathbb{C}G}(\mathbb{C}Ge, M) \longleftrightarrow eM$*

Proof. Indeed, we show that

$$\Phi : Hom_{\mathbb{C}G}(\mathbb{C}Ge, M) \longrightarrow eM$$

$$\theta \longmapsto \theta(e)$$

is an isomomorphism.

First, by taking an element $\theta \in Hom_{\mathbb{C}G}(\mathbb{C}Ge, M)$, then $\theta(e) = \theta(ee) = e\theta(e) \in eM$. So the map Φ is well-defined.

Now, suppose that $\theta, \theta^{'} \in Hom_{\mathbb{C}G}(\mathbb{C}Ge, M)$ s.t $\theta(e) = \theta^{'}(e)$. Then

$$\theta(re) = r\theta(e) = r\theta^{'}(e) = \theta^{'}(re)$$

for every $r \in \mathbb{C}G$.

So $\theta = \theta^{'}$ and hence Φ is 1-1.

Now, let $m \in M$. We define $\theta(r) := rm$, for $r \in \mathbb{C}Ge$. Then

$$\theta \in Hom_{\mathbb{C}G}(\mathbb{C}Ge, M) \quad \text{and} \quad \theta(e) = em = m$$

So $\Phi(\theta) = m$

Thus we have proved that the map $\theta \mapsto \theta(e)$ is bijective. □

Now, from the two previous claims we obtain that M appears as a component of $V = \mathbb{C}Ge$ if and only if $eM \neq 0$

- **STEP 3 :**

Consider the map $M \longrightarrow eM$, where M is irreducible $\mathbb{C}G-$module.
Now eM is an $e\mathbb{C}Ge-$module since $e(\mathbb{C}Gee)M \subseteq eM$.

We will show that eM is in fact irreducible:

Take $m \in eM$, $m \neq 0$. Then
$$e\mathbb{C}Gem = e\mathbb{C}Gm = eM$$
since M is irreducible and we get that $\mathbb{C}Gm = M$

So eM is an irreducible $e\mathbb{C}Ge-$module.

- **STEP 4 :**

Hence we could obtain that there exists a correspondence

$$\{\text{irreducible } \mathbb{C}G - \text{modules as components of the V }\} \longleftrightarrow \{\text{irreducible } e\mathbb{C}Ge - \text{modules}\}$$
$$M \longleftrightarrow eM$$

Note that this correspondence could be written also in the language of the representations:

Particularly, there is 1-1 correspondence:

$$\{\text{irreducible representations of G appearing in V} \longleftrightarrow \{\text{irreducible representations of } e\mathbb{C}Ge\}$$

We already know that if M appears in V then $eM \neq 0$ and eM is an irreducible $e\mathbb{C}Ge-$module. So it remains to show that this map is bijective.

- First, we show that the map $M \longmapsto eM$ is surjective.

  Let N be an irreducible $e\mathbb{C}Ge-$module. Then by previous lemma and proposition we have that
  $$N \cong e\mathbb{C}Ge \cdot n$$
  where $n \in e\mathbb{C}Ge$ and $n^2 = n$.

  Now since $n \in e\mathbb{C}Ge$ and $en = n$ we obtain that
  $$N \cong e\mathbb{C}Gn$$
  So it remains to show that $\mathbb{C}Gn$ is an irreducible $\mathbb{C}G-$module.
  Suppose not. Then by previous proposition n isn't primitive. So
  $$n = n_1 + n_2$$

where $n_1, n_2 \neq 0,\ {n_1}^2 = n_1,\ {n_2}^2 = n_2,\ n_1 n_2 = 0 = n_2 n_1$.

Then

$$n_1 = nn_1 = nn_1 n = enn_1 ne \in e\mathbb{C}Ge$$

Similarly $n_2 \in e\mathbb{C}Ge$

So

$$n_1, n_2 \in e\mathbb{C}Ge$$

So n isn't primitive in $e\mathbb{C}Ge$, which is contradiction due to

$$\begin{aligned} N \text{ is irreducible } e\mathbb{C}Ge - \text{module} &\Rightarrow (e\mathbb{C}Ge)n \text{ irreducible } e\mathbb{C}Ge - \text{module} \\ &\Rightarrow n \text{ is primitive} \end{aligned} \tag{8}$$

So $\mathbb{C}Gn$ is irreducible and $e\mathbb{C}Gn \cong N$. So the map is surjective.

– Now will show that the map is also injective.

Suppose $M^{'}$ is an irreducible $\mathbb{C}G-$module with $eM^{'} \cong N$ as left $e\mathbb{C}Ge-$modules.

We will show that $M^{'} \cong \mathbb{C}Gn$.

Let $n \in N$ so $n \in nN$ and thus $nN \neq 0$. Hence using the isomorphism $eM^{'} \cong N$ we get that $neM^{'} \neq 0$. But $ne = n$ so $nM^{'} \neq 0$. Thus there exist $m^{'} \in M^{'}$ with $nm^{'} \neq 0$.

Consider the map

$$\mathbb{C}Gn \longrightarrow M^{'}$$
$$x \longmapsto xm^{'}$$

This is a homomorphism of left $\mathbb{C}G-$modules (clear from the definition). Also it's non-zero by our choice of $m^{'}$. Both sides are irreducible. So we must have $\mathbb{C}Gn \cong M^{'}$.

So the map is bijective.

- **<u>STEP 5 :</u>**

Consider $dim\ eM = dim\ Hom_{\mathbb{C}G}(\mathbb{C}Ge, M)$ and $V = \mathbb{C}Ge$.

By Maschke's theorem $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$ where V_i are irreducible $\mathbb{C}G-$modules.

Let $\theta \in Hom_{\mathbb{C}G}(V, M)$. Then θ induces, by restriction, $\theta_i \in Hom(V_i, M)$.

If $V_i \ncong M$ then $\theta_i = 0$.
If $V_i \cong M$ then $Hom(V_i, M) \cong \mathbb{C}$ by Schur's Lemma.

So

$$dim\ Hom_{\mathbb{C}G}(\mathbb{C}Ge, M) = \#\text{ of } V_i's \text{ isomorphic to M}$$

i.e the multiplicity with which M appears as a component of V.

□

Corollary 5.4.1. *Take* $e = \frac{1}{|B|}\sum_{b\in B} b,\ e\mathbb{C}Ge \cong H$.

Then exists 1-1 correspondence that :

$$\{\textit{irreducible H-modules}\} \longleftrightarrow \{\textit{irreducible } \mathbb{C}G - \textit{modules appearing as components of } V = \mathbb{C}Ge\}$$

So in the language of the representation theory we get that:

$$\{\textit{irreducible representations of } G \textit{ appearing in } V \longleftrightarrow \{\textit{irreducible representations of } H\}$$

and

$$\textit{the dimension of irreps of } H \leftrightarrow \textit{ multiplicity of irreps of } G \textit{ appearing in } V$$

where with terminology irreps we mean the irreducible representations.

5.5 The Examples of $GL_2(q)$ and $GL_3(q)$

1. Let $G = GL_2(q)$ and B be the subgroup of triangular matrices. Then $W \cong S_2$, $|W| = 2$, and $dim\ H = 2$.
   By a combinatorial argument, in particular, for any matrix in G, we can take any vector, except the zero element, as the first row, and the second row can't be a multiple of the first. Hence
   $$|G| = (q^2-1)(q^2-q) = q(q-1)(q^2-1) = q(q-1)^2(q+1)$$
   By similarly argument, for any element in the subgroup B, we get that
   $$|B| = (q-1)(q^2-q) = q(q-1)^2$$
   since this time our choices for the first vector is further restricted, due to any element in B has the form $\begin{pmatrix} \star & \star \\ 0 & \star \end{pmatrix}$.
   So $dim\ V = [G : B] = q + 1$, where $V = \mathbb{C}Ge$, as defined above. Then $V = V_1 \oplus V_2$, where V_1 gives rise to an irreducible representation of order 1, and V_2 to one of order q called the Steinberg representation.
   H has dimension 2 so is generated by T_1, T_s, where
   $${T_s}^2 = qT_1 + (q-1)T_s$$
   since $s^2 = 1$, i.e $\ell(s^2) < \ell(s)$.
   So H has two irreducible representations by the above Correspondence Theorem, both of degree 1. Under these representations
   $$T_1 \mapsto 1,\ T_s \mapsto b$$
   , where $b^2 = q + (q-1)b \longrightarrow b^2 - (q-1)b - q = 0 \Longrightarrow (b-q)(b+1) = 0$.
   So the two irreducible representations of H are
   $$\rho : T_1 \mapsto 1, T_s \mapsto -1, \quad \text{and} \quad \sigma : T_1 \mapsto 1, T_s \mapsto q$$

2. Let $G = GL_3(q)$. Similarly to above we have
   $$|G| = (q^3-1)(q^3-q)(q^3-q^2) = q^3(q-1)(q^2-1)(q^3-1)$$
   and
   $$|B| = (q-1)(q^2-q)(q^3-q^2) = q^3(q-1)^3$$
   So $dim\ V = [G : B] = (q+1)(q^2+q+1) = q^3+2q^2+2q+1$
   Also $W \cong S_3$, and so $dim\ H = 6$
   In fact H has irreducible representations of degrees 1,1,2 and $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$, where V_i irreps and $V_2 \cong V_3$, giving representations of dimension 1, (q^2+q), (q^2+q), q^3 respectively. The q^3 representation called the Steinberg representation.

5.6 The Reflection representation of the Hecke algebra $H(G, B, 1_B)$

Let G be a finite group with BN-pair, W be the Weyl group of G, and $S = \{s_1, \cdots, s_n\}$ the distinguished generators of W. Then let $H = H(G, B, 1_B)$ be the associated Hecke algebra of G with respect to the subgroup B. As we have already seen, the Hecke algebra H has a presentation with generators $\{T_{s_1}, \cdots, T_{s_n}\}$ and satisfying the homogeneous and quadratic relations. i.e
$${T_{s_i}}^2 = q_iT_1 + (q_i-1)T_{s_i},\ \ 1 \le i \le n$$
and
$$\begin{aligned} (T_{s_i}T_{s_j})^{m_{ij}} &= (T_{s_j}T_{s_i})^{m_{ij}} && \text{if } m_{ij} = \text{ even} \\ (T_{s_i}T_{s_j})^{m_{ij}}T_{s_i} &= (T_{s_j}T_{s_i})^{m_{ij}}T_{s_j} && \text{if } m_{ij} = \text{ odd} \end{aligned}$$
where m_{ij} is the order of the elements s_is_j in W.

We have seen that the reflection representation of the Coxeter group W is irreducible if and only if the Coxeter system (W,S) is irreducible. So we can make the assumption that (W,S) is irreducible, and thus the Coxeter graph of (W,S) is tree.

Let $V = \bigoplus_{1\le i\le n} \mathbb{C}v_i$ be an n-dimensional $\mathbb{C}-$vector space, where $|S| = n$. Also, let $\{c_{ij}\}_{1\le i,j\le n} \in \mathbb{C}$ such that:

- $c_{ii} = q_i + 1$
- $c_{ij} = c_{ji} = 0 \quad \text{if } m_{ij} = 2$
- $c_{ij}c_{ji} = q_i + q_j + 2\sqrt{q_i q_j} cos\frac{2\pi}{m_{ij}} \quad \text{if } m_{ij} > 2$

Theorem 5.6.1. *Let (W,S) be an irreducible Coxeter system associated with a finite group G with BN-pair, and let be the corresponding Hecke algebra $H = H(G, B, 1_B)$. There exists a representation $\rho : H \longrightarrow End_{\mathbb{C}}V$ such that*

$$\rho(T_{s_i})v = q_i v - \frac{(q_i+1)B(v_i, v)}{B(v_i, v_i)} v_i, \quad \textit{for all} \quad v \in V, \ s_i \in S$$

The representation ρ is irreducible, and the bilinear form B is nondegenerate on V.

For the proof we will need the following lemmas.

Lemma 5.6.1. *For each $1 \le i, j \le n$ and the above complex numbers c_{ij}, there exists a symmetric bilinear form B on V, such that $B(v_i, v_i) \neq 0$ for $i = 1, \cdots, n$ and it's satisfying the following property:*

$$-c_{ij} = \frac{(q_i+1)B(v_i, v_j)}{B(v_i, v_i)}, \quad \textit{for} \ 1 \le i, j \le n$$

Lemma 5.6.2. *For $i \neq j$, we take the subspace of V generated by a elements v_i and v_j, i.e the subspace $< v_i, v_j >$, then the restriction $B|_{<v_i,v_j>}$ is nondegenerate.*

Proof of the Theorem . First we have to show that the map ρ, as defined in the above theorem, preserves the defining relations of the Hecke algebra H, i.e that on the generators T_{s_i} the map ρ preserves the quadratic and homogeneous relations, for each $i \in \{1, \cdots, n\}$

-So for the quadratic relation:

For each i, with $1 \le i \le n$ since $B(v_i, v_i) \neq 0$ we obtain $V =< v_i > \oplus < v_i >^{\perp}$, where $< v_i >^{\perp} = \{v \in V : B(v, v_i) = 0\}$. Then by simple calculations of the act of the map ρ on the subspaces $< v_i >, < v_i >^{\perp}$, we get that ρ acts as scalar multiplication by q_i on $< v_i >^{\perp}$ and by -1 on $< v_i >$. So it follows that

$$\rho(T_{s_i})^2 = q_i T_1 + (q_i - 1)\rho(T_{s_i})$$

and thus the quadratic relations hold.

-For the homogeneous relation:

By the above lemma we have that the restriction $B|_{<v_i,v_j>}$, with $i \neq j$, is nondegenerate. So we can decompose the space V as

$$V =< v_i, v_j > \oplus < v_i, v_j >^{\perp}$$

Now as in the previous case of the quadratic relation, by simple calculations we get that, the $\rho(T_{s_i})$ and $\rho(T_{s_j})$ act as scalar multiplications by q_i and q_j on the subspace $< v_i, v_j >^{\perp}$, respectively. So it occurs that the homogeneous relations for $\rho(T_{s_i})$ and $\rho(T_{s_j})$ hold on $< v_i, v_j >^{\perp}$. Now on $< v_i, v_j >$ the $\rho(T_{s_i})$ and $\rho(T_{s_j})$ acts by the matrices

$$R_{s_i} = \begin{pmatrix} -1 & c_{ij} \\ 0 & q_i \end{pmatrix} \quad \text{and} \quad R_{s_j} = \begin{pmatrix} q_j & 0 \\ c_{ji} & -1 \end{pmatrix}$$

respectively, by the lemma 5.6.1. Thus the homogeneous relations also hold on $< v_i, v_j >$, since by multiplying from both sides the above matrices $R_{s_i}.R_{s_j}$, we obtain that the products $R_{s_i}R_{s_j}$ and $R_{s_j}R_{s_i}$ have the same characteristic polynomial of degree 2, and hence we can diagonaziable them. So the map ρ can be extended to a representation of H, by the presentation of the Hecke algebra.

Now we have to show that ρ is irreducible representation and B is nondegenerated bilinear form. We apply induction on n. So we assume that the result holds for irreducible Coxeter systems of rank less than n. Now, choose a subset $J \subseteq S$, such that $|J| = n - 1$ and let $V_J =< v_j : s_j \in J >$. By the induction hypothesis we have that B_{V_J} is nondegenerate, and so we can decompose the space V into $V = V_J \oplus {V_J}^{\perp}$ and $dim_{\mathbb{C}} {V_J}^{\perp} = 1$. Also let H_J be the subalgebra of H generated by the set $\{T_{s_j} : s_j \in J\}$. By induction, we get that the restriction of the representation ρ to the H_J, $\rho|_{H_J}$, is irreducible on V_J, while the subspace ${V_J}^{\perp}$ affords the index representation, since from the first part of the proof, we have seen that the map ρ acts on ${V_J}^{\perp}$ as a scalar multiplication by q_j, where $s_j \in J$. So $V_J, {V_J}^{\perp}$ as H_J-modules are simple and also are non-isomorphic, for every $n > 1$, since for $n > 3$ they have different dimensions, and for $n = 2$ the V_J affords the sign representation, again by first part of the proof.

Now in order to finally get that ρ is irreducible, we assume the opposite for contradiction, i.e by consider the V as H-module, we assume that V is not simple. So since H is semisimple there exists non-trivial H-submodules V_1, V_2, such that $V = V_1 \oplus V_2$. Then both V_1, V_2 are also H_J-modules and so they coincide with $V_J, {V_J}^{\perp}$, by the above. But by the definition of J and the representation ρ, we get that the V_J is not a H-submodule of V, and we have a contradiction. Thus the representation ρ is irreducible.

Furthermore, we conclude that B is non-degenerate on V. Since if we assume that this is not true, so $V^{\perp} \neq 0$, we get that ρ acts on $V^{\perp}$ as a scalar multiplication by q_i, for each $1 \leq i \leq n$, and thus the subspace $< v >$, for $v \neq 0$ with $v \in V^{\perp}$, is a proper H-submodule of V. The last statement contradict the fact that ρ is irreducible. So the bilinear form B is non-degenerate on V. $\square$

6 The Generic Hecke Algebra

The Hecke algebra introduced so far is a $\mathbb{C}-$algebra with basis T_w, $w \in W$, and with multiplication defined by the rules :

$$T_s T_w = \begin{cases} T_{sw} & \text{, if } \ell(sw) = \ell(w) + 1 \\ qT_{sw} + (q-1)T_w & \text{, if } \ell(sw) = \ell(w) - 1 \end{cases}$$

where $s \in S$, $w \in W$, and q is a prime power, e.g p^e.

In order to obtain the generic Hecke algebra, we will begin with a general construction of associative algebra over a commutative ring A, with 1. In particular, we will see that such an algebra will be a free A-module, with the elements of the basis be parameterized by the elements of W, and with multiplication law which in a sense reflects the multiplication in W. Also the construction which we will illustrate below it will depend on some parameters $a_s, b_s \in A$, for $s \in S$, with the property $a_s = a_t$ and $b_s = b_t$ whenever s and t are conjugate in W. The starting point of this construction it will be a free A-module $\mathcal{E}$ on W, such that the basis elements be the T_w, with $w \in W$.

6.1 The Construction of Generic Algebras

Theorem 6.1.1. *Let A be a commutative ring with 1 and elements $a_s, b_s \in A$, $s \in S$, such that $a_s = a_t$ and $b_s = b_t$ whenever s and t are conjugate in W, where (W, S) is a Coxeter system. Then there exists a unique structure of associative A-algebra on the free A-module $\mathcal{E}$, with T_1 acting as the identity, such that the following conditions hold for all $s \in S$, $w \in W$:*

$$T_s T_w = \begin{cases} T_{sw} & \text{, if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & \text{, if } \ell(sw) < \ell(w) \end{cases}$$

Proof. • **Existence:** The idea behind the proof of the existence of an algebra structure as described in the theorem it is not to introduced directly into the A-module $\mathcal{E}$, but instead we exploit the existing ring structure in *End* $\mathcal{E}$, the algebra of all A-module endomorphisms of $\mathcal{E}$. Now if $\mathcal{E}$ has an algebra structure, the left multiplication operators corresponding to the elements of $\mathcal{E}$ will generate an isomorphic copy of this algebra inside *End* $\mathcal{E}$. Then by finding the appropriate subalgebra of *End* $\mathcal{E}$ and via an isomorphism we will transfer the algebra structure of the subalgebra to $\mathcal{E}$.

Now we define the endomorphisms (left and right multiplication operators) $\lambda_s = \lambda(T_s)$ and $\rho_s = \rho(T_s)$ of A, $\forall s \in S$, by

$$\lambda(T_s) T_w = \begin{cases} T_{sw} & \text{, if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & \text{, if } \ell(sw) < \ell(w) \end{cases}$$

and

$$\rho(T_s) T_w = \begin{cases} T_{ws} & \text{, if } \ell(ws) > \ell(w) \\ a_s T_w + b_s T_{ws} & \text{, if } \ell(ws) < \ell(w) \end{cases}$$

in order to conform with the multiplication rule of the elements T_s, T_w, as described above.

Also let $\mathcal{L}$ be the algebra of endomorphisms of A generated by $\lambda(T_s)$, for all $s \in S$ and $\mathcal{R}$ be the algebra of endomorphisms of A generated by $\rho(T_s)$, for all $s \in S$.

Lemma 6.1.1. *Every operator λ_s commutes with every operator ρ_t, $\forall s, t \in S$*

To prove the above lemma, we will need first the below lemma from the theory of Coxeter groups.

Lemma 6.1.2. *Let $w \in W$ and $s, t \in S$. If $\ell(swt) = \ell(w)$ and $\ell(sw) = \ell(wt)$, then $sw = wt$ (or equivalently, $swt = w$).*

Proof. Let $w = s_1 ... s_r$ be a reduced expression of w, so $\ell(w) = r$. Then we have two cases :

1. $\ell(sw) > \ell(w)$: Then $\ell(sw) = r + 1$ and thus the expression $s_0 s_1 \cdots s_r$, where $s_0 = s$, is a reduced expression for the element sw. Now, also, $\ell(w) = \ell((sw)t) < \ell(sw)$, so we can apply the Exchange condition to the pair sw, t and so we have that there exists $0 \leq i \leq r$ such that
$$swt = s_0 \cdots \hat{s_i} \cdots s_r$$
But we cannot have $1 \leq i \leq r$, because then
$$wt = ss_0 s_1 \cdots \hat{s_i} \cdots s_r = s_1 \cdots \hat{s_i} \cdots s_r$$
which contradicts the fact that $\ell(wt) = \ell(sw) = \ell(w) + 1$. So $i = 0$. Then
$$swt = s_1 \cdots s_r = w \Longleftrightarrow sw = wt$$
2. $\ell(sw) < \ell(w)$: Then $\ell(sw) < \ell(w) = \ell(s(sw))$. Now, by observe that the hypothesis of the lemma is satisfied by the element sw in place of w, we could apply the result of the case (1) to sw. So, we obtain that, $s(sw)t = sw$, i.e $wt = sw$.

□

Proof. Now we will prove the lemma 6.1.1 :

Let $w \in W$ and compare the action of the operators $\lambda_s \rho_t$ and $\rho_t \lambda_s$ on T_w. This is equivalent an associativity condition, i.e $(T_s T_w) T_t = T_s (T_w T_t)$. We know that multiplication by s or t changes the length by 1, so there are six possibilities for the relative lengths of sw, wt, swt, w, since for example, it is impossible for all of these to have distinct lengths. We distinguish the following cases:

1. $\ell(w) < \ell(wt) = \ell(sw) < \ell(swt)$:
   Then by the description of the operators above we have that $\lambda_s \rho_t (T_w) = T_{swt} = \rho_t \lambda_s (T_w)$
2. $\ell(swt) < \ell(wt) = \ell(sw) < \ell(w)$:
   By direct calculation $\lambda_s \rho_t (T_w) = \lambda_s (a_t T_w + b_t T_{wt}) = a_t \lambda_s (T_w) + b_t \lambda_s (T_{wt}) = a_t (a_s T_w + b_s T_{sw}) + b_t (a_s T_{wt} + b_s T_{swt}) = a_t a_s T_w + a_t b_s T_{sw} + b_t a_s T_{wt} + b_t b_s T_{swt}$. By similar calculations of $\rho_t \lambda_s (T_w)$ yields the same result.
3. $\ell(wt) = \ell(sw) < \ell(swt) = \ell(w)$:
   In this case the hypotheses of the above lemma are satisfied, so we get $sw = wt$, which says that s, t conjugate in W and thus $a_s = a_t$ and $b_s = b_t$. Now by direct calculation $\lambda_s \rho_t (T_w) = a_t a_s T_w + a_t b_s T_{sw} + b_t T_{swt}$ and $\rho_t \lambda_s (T_w) = a_s a_t T_w + a_s b_t T_{wt} + b_s T_{swt}$. From the fact that $a_s = a_t$ and $b_s = b_t$ we get equality between the above relations of the operators.
4. $\ell(wt) < \ell(w) = \ell(swt) < \ell(sw)$:
   Here it's obvious since, $\lambda_s \rho_t (T_w) = a_t T_{sw} + b_t T_{swt} = \rho_t \lambda_s (T_w)$
5. $\ell(sw) < \ell(w) = \ell(swt) < \ell(wt)$:
   Similarly as in the previous case, we get $\lambda_s \rho_t (T_w) = a_s T_{wt} + b_s T_{swt} = \rho_t \lambda_s (T_w)$
6. $\ell(w) = \ell(swt) < \ell(wt) = \ell(sw)$:
   Just as in the previous case we have $\lambda_s \rho_t (T_w) = a_s T_{wt} + b_s T_{swt}$. But $\rho_t \lambda_s (T_w) = a_t T_{sw} + b_t T_{swt}$. Again the above lemma satisfied and so we get $sw = wt$, which says that $a_s = a_t$ and $b_s = b_t$.

   So we have proved that in every case the equality holds, and thus $\lambda_s \rho_t = \rho_t \lambda_s$, for every $s, t \in S$.

□

Now we prove that there exists a isomorphism of A-modules, between the algebras $\mathcal{L}$ and $\mathcal{E}$. Indeed, we define a map $\phi : \mathcal{L} \longrightarrow \mathcal{E}$ by $\lambda \mapsto \lambda_1 = \lambda(T_1)$, thus sending 1 to T_1 and λ_s to T_s, for all $s \in S$. It is obvious that ϕ is an A-module map. Moreover, is surjective and injective.

- ϕ is surjective: Let a basis element T_w of the free A-module $\mathcal{E}$ and let $w = s_1 \cdots s_r$ reduced expression of w. Then $T_w = T_{s_1} \cdots T_{s_r}$. We have $\phi(\lambda_{s_1} \cdots \lambda_{s_r}) = (\lambda_{s_1} \cdots \lambda_{s_r})(T_1) = T_{s_1} \cdots T_{s_r} = T_w$
- ϕ is injective: Suppose $\phi(\lambda) = 0$, i.e $\lambda(T_1) = 0$. Let $a \in \mathcal{E}$ we will show that $\lambda(a) = 0$. First we define the surjective A-module map $\psi : \mathcal{R} \longrightarrow \mathcal{E}$ with $\psi(\rho) = \rho(T_1)$, thus sending 1 to T_1 and ρ_t to T_t for all $t \in S$. So since ψ is surjective $a = \psi(\rho) = \rho(T_1)$, for some $\rho \in \mathcal{R}$.Thus, $\lambda(a) = \lambda(\rho(T_1)) = \rho(\lambda(T_1)) = \rho(0) = 0$. So ϕ is injective.

So ϕ is bijective.

Now since ϕ is an isomorphism of A-modules, it follows that $\mathcal{L}$ has a free A-basis consisting of all $\lambda_w := \lambda_{s_1} \cdots \lambda_{s_r}$, where $w = s_1 \cdots s_r$ is reduced and the endomorphism λ_w is independent of the choice of reduced expression. (Here λ_1 is the identity element on $\mathcal{E}$.) Furthermore, the algebra structure on $\mathcal{L}$ can be transferred to $\mathcal{E}$ and make it into an algebra via the map ϕ.

It remains only to show that this structure satisfies the relations

$$T_s T_w = \begin{cases} T_{sw} & ,\text{if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & ,\text{if } \ell(sw) < \ell(w) \end{cases}$$

In this direction we have the following remark.

Remark 6.1.1. *The relations*

$$T_s T_w = \begin{cases} T_{sw} & ,\textit{if } \ell(sw) > \ell(w) \\ a_s T_w + b_s T_{sw} & ,\textit{if } \ell(sw) < \ell(w) \end{cases}$$

are equivalent to the relations

$$T_s T_w = T_{sw} \textit{ if } \ell(sw) > \ell(w) \tag{9}$$

$$T_s{}^2 = a_s T_s + b_s T_1 \tag{10}$$

Indeed, since $s^2 = 1$ we have that $\ell(s^2) = 0 = \ell(s) - 1$ so $T_s{}^2 = T_s T_s = a_s T_s + b_s T_1$. Conversely, if we have shown that the algebra $\mathcal{E}$ satisfying the relations (9) and (10) we will show that in case $\ell(sw) < \ell(w)$ we get the relation $T_s T_w = a_s T_w + b_s T_{sw}$. Note that, when $\ell(w) = 1$, we must have $w = s$, so we have that $T_s T_s = a_s T_s + b_s T_{ss} = a_s T_s + b_s T_1 = T_s{}^2$. Thus in that case we have the result. In general, we have $\ell(s(sw)) > \ell(sw)$ so by the relation (9) we get $T_s T_{sw} = T_{ssw} = T_w$. Then by the relation (10) we obtain that $T_s T_w = T_s{}^2 T_{sw} = (a_s T_s + b_s T_1) T_{sw} = a_s T_s T_{sw} + b_s T_{sw} = a_s T_w + b_s T_{sw}$, as required.

So by the above remark we just have to show that the algebra structure we have construct on $\mathcal{E}$ satisfies the relations (9) and (10). Indeed, first let $\ell(sw) > \ell(w)$. We have to verify that $\lambda_s \lambda_w = \lambda_{sw}$. So by taking a reduced expression $s_1 \cdots s_r$ for the element w, we get a reduced expression $s s_1 \cdots s_r$ for the element sw. Now $\lambda_s \lambda_w = \lambda_s \lambda_{s_1} \cdots \lambda_{s_r}$ agrees with the definition of λ_{sw}. Now it remains to show that $\lambda_s{}^2 = a_s \lambda_s + b_s \lambda_1$ in order to show that the algebra satisfies the relation (10). Moreover it is enough to check sides at a basis element T_w of $\mathcal{E}$. In case $\ell(sw) > \ell(w)$, we have :

$$\lambda_s{}^2(T_w) = \lambda_s(T_{sw}) = a_s T_{sw} + b_s T_w = (a_s \lambda_s + b_s \lambda_1)(T_w)$$

In case $\ell(sw) < \ell(w)$ we get :

$$\lambda_s{}^2(T_w) = \lambda_s(a_s T_w + b_s T_{sw}) = a_s \lambda_s(T_w) + b_s T_s T_{sw} = (a_s \lambda_s + b_s \lambda_1)(T_w)$$

Thus, from all the above we have prove that there exists an algebra structure on the free A-module $\mathcal{E}$ given by multiplication

$$T_sT_w = \begin{cases} T_{sw} & \text{, if } \ell(sw) > \ell(w) \\ a_sT_w + b_sT_{sw} & \text{, if } \ell(sw) < \ell(w) \end{cases}$$

- **Uniqueness:** Let $w = s_1 \cdots s_r$ be a reduced expression of the element w, by iteration of the multiplication rule on $\mathcal{E}$ we obtain that $T_w = T_{s_1} \cdots T_{s_r}$. So $\mathcal{E}$ is in fact generated as an algebra by the T_s, with $s \in S$, together with $1 = T_1$. Now by consecutive applications of the multiplications rules we can write down the full multiplication table for the basis elements T_w of $\mathcal{E}$. So we get the requested uniqueness.

$\square$

The algebra constructed in the above theorem, we will be denoted by $\mathcal{E}_A(a_s, b_s)$, and will be called a generic algebra.

Remark 6.1.2. *The set of generators $\{T_s : s \in S\}$ together with the defining relations R :*

$$T_s{}^2 = a_sT_s + b_sT_1 \quad \textit{(quadratic relations)}$$

and the homogeneous relations:

$$(T_sT_t)^q = (T_tT_s)^q \;\; \textit{if} \;\; m(s,t) = 2q < \infty \quad (\textit{i.e } (st)^{2q} = 1)$$

$$(T_sT_t)^qT_s = (T_tT_s)^qT_t \;\; \textit{if} \;\; m(s,t) = 2q+1 < \infty \quad (\textit{i.e } (st)^{2q+1} = 1)$$

where $s, t \in S$ and $m(s,t)$ is the order of st in W, form a presentation for the algebra $\mathcal{E}$.

Proof. Let $\mathcal{F}$ be a A-algebra with generators the set $\{\overline{T}_s : s \in S\}$ and defining relations $\overline{R}$ (as in R, but replacing T_s by $\overline{T}_s$). Then there exists a canonical epimorphism $\pi : \mathcal{F} \longrightarrow \mathcal{E}$ given by $\pi(\overline{T}_s) = T_s$, $s \in S$. Now by Matsumoto's Theorem, there exists a map $f : W \longrightarrow \mathcal{F}$ such that $f(s) = \overline{T}_s$, for $s \in S$ and $f(s_1 \cdots s_r) = T_{s_1} \cdots T_{s_r}$ for every reduced expression of an element of W. Since the elements $\{T_s\}_{s \in S}$ form a basis for $\mathcal{E}$, there exists a linear map $g : \mathcal{E} \longrightarrow \mathcal{F}$ such that $g(T_w) = \overline{T}_{s_1} \cdots \overline{T}_{s_r}$, for every reduced expression $s_1 \cdots s_r$ for $w \in W$ and $g|_S = f$.
Moreover g is a homomorphism, i.e we will show that $g(T_wT_{w'}) = g(T_w)g(T_{w'})$. Although, by the definition of the map g it is sufficient to prove that

$$g(T_sT_w) = g(T_s)g(T_w)$$

for all $s \in S$ and $w \in W$. In this direction we consider the following cases:

1. If $\ell(sw) > \ell(w)$, then $T_sT_w = T_{sw}$, by the multiplication rules in the algebra $\mathcal{E}$ and the expression $ss_1 \cdots s_r$ is reduced for the element sw. Thus $g(T_sT_w) = g(T_{sw}) = \overline{T}_s\overline{T}_{s_1} \cdots \overline{T}_{s_r} = f(T_s)g(T_w) = g(T_s)g(T_w)$.

2. If $\ell(sw) < \ell(w)$. Put $w' = sw \Longrightarrow w = sw'$. Then $\ell(sw') > \ell(w')$, and by the previous case of the proof, we have that

$$g(T_sT_{w'}) = g(T_s)g(T_{w'}) \Longleftrightarrow g(T_sT_{sw}) = g(T_s)g(T_{sw})$$

but $T_sT_{sw} = T_w$, so

$$g(T_w) = g(T_s)g(T_{sw}) = f(T_s)g(T_w)$$

and thus

$$f(T_s)g(T_w) = g(T_s)g(T_w) = [g(T_s)]^2g(T_{sw})$$

Now by assumption, the elements $g(T_s)$ also satisfy the quadratic relations and hence

$$\begin{aligned} f(T_s)g(T_w) &= [g(T_s)]^2g(Tsw) \\ &= (a_sg(T_s) + b_sg(T_1))g(T_{sw}) \\ &= a_sg(T_s)g(T_{sw}) + b_sg(T_{sw}) \\ &= a_sg(T_w) + b_sg(T_{sw}) \end{aligned}$$

But $T_sT_w = a_sT_w + b_sT_{sw}$, so by applying g to the last relation we get that

$$g(T_sT_w) = a_sg(T_w) + b_sg(T_{sw})$$

By comparing the last formula with the previous we get $g(T_s)g(T_w) = g(T_sT_w)$. So g is homomorphism.
Now g is surjective. Indeed, since the Img is a subalgebra of $\mathcal{F}$ (because g is a homomorphism) we have that Img is closed under multiplication. Also note that Img contains all the generators. So $\mathcal{F} \subseteq Img$, and thus $Img = \mathcal{F}$,i.e g is surjective.
Moreover, $\pi \ o \ g = 1_{\mathcal{E}}$ and so g is 1-1.
Thus, g is an isomorphism. So $\mathcal{E} \cong \mathcal{F}$. This completes the proof.

□

6.2 The Generic Hecke Algebra $\mathcal{H}$

We are ready now to define the generic Hecke algebra $\mathcal{H}$ of W. But first let's make the following observation. As we already have stated in the beginning of this chapter the Hecke algebra $H = H(G, B, 1_B)$ is a $\mathbb{C}$-algebra with basis T_w, $w \in W$, and with multiplication rules defined by

$$T_sT_w = \begin{cases} T_{sw} & ,\text{if } \ell(sw) = \ell(w) + 1 \\ qT_{sw} + (q-1)T_w & ,\text{if } \ell(sw) = \ell(w) - 1 \end{cases}$$

Since $s^2 = 1$ we get that ${T_s}^2 = qT_1 + (q-1)T_s$ (the quadratic relation) then by simple calculations we have

$${T_s}^2 - (q-1)T_s - qT_1 - 0 \Longrightarrow (T_s - qT_1)(T_s + T_1) = 0 \Longrightarrow (q^{-\frac{1}{2}}T_s - q^{\frac{1}{2}}T_1)(q^{-\frac{1}{2}}T_s + q^{-\frac{1}{2}}T_1) = 0$$

So it's obvious from the above that in order to be things nice we would like the elements $q^{-\frac{1}{2}}, q^{\frac{1}{2}}$ to be in the base ring.

Now, by combining the existing construction of generic algebras $\mathcal{E}_A(a_s, b_s)$ over a commutative ring A, with the above observation we understand that by an appropriate choice for the algebra A and for the parameters $a_s, b_s \in A$, we will obtain the generic Hecke algebra. Specifically, we choose the ring $A = \mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]$ be the ring of Laurent polynomials over $\mathbb{Z}$ in the indeterminate $q = u$. Also with the further convention that $a_s = u$ and $b_s = u - 1$, for all $s \in S$, we get the generic Hecke algebra of W, denoted by $\mathcal{H} = \mathcal{H}_{\mathbb{Z}[u^{\frac{1}{2}}, u^{-\frac{1}{2}}]}(u)$. The relations (9) and (10) from the previous construction now become:

$$T_sT_w = T_{sw} \ \text{ if } \ell(sw) > \ell(w)$$
$${T_s}^2 = (u-1)T_s + uT_1$$

As we will see in a later chapter Tit's proved that if we specialize $u \mapsto q$, where q is a prime power, and W finite, then $\mathcal{H}_{\mathbb{C}}(q) \cong \mathbb{C}W$, where $\mathbb{C}W$ is the group algebra, which we can also think of it as an example of a generic algebra with a particularly choice of the parameters a_s, b_s to be $a_s = 0$ and $b_s = 1$.

Iawhori conjectured that $\mathcal{H}_{\mathbb{Q}}(q) \cong \mathbb{Q}W$, but this turned out to be false. Although, Lusztig finally proved that $\mathcal{H}_{\mathbb{Q}[q^{\frac{1}{2}}]}(q) \cong \mathbb{Q}[q^{\frac{1}{2}}]W$.

6.3 Bruhat ordering

In this section we introduce the very important concept of the Bruhat ordering on a Coxeter group W, which will play a key role in our future study of some special properties that have the generic Hecke algebra.

So let W be a Coxeter group, with fundamental reflections $S = \{s_1, \cdots, s_n\}$. We define the set of reflections of W to be $T = \bigcup_{w \in W} wSs^{-1}$. By using this, we write $u \leq v$ if we can build the element v from u through a series of multiplications by reflections, with length increasing by one each time. More formally, we have the following definition of the Bruhat order.

Definition 6.3.1. *We say that $u \leq v$ if there exists $t_1, \cdots, t_r \in T$ such that*

$$v = ut_1 \cdots t_r$$

and for each $1 \leq i \leq r-1$ we have $\ell(ut_1 \cdots t_i) < \ell(ut_1 \cdots t_{i+1})$. The resulting relation is a partial ordering of W, and we will call it Bruhat ordering.

Remark 6.3.1. *(i) It is convenient sometimes to have the alternative characterized of the Bruhat order by the use of the subexpressions. More precisely, if $w = s_1 \cdots s_r$ is a reduced expression for w, then $x \leq w$ if and only if $x = s_{i_1} \cdots s_{i_q}$ where $1 \leq i_1 < i_2 < \cdots < i_q \leq r$. In other words $x \leq w$ if and only if we have a reduced expression for x which is a subexpression of w.*

(ii) Also we can make the following definitions that lead us to the Bruhat graph. In particular, we write $u \to v$ if $ut = v$, for some $t \in T$, with $\ell(v) > \ell(u)$. Then we can make the alternative definition of the Bruhat ordering by saying $u \leq v$ if there exists a sequence $u = w_0 \to w_1 \to \cdots \to w_m = v$. Now the Bruhat graph is a directed graph related to the Bruhat order, by the vertices it will be the set of elements of the Coxeter group and the edges consists of directed edges (u,v) whenever $u \to v$.

Example 6.3.1. *Let S_3 be the symmetric group with generators s_1, s_2. Then $T = \{s_1, s_2, s_1s_2s_1\}$ Then by further calculations we can get the following Bruhat graph:*

$$\begin{tikzcd}
& s_1s_2s_1 & \\
s_2s_1 \arrow[ur] & & s_1s_2 \arrow[ul] \\
s_1 \arrow[u] \arrow[urr] & & s_2 \arrow[u] \arrow[ull] \\
& 1 \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

6.4 Inverses and θ-map of Hecke algebra

Now turning back our point of interest to the generic Hecke algebra, where $\mathcal{H} = \mathcal{H}_A(u)$ with $A = \mathbb{Z}[u^{\frac{1}{2}}, u^{\frac{1}{2}}]$. Then the elements of A have the form $\sum_{i \in \mathbb{Z}} a_i u^{\frac{1}{2}i}$, $a_i \in \mathbb{Z}$ and $a_i = 0$ for all but finitely many i.

We define a map from A to A, by $a \mapsto \overline{a}$, i.e such that $\sum a_i u^{\frac{1}{2}i} \mapsto \overline{\sum a_i u^{\frac{1}{2}} i} = \sum a_i u^{-\frac{1}{2}i}$
This is an automorphism of the ring A of order 2. Then we can define, as we will see in the next proposition, an involution $\theta : \mathcal{H} \longrightarrow \mathcal{H}$ by

$$\theta(\sum_{w \in W} a_w T_w) = \sum_{w \in W} \overline{a_w} (T_{w^{-1}})^{-1}$$

But before these let examine some special features that emerge from the way we have constructed the generic Hecke algebra. In particular as the following Lemma shows for the algebra $\mathcal{H}$, is the existence of inverses for the basis elements T_w, due to the presence of the indeterminate u^{-1} in the base ring.

Lemma 6.4.1. *The basis elements T_w is invertible in $\mathcal{H}$*

Proof. Let $w = s_1 \cdots s_r$ be a reduced expression of w. Then we know that $T_w = T_{s_1} \cdots T_{s_r}$. So it's sufficient to show that each T_s is invertible. Indeed, by the quadratic relation we have :

$${T_s}^2 = uT_1 + (u-1)T_s$$

$$u^{-1}{T_s}^2 = T_1 + (1 - u^{-1})T_s$$

$$T_s(u^{-1}T_s + (u^{-1} - 1)T_1) = T_1$$

where T_1 is the identity element of $\mathcal{H}$. So

$$T_s^{-1} = u^{-1}T_s + (u^{-1} - 1)T_1$$

□

Proposition 6.4.1. *For every $h, h_1, h_2 \in \mathcal{H}$ and $a \in A$*

1. $\theta(ah) = \overline{a}\theta(h)$, *i.e θ is semilinear*
2. $\theta(h_1 + h_2) = \theta(h_1) + \theta(h_2)$
3. $\theta(h_1 h_2) = \theta(h_1)\theta(h_2)$
4. $\theta^2 = 1$, *i.e θ is an involution.*

Proof. 1. Obvious from the definition of θ

2. Obvious from the definition of θ

3. **STEP 1:** First show $\theta(T_sT_w) = \theta(T_s)\theta(T_w)$.
Suppose first $\ell(sw) = \ell(w) + 1$. Then $T_sT_w = T_{sw}$ so

$$\theta(T_sT_w) = \theta(T_{sw}) = {T_{sw^{-1}}}^{-1} = {T_{w^{-1}s}}^{-1} = (T_{w^{-1}}T_s)^{-1} = T_s^{-1}{T_{w^{-1}}}^{-1} = \theta(T_s)\theta(T_w)$$

Now suppose $\ell(sw) = \ell(w) - 1$. Let $w' = sw \Longrightarrow w = sw'$ and $\ell(sw') = \ell(w) + 1$.

$$\begin{aligned}
\theta(T_sT_w) &= \theta(T_sT_sT_{w'}) \\
&= \theta((uT_1 + (u-1)T_s)T_{w'}) \\
&= \theta(uT_{w'} + (u-1)T_sT_{w'}) \\
&= u^{-1}\theta(T_{w'} + (u^{-1} - 1)\theta(T_sT_{w'}), \text{ by (1) and (2) of this proposition} \\
&= (u^{-1}T_1 + (u^{-1} - 1)\theta(T_s))\theta(T_{w'}) \\
&= (u^{-1}T_1 + (u^{-1} - 1)T_s^{-1})\theta(T_{w'})
\end{aligned}$$

Now by calculate $\theta(T_s)\theta(T_w)$ we get:

$$\begin{aligned}
\theta(T_s)\theta(T_w) &= \theta(T_s)\theta(T_sT_{w'}) \\
&= \theta(T_s)\theta(T_s)\theta(T_{w'}), \text{ by first case} \\
&= T_s^{-2}\theta(T_{w'}) \\
&= (u^{-1}T_1 + (u^{-1} - 1)T_s^{-1})\theta(T_{w'})
\end{aligned}$$

Since $T_s^{-1} = u^{-1}T_s + (u^{-1} - 1)T_1$ so multiplying by T_s^{-1} gives

$$T_s^{-2} = u^{-1}T_1 + (u^{-1} - 1)T_s^{-1}$$

So

$$\theta(T_sT_w) = \theta(T_s)\theta(T_w)$$

STEP 2: We will show that $\theta(T_sh) = \theta(T_s)\theta(h)$, for every $h \in \mathcal{H}$. Write $h = \sum a_wT_w$ so

$$\begin{aligned}
\theta(T_sh) &= \theta(\sum a_wT_sT_w) \\
&= \sum \overline{a_w}\theta(T_sT_w), \text{ by (1) and (2)} \\
&= \sum \overline{a_w}\theta(T_s)\theta(T_w), \text{ by STEP 1} \\
&= \theta(T_s)\sum \overline{a_w}\theta(T_w) \\
&= \theta(T_s)\theta(h)
\end{aligned}$$

STEP 3 : Now show that $\theta(T_w h) = \theta(T_w)\theta(h), \quad \forall h \in \mathcal{H}$.
For, let $w = s_1 \cdots s_r$ be reduced. Then $T_w = T_{s_1} \cdots T_{s_r}$ so

$$\begin{aligned}\theta(T_w h) &= \theta(T_{s_1} \cdots T_{s_r} h) \\ &= \theta(T_{s_1})\theta(T_{s_2} \cdots T_{s_r} h) \\ &= \cdots = \theta(T_{s_1}) \cdots \theta(T_{s_r})\theta(h) \\ &= \theta(T_w)\theta(h)\end{aligned}$$

STEP 4 : Finally show $\theta(h'h) = \theta(h')\theta(h)$, for every $h', h \in \mathcal{H}$.
Let $h' = \sum a_w T_w$ so

$$\begin{aligned}\theta(h'h) &= \theta(\sum a_w T_w h) = \sum \overline{a_w}\theta(T_w h), \text{ by (1) and (2)} \\ &= \sum \overline{a_w}\theta(T_w)\theta(h) = \theta(h')\theta(h)\end{aligned}$$

4. $\theta^2(ah) = \theta(\overline{a}\theta(h)) = \overline{\overline{a}}\theta^2(h) = a\theta^2(h)$, since $\overline{\overline{a}} = a$.
So θ is linear. We must show it fixes all T_w, for all T_w. By calculations we get:

$$\theta^2(T_w) = \theta(\theta(T_w)) = \theta({T_{w^{-1}}}^{-1})$$

Now $T_{w^{-1}}{T_{w^{-1}}}^{-1} = T_1$ so $\theta(T_{w^{-1}})\theta({T_{w^{-1}}}^{-1}) = T_1$. So $\theta({T_{w^{-1}}}^{-1}) = \theta(T_{w^{-1}})^{-1}$

$$\theta^2(T_w) = \theta({T_{w^{-1}}}^{-1}) = \theta(T_{w^{-1}})^{-1} = ({T_w}^{-1})^{-1} = T_w$$

So since θ^2 is linear and it fixes all T_w it must be the identity.

$\square$

6.5 The R-polynomials

Now let examine a little deeper the inverses ${T_w}^{-1}$ for the basis elements $\{T_w\}_{w \in W}$. Suppose that $w \in W$, and that $w = s_1 \cdots s_r$ is a reduced word. Then from the defining relations of the Hecke algebra, we have that

$$T_w = T_{s_1} \cdots T_{s_r}$$

So in order to calculate the inverse ${T_w}^{-1}$ we could make it by

$${T_w}^{-1} = {T_{s_1}}^{-1} \cdots {T_{s_r}}^{-1}$$

But it's clear that as the length $\ell(w)$ is increases it will become unmanageable and progressively more complicated to compute the inverses from this formula. In this direction, by using the Bruhat ordering, we will see that the inverses can be written as a linear combination of T_x, for which $x \leq w$, in the sense of the Bruhat ordering, where the coefficients can be non zero only if $x \leq w$ and in fact these coefficients are polynomials in u, which we will called R-polynomials. More precisely, we would like to shown the following main Theorem:

Theorem 6.5.1. *For all $x, w \in W$ we have that the basis elements T_w of the Hecke algebra $\mathcal{H}$ is invertible and for the inverse of a typical $T_{w^{-1}}$ we have:*

$$T_{w^{-1}}^{-1} = \sum_{x \leq w} {u_x}^{-1}\overline{R_{x,w}}T_x$$

where $\overline{R_{x,w}} = \epsilon_x\epsilon_w u_x {u_w}^{-1} R_{x,w}$, with $u_x = u^{\ell(x)}$,$R_{x,w} \in \mathbb{Z}[u]$ is a polynomial in u of degree $\ell(w) - \ell(x)$ (Note $R_{x,w} = 0$ whenever $x \nleq w$) and $R_{w,w} = 1$
Also $\sum_{x \leq y \leq w} \epsilon_x\epsilon_y R_{x,y}R_{y,w} = \delta_{x,w}$

Now, we will quote the following propositions, from which we will obtain an algorithm for computing the R-polynomials, and also lead us to the proof of the above Theorem.

Proposition 6.5.1. *Suppose $\ell(sy) = \ell(y) - 1$, $s \in S$, $y \in W$. Then*

$$R_{x,y} = \begin{cases} R_{sx,sy} & \text{if } \ell(sx) = \ell(x) - 1 \\ uR_{sx,sy} + (u-1)R_{x,sy} & \text{if } \ell(sx) = \ell(x) + 1 \end{cases}$$

Proof. $\ell(sy) = \ell(y) - 1$, $y' = sy$, so $y = sy'$ and $y^{-1} = y'^{-1}s$. Also $T_{y^{-1}} = T_{y'^{-1}}T_s$

$$\begin{aligned} {T_{y^{-1}}}^{-1} &= {T_s}^{-1}{T_{y'^{-1}}}^{-1} \\ &= (u^{-1}T_s + (u^{-1}-1)T_1)(\sum_x \overline{R_{x,y'}}u^{-\ell(x)}T_x) \\ &= \sum_x \overline{R_{x,y'}}u^{-(\ell(x)+1)}T_sT_x + \sum_x \overline{R_{x,y'}}(u^{-1}-1)u^{-\ell(x)}T_x \\ &= \sum_{x,sx>x} \overline{R_{x,y'}}u^{-(\ell(x)+1)}T_{sx} + \sum_{x,sx<x} \overline{R_{x,y'}}u^{-(\ell(x)+1)}(uT_{sx} + (u-1)T_x) \\ &+ \sum_x \overline{R_{x,y'}}(u^{-1}-1)u^{-\ell(x)}T_x \end{aligned}$$

So

$$\begin{aligned} {T_{y^{-1}}}^{-1} = &\sum_{x,sx>x} (\overline{R_{x,y'}}u^{-\ell(x)}(u^{-1}-1) + \overline{R_{sx,y'}}u^{-(\ell(sx)+1)}u)T_x \\ &+ \sum_{x,sx<x} (\overline{R_{x,y'}}u^{-\ell(x)}(u^{-1}-1) + \overline{R_{sx,y'}}u^{-(\ell(sx)+1)} + \overline{R_{x,y'}}u^{-(\ell(sx)+1)}(u-1))T_x \end{aligned}$$

Also

$${T_{y^{-1}}}^{-1} = \sum_x \overline{R_{x,y}}u^{-\ell(x)}T_x$$

Now comparing the coefficients of T_x from the above relations gives :

$$\overline{R_{x,y}} = \begin{cases} \overline{R_{sx,sy}} & \text{if } sx < x \\ \overline{R_{x,sy}}(u^{-1}-1) + \overline{R_{sx,sy}}u^{-1} & \text{if } sx > x \end{cases}$$

Applying the involution once more gives us what we want. □

Proposition 6.5.2. *If $\ell(ys) = \ell(y) - 1$, $s \in S$, $y \in W$. Then*

$$R_{x,y} = \begin{cases} R_{xs,xy} & \text{if } xs < x \\ uR_{xs,ys} + (u-1)R_{x,ys} & \text{if } xs > x \end{cases}$$

Proof. Using the previous proposition and inverses we get the result we want. □

Proposition 6.5.3.

$$R_{x,y} \neq 0 \text{ then } x \leq y$$

Proof. By induction on $\ell(y)$:
If $\ell(y) = 0$ then $y = 1$ and so we have the result from the above working.
So suppose $\ell(y) > 0$. Choose $s \in S$ with $sy < y$.
Suppose first that $sx < x$. Then by the previous proposition $R_{x,y} = R_{sx,sy}$. Suppose $R_{x,y} \neq 0$. Then $R_{sx,sy} \neq 0$ and so $sx \leq sy$ by the induction, but $x = s(sx)$, $y = s(sy)$ with the length increasing in both cases. So $x \leq y$.
Now suppose instead that $sx > x$, $R_{x,y} \neq 0$. Then

$$R_{x,y} = uR_{sx,sy} + (u-1)R_{x,sy}$$

by previous proposition.
So we have $R_{sx,sy} \neq 0$ or $R_{x,sy} \neq 0$ or both.
If $R_{sx,sy} \neq 0$ we get $sx \leq sy$, i.e $x < sx \leq sy <$ and so $x < y$.
If $R_{x,sy} \neq 0$ we have $x \leq sy$ by induction, but $x \leq sy < y$. So $x < y$ □

Proposition 6.5.4. *If $x \leq y$ then $R_{x,y}$ is a polynomial in u of degree $\ell(y) - \ell(x)$.*

Proof. By induction on $\ell(y)$:
If $\ell(y) = 0$, then $y = 1$ so $R_{1,1} = 1$, $R_{x,1} = 0$ for $x \neq 1$. So we are ok.
Suppose $\ell(y) > 0$. Choose $s \in S$ with $sy < y$.

Case 1: Suppose $sx < x$. Put $x' = sx$. Also $R_{x,y} = R_{sx,sy}$. Now we have that $x' \leq y$, $sx' > x'$, $sy < y$ so by a lemma in bruhat ordering we get $x' \leq sy$. Moreover $sx \leq sy$. By induction $R_{sx,sy}$ is a polynomial in u of degree $\ell(sy) - \ell(sx) = \ell(y) - \ell(x)$ since both increasing by 1.
Case 2: Suppose this time $sx > x$. Then

$$R_{x,y} = uR_{sx,sy} + (u-1)R_{x,sy}$$

and we have $x \leq y$, $sx > x$, $sy < y$ so again by the same lemma as previous we get that $x \leq sy$. So by induction $R_{x,sy}$ is a polynomial in u of degree $\ell(sy) - \ell(x) = \ell(y) - \ell(x) - 1$. So when we multiply it by $(u-1)$ the degree goes up by 1, i.e to $\ell(y) - \ell(x)$.
Also by induction $R_{sx,sy}$ is either 0 if $sx \not\leq sy$ or a polynomial in u of degree $\ell(sy) - \ell(sx) = \ell(y) - \ell(x) - 2$. So even when we multiply by u we just get degree $\ell(y) - \ell(x) - 1$ (so doesn't affect the overall degree of it).
Thus $R_{x,y}$ is a polynomial in u of degree $\ell(y) - \ell(x)$. □

Proposition 6.5.5. *Suppose $x \leq y$ and $\ell(y) - \ell(x) \leq 2$. Then*

$$R_{x,y} = (u-1)^{\ell(y)-\ell(x)}$$

Proof. 1. Suppose $\ell(y) - \ell(x) = 0$. Since $x \leq y$ this means $x = y$. Now let $x = s_1 \cdots s_r$ be reduced. Then $R_{x,x} = R_{s_1\cdots s_r, s_1 \cdots s_r}$.
So taking $s = s_1$, we get that $sx < x$, and thus by a previous proposition

$$R_{x,x} = R_{s_1\cdots s_r, s_1\cdots s_r} = R_{s_2\cdots s_r, s_2 \cdots s_r}$$

By repeatedly applying the proposition, varying s over $s_1, \cdots, s_r$, gives

$$R_{x,x} = R_{s_1\cdots s_r, s_1\cdots s_r} = \cdots = R_{1,1} = 1 = (u-1)^0$$

2. Suppose $\ell(y) - \ell(x) = 1$
Then exists reduced expression $y = s_1 \cdots s_r$ with $x = s_1 \cdots \hat{s_i} \cdots s_r$. So

$$\begin{aligned}
R_{x,y} &= R_{s_1\cdots \hat{s_i} \cdots s_r, s_1 \cdots s_r} \text{ since both expressions are reduced} \\
&= R_{s_2\cdots \hat{s_i} \cdots s_r, s_2 \cdots s_r} \text{ by applying the proposition as in the previous case} \\
&\vdots \\
&= R_{s_{i+1}\cdots s_r, s_i \cdots s_r} \\
&= R_{s_{i+1}\cdots s_{r-1}, s_i \cdots s_{r-1}} \text{ by applying the analogous proposition as previous for the right handed version} \\
&\vdots \\
&= R_{1,s_i} \\
&= uR_{s_i,1} + (u-1)R_{1,1} \text{ since length increasing} \\
&= u \cdot 0 + (u-1) \cdot 1 \text{ since } s_i \not\leq 1 \text{ and } R_{1,1} = 1 \\
&= u - 1 = (u-1)^1
\end{aligned}$$

3. Finally suppose $\ell(y) = \ell(x) + 2$ and $x \leq y$. Then exists reduced expression $y = s_1 \cdots s_r$ s.t $x = s_1 \cdots \hat{s_i} \cdots s_r$. So

$$\begin{aligned}
R_{x,y} &= R_{s_1\cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r, s_1 \cdots s_r} \\
&= R_{s_{i+1}\cdots \hat{s_j} \cdots s_r, s_i \cdots s_r} \text{ by repeatedly use of the proposition as previous} \\
&= R_{s_{i+1}\cdots s_{j-1}, s_i \cdots s_j} \text{ by repeatedly use of the right handed version of the proposition}
\end{aligned}$$

So

$$R_{x,y} = uR_{s_i s_{i+1}\cdots s_{j-1}, s_{i+1}\cdots s_j} + (u-1)R_{s_{i+1}\cdots s_{j-1}, s_{i+1}\cdots s_j}$$

Now the elements $s_i \cdots s_{j-1}$ and $s_{i+1} \cdots s_j$ have the same length, so if $s_i \cdots s_{j-1} \leq s_{i+1} \cdots s_j$ we must have equality. But if $s_i \cdots s_{j-1} = s_{i+1} \cdots s_j$ then $s_1 \cdots s_r = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r$ contradicting the fact that $s_1 \cdots s_r$ is reduced.
So $s_1 \cdots s_{j-1} \not\leq s_{i+1} \cdots s_j$ and thus $R_{s_i \cdots s_{j-1}, s_{i+1}\cdots s_j} = 0$
So

$$\begin{aligned} R_{x,y} &= (u-1)R_{s_{i+1}\cdots s_{j-1}, s_{i+1}\cdots s_j} \\ &= (u-1)(u-1) \text{ by the part (2) of the proof} \\ &= (u-1)^2 \end{aligned}$$

□

Example 6.5.1. *Let $W = W(A_2) \cong S_3$.*

$$W =< s_1, s_2 | {s_1}^2 = 1 = {s_2}^2, (s_1 s_2)^3 = 1 >= \{1, s_1, s_2, s_1 s_2, s_2 s_1, w_o\}$$

where $w_o = s_1 s_2 s_1 = s_2 s_1 s_2$. The table for $R_{x,y}$ looks like :

y / x	1	s_1	s_2	$s_1 s_2$	$s_2 s_1$	w_o
1	1	u-1	u-1	$(u-1)^2$	$(u-1)^2$	$(u-1)(u^2-u+1)$
s_1	0	1	0	u-1	u-1	$(u-1)^2$
s_2	0	0	1	u-1	u-1	$(u-1)^2$
$s_1 s_2$	0	0	0	1	0	u-1
$s_2 s_1$	0	0	0	0	1	u-1
w_o	0	0	0	0	0	1

Because from the previous propositions we have that:

- *For every $y \leq x$ we have that $R_{x,y} = 0$, so all the elements under the main diagonal of the above table have to be 0.*
- *If $x = y$, then $R_{x,x} = R_{1,1} = 1$, so in the main diagonal we will have only 1.*
- *For every elements of W such that $\ell(y) - \ell(x) = 1$, we will have that $R_{x,y} = u - 1$, by the case (2) of the proposition 6.5.5*
- *Similarly by the case (3) of the proposition 6.5.5, if $\ell(y) - \ell(x) = 2$ we have that $R_{x,y} = (u-1)^2$.*
- *So the only R-polynomial we have to compute is the R_{1,w_o}.*

$$R_{1,w_o} = R_{1,s_1 s_2 s_1} = uR_{s_1, s_2 s_1} + (u-1)R_{1,s_2 s_1} = u(u-1) + (u-1)(u-1)^2$$

from the above statements. Finally $R_{1,w_o} = (u-1)(u^2-u+1)$

Definition 6.5.1. *Let $\epsilon_w = (-1)^{\ell(w)}$ and $u_w = u^{\ell(w)}$*

Proposition 6.5.6.

$$\overline{R_{x,y}} = \epsilon_x \epsilon_y u_x {u_y}^{-1} R_{x,y}$$

Proof. By induction on $\ell(y)$:
If $\ell(y) = 0$. Then $y = 1$ and $R_{x,1} = 0$, for $x \neq 1$, $R_{1,1} = 1$. So in this case we have the result.
Assume $\ell(y) > 0$. Choose $s \in S$ with $sy < y$.

<u>**Case 1:**</u> Suppose $sx < x$. Then $R_{x,y} = R_{sx,sy}$.
By induction

$$\overline{R_{sx,sy}} = \epsilon_{sx}\epsilon_{sy} u_{sx} {u_{sy}}^{-1} R_{sx,sy}$$

So

$$\overline{R_{x,y}} = \epsilon_x \epsilon_y u_x {u_y}^{-1} R_{x,y}$$

Case 2: Suppose $sx > x$. Then $R_{x,y} = uR_{sx,sy} + (u-1)R_{x,sy}$ So

$$\begin{aligned}
\overline{R_{x,y}} &= u^{-1}\overline{R_{sx,sy}} + (u^{-1}-1)\overline{R_{x,sy}} \\
&= \epsilon_{sx}\epsilon_{sy} u_{sx} {u_{sy}}^{-1} R_{sx,sy} + (u^{-1}-1)\epsilon_x\epsilon_y u_x {u_y}^{-1} R_{x,sy} \text{ by induction} \\
&= \epsilon_x\epsilon_y u_x {u_y}^{-1}(uR_{sx,sy} - (u^{-1}-1)uR_{x,sy}) \\
&= \epsilon_x\epsilon_y u_x {u_y}^{-1}(uR_{sx,sy} + (u-1)R_{x,sy}) \\
&= \epsilon_x\epsilon_y u_x {u_y}^{-1} R_{x,y}
\end{aligned}$$

□

Proposition 6.5.7. *If W is finite, then $R_{x,w} = R_{w_o w, w_o x}$ for all $x \leq w$. Where w_o is the longest element of W.*

Remark 6.5.1. *Recall that it satisfies $\ell(w_o w) = \ell(w_o) - \ell(w)$ for all $w \in W$ and that $x < w$ if and only if $w_o w < w_o x$.*

Proof. By induction on $\ell(w)$:
If $\ell(w) = 1$ then $w = 1$ and so $R_{1,1} = R_{w_o,w_o} = 1$.
Assume $\ell(w) > 0$, choose $s \in S$ such that $ws < w$. There are two cases to consider:
Case 1: If $xs < x$ then also $w_o x < w_o xs$. So we can apply the right handed version of proposition to each of these situations and then by induction we get :

$$R_{x,w} = R_{xs,sw} = R_{w_o ws, w_o xs} = R_{w_o w, w_o x}$$

Case 2: If $x < xs$ then again by the proposition we have

$$R_{x,w} = uR_{xs,ws} + (u-1)R_{x,ws}$$

Now by induction we get that

$$uR_{xs,ws} + (u-1)R_{x,ws} = uR_{w_o ws, w_o xs} + (u-1)R_{w_o ws, w_o x}$$

So

$$R_{x,w} = uR_{w_o ws, w_o xs} + (u-1)R_{w_o ws, w_o x} = uR_{w_o ws, w_o xs} + (u-1)R_{w_o w, w_o xs}$$

where the second term replaced by $(u-1)R_{w_o w, w_o xs}$ due to the proposition. By applying again the same proposition we obtain that

$$R_{x,w} = uR_{w_o ws, w_o xs} + (u-1)R_{w_o w, w_o xs} = R_{w_o w, w_o x}$$

, as required.

□

Theorem 6.5.2. *For all $x, w \in W$ we have that the basis elements T_w of the Hecke algebra $\mathcal{H}$ is invertible and for the inverse of a typical $T_{w^{-1}}$ we have:*

$$T_{w^{-1}}^{-1} = \sum_{x \leq w} {u_x}^{-1} \overline{R_{x,w}} T_x$$

where $\overline{R_{x,w}} = \epsilon_x \epsilon_w u_x {u_w}^{-1} R_{x,w}$, $R_{x,w} \in \mathbb{Z}[u]$ is a polynomial in u of degree $\ell(w) - \ell(x)$ (Note $R_{x,w} = 0$ whenever $x \nleq w$) and $R_{w,w} = 1$
Also $\sum_{x \leq y \leq w} \epsilon_x \epsilon_y R_{x,y} R_{y,w} = \delta_{x,w}$

6.6 The Kazhdan-Lusztig basis of Hecke algebra $\mathcal{H}$

Our goal now is to find a new basis $\{C_w\}$ for the generic Hecke algebra $\mathcal{H}$, such that indexed again by the elements of W, but consisting of elements fixed by the involution θ, i.e we will construct a basis for $\mathcal{H}$ such that consisting of $\theta-$invariant elements. Recall that the involution $\theta : \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\sum a_w T_w \longmapsto \sum \overline{a_w} (T_{w^{-1}})^{-1}$$

Note also that, as in the case of the calculation of the inverse elements of the basis T_w of $\mathcal{H}$, we had to introduce the concept of P-polynomials, similarly in order to construct the new basis C_w of $\mathcal{H}$, we will simultaneously obtain a new family of polynomials, which we will call them Kazhdan-Lusztig polynomials, denoted by $P_{x,w}$, for $x, w \in W$, and which will be reminiscent of R-polynomials but subtler, and therefore easier to calculate them.

Last but not least, this basis it will play a crucial role in the construction, that we will see in a later chapter, of relatively low-dimensional representations of $\mathcal{H}$ arising from the structure of the underlying Coxeter group W.

Now, we present the idea behind this construction. More precisely, by experimenting with the relation:

$$T_s{}^{-1} = u^{-1}T_s - (1 - u^{-1})T_1$$

It is easy to see, by simple calculations of the act of the involution θ, that θ sends $T_s - uT_1$ to $u^{-1}(T_s - uT_1)$. Now by multiplying the first expression by $u^{-\frac{1}{2}}$ and act again with the map θ, we can see that this element is $\theta-$invariant. So we could define

$$C_s := u^{-\frac{1}{2}}(T_s - uT_1)$$

Now we could continue to construct further $\theta-$invariants elements of $\mathcal{H}$, by multiplying various C_s, for $s \in S$, in the spirit of the way the original basis elements T_w of $\mathcal{H}$ are built out of the T_s. For example, if $s \neq t$, by multiplying C_s with C_t, we get

$$C_sC_t = u^{-1}(T_{st} - uT_s - uT_t + u^2T_1)$$

and we could define this element as C_{st}. At this step of the process, if we consider the only other possible reduced expression of length 2, i.e the element ts, and we assume that $st = ts$, then if we make the multiplication C_tC_s, we would have obtained again the same element, as we wanted, from the previous multiplication of C_sC_t. But as the length of the elements is increasing, this approach won't work out because the following problems arise: For example, let an element $w \in W$, such that $\ell(w) = 3$, e.g the element sts, then by calculations we get

$$C_sC_tC_s = u^{-\frac{3}{2}}(T_{sts} - uT_{st} - uT_{ts} + u^2(1 + u^{-1})T_s + u^2T_t - u^3(1 + u^{-1})T_1)$$

If we label this element by C_{sts} we have that

- **1st Problem:** In the case that $sts = tst$, by calculations we get that $C_{sts} \neq C_{tst}$, which lead to obvious ambiguity, that is occurs from the way we have until now construct the new basis.

- **2nd Problem:** The polynomials that are appearing as coefficients of T_s and T_1 in the above expressions for the elements C_{sts}, etc are way more complicated than we would like to be, in order to be easy to compute them.

But, in order to avoid such situations, we will see below that the expressions such as $C_sC_tC_s - C_s$ are $\theta-$invariants and also a linear combination of T_x, where $x \leq sts$ (by the Bruhat order), with the coefficients being quite simpler to calculate. So we conclude to the following very important Theorem.

Theorem 6.6.1. *There is a unique element $C_w \in \mathcal{H}$ such that $\theta(C_w) = C_w$ (i.e C_w is $\theta-$invariant) and*

$$C_w = \epsilon_w u_w{}^{\frac{1}{2}} \sum_{x \in W : x \leq w} \epsilon_x u_x{}^{-1} \overline{P_{x,w}} T_x$$

where $P_{w,w} = 1$, $P_{x,w} \in \mathbf{Z}[u]$ has degree $\leq \frac{1}{2}(\ell(w) - \ell(x) - 1)$ if $x < w$.
The $P_{x,w}(u)$ is called a Kazhdan-Lusztig polynomial.

Proof. • **Uniqueness:**

Let $w \in W$ we write

$$C_w = \sum_{y \leq w} \alpha(y,w)\overline{P_{y,w}}T_y, \quad \text{where} \quad \alpha(y,w) = \epsilon_y \epsilon_w {u_w}^{\frac{1}{2}} {u_y}^{-1}$$

and assuming that C_w has the properties we have state in the theorem.

$$\begin{aligned}\theta(C_w) &= \sum_{y:\, y \leq w} u^{-\frac{1}{2}\ell(w)+\ell(y)} P_{y,w} {T_{y^{-1}}}^{-1} \\ &= \sum_{y:\, y \leq w} \epsilon_y \epsilon_w u^{-\frac{1}{2}\ell(w)+\ell(y)} P_{y,w} \sum_{x:\, x \leq y} \overline{R_{x,y}} u^{-\ell(x)} T_x \\ &= \sum_{x:\, x \leq w} \Big(\sum_{y:\, x<y \leq w} \epsilon_y \epsilon_w u^{-\frac{1}{2}\ell(w)+\ell(y)-\ell(x)} \overline{R_{x,y}} P_{y,w} \Big) T_x\end{aligned}$$

For $\theta(C_w) = C_w$ need

$$\begin{aligned}\epsilon_x \epsilon_w u^{\frac{1}{2}\ell(w)-\ell(x)} \overline{P_{x,w}} &= \sum_{y:\, x<y \leq w} \epsilon_y \epsilon_w u^{-\frac{1}{2}\ell(w)+\ell(y)-\ell(x)} \overline{R_{x,y}} P_{y,w} \\ u^{\frac{1}{2}\ell(w)-\ell(x)} \overline{P_{x,w}} &= \sum_{y:\, x<y \leq w} \epsilon_x \epsilon_y u^{-\frac{1}{2}\ell(w)+\ell(y)-\ell(x)} \overline{R_{x,y}} P_{y,w}\end{aligned}$$

Know we will show that the $P_{y,w}$'s are uniquely determined by induction on $\ell(w) - \ell(y)$
If $n = 0$ then $P_{w,w} = 1$ having been settled by hypothesis.
We then assume $P_{y,w}$ are all uniquely determined where $x \leq y \leq w$, and we will show that this forces the choice of $P_{x,w}$ to be uniquely determined.

By multiplying both sides of the above equation with $u^{\frac{1}{2}\ell(x)}$ and move the term for $y = x$ from the r.h.s to the left (using the fact that $R_{x,x} = 1$) yields :

$$u^{\frac{1}{2}\ell(w)-\frac{1}{2}\ell(x)} \overline{P_{x,w}} - u^{-\frac{1}{2}\ell(w)+\frac{1}{2}\ell(x)} = \sum_{y:\, x<y \leq w} \epsilon_x \epsilon_y u^{-\frac{1}{2}\ell(w)+\ell(y)-\frac{1}{2}\ell(x)} \overline{R_{x,y}} P_{y,w} \tag{11}$$

Now by induction all $P_{y,w}$ are already uniquely determined for $x < y \leq w$, so everything on the r.h.s of the equation is known and is unique, so we have to argue that $P_{x,w}$ is also uniquely determined.

Since $x < w$, the degree assumption of $\overline{P_{x,w}}$, from the theorem, implies that the first term of l.h.s of the equation is a polynomial in $u^{\frac{1}{2}}$ with no constant term (otherwise $P_{x,w}$ would have to contain $u^{\frac{1}{2}}$, which is contradiction due to $P_{x,w} \in \mathbb{Z}[u]$). Similarly, the term $u^{-\frac{1}{2}\ell(w)+\frac{1}{2}\ell(x)} P_{x,w}$ is a polynomial in $u^{-\frac{1}{2}}$ with no constant term. So no cancellation occurs, and there is at most one choice for $P_{x,w}$ satisfying the equation (6). So $P_{x,w}$ is uniquely determined.

- **Existence:**

If $w = 1$ then $C_1 = T_1$ Q.E.D
Now we use induction on $\ell(w)$. So assume $w \neq 1$. Also assume $\forall w' \in W$ with $\ell(w') < \ell(w)$. Let $w = sv$, where $\ell(v) = \ell(w) - 1$. So C_v exists by induction and thus $P_{z,v}$ is defined for every $z \leq v$. Let $\mu(z,v)$ be the coefficient of the highest power of u in $P_{z,v}$. i.e is the coefficient of $u^{\frac{1}{2}(\ell(v)-\ell(z)-1)}$ in $P_{z,v}$. We define:

$$C_w := C_s C_v - \sum_{z:\, z \prec v, sz<z} \mu(z,v) C_z,$$

where $C_s = u^{-\frac{1}{2}}T_s - u^{\frac{1}{2}}T_1$ and by $z \prec v$ we mean that $z < v$ and $P_{z,v}$ is Kazhdan-Lusztig polynomial of the maximum possible degree, so

$$C_w = (u^{-\frac{1}{2}}T_s - u^{\frac{1}{2}}T_1)C_v - \sum_{z:\ z \prec v, sz<z} \mu(z,v)C_z$$

Now we will show that C_w satisfies the required conditions:

First, C_w is $\theta-$invariant :
Indeed,

$$\theta(C_w) = (u^{\frac{1}{2}}{T_s}^{-1} - u^{-\frac{1}{2}}T_1)\theta(C_v) - \sum_{z:\ z<v, sz<z} \mu(z,v)\theta(C_z)$$

But $\theta(C_v) = C_v$ and $\theta(C_z) = C_z$ by induction. Also

$$u^{\frac{1}{2}}{T_s}^{-1} - u^{-\frac{1}{2}}T_1 = u^{\frac{1}{2}}(u^{-1}T_s + (u^{-1}-1)T_1) - u^{-\frac{1}{2}}T_1 = u^{-\frac{1}{2}}T_s - u^{\frac{1}{2}}T_1$$

So

$$\theta(C_w) = (u^{-\frac{1}{2}}T_s - u^{\frac{1}{2}}T_1)C_v - \sum_{z:\ z<v, sz<z} \mu(z,v)C_z = C_w$$

Now we calculate the coefficients of C_w:

$$C_w = (u^{-\frac{1}{2}}T_s - u^{\frac{1}{2}}T_1) \sum_{y:\ y\le v} \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)}\overline{P_{y,v}}T_y - \sum_{z:\ z<v, sz<z} \mu(z,v) \sum_{y:\ y\le z} \epsilon_y\epsilon_z u^{\frac{1}{2}\ell(z)-\ell(y)}\overline{P_{y,z}}T_y$$

Then

$$\begin{aligned} C_w = & \sum_{y:\ y\le v} \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{y,v}}T_sT_y \\ & - \sum_{y:\ y\le v} \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)+\frac{1}{2}}\overline{P_{y,v}}T_y \\ & - \sum_{y:\ y<v} \sum_{z:\ y\le z<v, sz<z} \mu(z,v)\epsilon_y\epsilon_z u^{\frac{1}{2}\ell(z)-\ell(y)}\overline{P_{y,z}}T_y \end{aligned}$$

Foe the 1st term in the above relation we have that:

$$\begin{aligned} 1st\ term = & \sum_{y:\ y\le v, sy>y} \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{y,v}}T_{sy} \\ & + \sum_{y:\ y\le v, sy<y} \epsilon_y\epsilon_v u^{\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{y,v}}(uT_{sy} + (u-1)T_y) \\ = & \sum_{y:\ sy\le v, sy<y} -\epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)+\frac{1}{2}}\overline{P_{sy,v}}T_y + \sum_{y:\ y\le v, sy<y} \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}(u-1)\overline{P_{y,v}}T_y \\ & + \sum_{y:\ sy\le v, sy>y} -\epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{sy,v}}T_y \end{aligned}$$

So the coefficient of T_y in C_w is as follows:
All y appearing have $y \le w$, (since for the 1st term $sy \le v$, $sy < y$, then $y = s(sy)$, $w = sv \Longrightarrow y \le w$, and similarly for the 2nd term $y \le v < w$).

Now suppose $y \le w$ and $sy < y$. Then the coefficient is:

$$-\epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)+\frac{1}{2}}\overline{P_{sy,v}} - \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{y,v}} - \sum_{z:\ y\le z<v, sz<z} \mu(z,v)\epsilon_y\epsilon_z u^{\frac{1}{2}\ell(z)-\ell(y)}\overline{P_{y,z}}$$

In the other hand, suppose $y \le w$ and $sy > y$. Then the coefficient of T_y in C_w is:

$$-\epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)-\frac{1}{2}}\overline{P_{sy,v}} - \epsilon_y\epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)+\frac{1}{2}}\overline{P_{y,v}} - \sum_{z:\ y\le z<v, sz<z} \mu(z,v)\epsilon_y\epsilon_z u^{\frac{1}{2}\ell(z)-\ell(y)}\overline{P_{y,z}}$$

We would like to put both of these into one formula, so we define:

$$c = \begin{cases} 1, & \text{if } sy < y \\ 0, & \text{if } sy > y \end{cases}$$

Then the coefficient of T_y in C_w for $y \leq w$ is

$$-\epsilon_y \epsilon_v u^{\frac{1}{2}\ell(v)-\ell(y)+\frac{1}{2}}((u^{-1})^{1-c}\overline{P_{sy,v}} + (u^{-1})^{c}\overline{P_{y,v}}) - \sum_{z:\, y \leq z < v, sz < z} \mu(z,v)\epsilon_y \epsilon_z u^{\frac{1}{2}\ell(z)-\ell(y)}\overline{P_{y,z}}$$

But this coefficient must be, by definition

$$\epsilon_y \epsilon_w u^{\frac{1}{2}\ell(w)-\ell(y)\overline{P_{y,w}}}$$

So

$$\overline{P_{y,w}} = (u^{-1})^{1-c}\overline{P_{sy,v}} + (u^{-1})^{c}\overline{P_{y,v}} - \sum_{z:\, y \leq z < v, sz < z} \mu(z,v)\epsilon_w \epsilon_z u^{-\frac{1}{2}(\ell(w)-\ell(z))}\overline{P_{y,z}}$$

Hence

$$P_{y,w} = u^{1-c}P_{sy,v} + u^{c}P_{y,v} - \sum_{z:\, y \leq z < v, sz < z} \mu(z,v)u^{\frac{1}{2}(\ell(w)-\ell(z))}P_{y,z}$$

So by induction on $\ell(w)$ the above formula determines the $P_{y,w}$ and all the P 's in r.h.s lie in $\mathbb{Z}[u]$. Hence $P_{y,w} \in \mathbb{Z}[u]$ (since $\mu(z,v) \neq 0 \Longrightarrow \frac{1}{2}(\ell(v) - \ell(z) + 1) \in \mathbb{Z}$). It remains to check the degree hypothesis is satisfied by the above $P_{y,w}$.
By induction we get:

$$P_{sy,v} \textit{ has degree } \leq \frac{1}{2}(\ell(v) - \ell(sy) - 1) \textit{ when } sy < v$$
$$P_{y,v} \textit{ has degree } \leq \frac{1}{2}(\ell(v) - \ell(y) - 1) \textit{ when } y < v$$
$$P_{y,z} \textit{ has degree } \leq \frac{1}{2}(\ell(z) - \ell(y) - 1) \textit{ when } y < z$$

Now suppose first that $sy > y$. Then

$$P_{y,w} = uP_{sy,v} + P_{y,v} - \sum_{z:\, y \leq z < v, sz < z} \mu(z,v)u^{\frac{1}{2}(\ell(v)-\ell(z)+1)}P_{y,z}$$

$y \neq z$ since $sy > y$ but $sz < z$ so

$$\textit{Degree of } P_{y,z} \leq \frac{1}{2}(\ell(z) - \ell(y) - 1)$$
$$\textit{Degree of last term } \leq \frac{1}{2}(\ell(v) - \ell(y)) = \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

$v \neq sy$ since $sv > v$(if $v = sy \Longrightarrow sv = y \Longrightarrow w = y$, but we assume $sy > y$, i.e $sw > w$, which is contradiction since $v = sw < w$). Also $ssy < sy$ (because $y < sy$) so

$$\textit{Degree of } P_{sy,v} \leq \frac{1}{2}(\ell(v) - \ell(sy) - 1) = \frac{1}{2}(\ell(v) - \ell(y) - 2)$$
$$\textit{Degree of } uP_{sy,v} \leq \frac{1}{2}(\ell(v) - \ell(y)) = \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

Moreover, whether $y = v$ or not

$$\textit{Degree of } P_{y,v} \leq \frac{1}{2} = \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

So

$$deg_{y \neq w} P_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

Now suppose $sy < y$ and $y < w$. Then

$$P_{y,w} = P_{sy,v} + uP_{y,v} - \sum_{z:\ y \leq z < v, sz < z} \mu(z,v) u^{\frac{1}{2}(\ell(v)-\ell(z)+1)} P_{y,z}$$

$sy \neq v$ since $y \neq w$. So if $P_{sy,v} \neq 0$, then

$$degP_{sy,v} \leq \frac{1}{2}(\ell(v) - \ell(sy) - 1) = \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

$y \neq v$ since $sy < y$, $sv > v$. So if $P_{y,v} \neq 0$

$$\begin{aligned} degP_{y,v} &\leq \frac{1}{2}(\ell(v) - \ell(y) - 1) \\ deguP_{y,v} &\leq \frac{1}{2}(\ell(w) - \ell(y)) \end{aligned}$$

If $z \neq y$ then by induction

$$\begin{aligned} degP_{y,z} &\leq \frac{1}{2}(\ell(z) - \ell(y) - 1) \\ deg(\mu(z,v) u^{\frac{1}{2}(\ell(v)-\ell(z)+1)} P_{y,z}) &\leq \frac{1}{2}(\ell(v) - \ell(y)) = \frac{1}{2}(\ell(w) - \ell(y) - 1) \end{aligned}$$

If $z = y$ then

$$deg(\mu(z,v) u^{\frac{1}{2}(\ell(v)-\ell(z)+1)} P_{y,z}) = \frac{1}{2}(\ell(v) - \ell(z) + 1) = \frac{1}{2}(\ell(w) - \ell(y))$$

So $P_{y,w}$ has degree $\leq \frac{1}{2}(\ell(w) - \ell(y))$, but we need to show that

$$degP_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1)$$

. For that, consider the terms of degree $\frac{1}{2}(\ell(w) - \ell(y))$, and we will show they cancel. These terms are:

$$\begin{aligned} \textit{leading term of } uP_{y,v} &= \mu(y,v) u^{\frac{1}{2}(\ell(v)-\ell(y)-1)+1} \\ &= \mu(y,v) u^{\frac{1}{2}(\ell(v)-\ell(y)+1)} \\ &= \mu(y,v) u^{\frac{1}{2}(\ell(w)-\ell(y))} \end{aligned}$$

and from $z = y$ is the term $-\mu(y,v) u^{\frac{1}{2}(\ell(w)-\ell(y))}$. So the above terms cancel each other out. Thus by induction

$$degP_{y,w} \leq \frac{1}{2}(\ell(w) - \ell(y) - 1) \text{ if } y \neq w$$

□

Definition 6.6.1. *The polynomial $P_{y,w}(u)$ are called the Kazhdan-Lusztig polynomials and the set $\{C_w\}$ is called Kazhdan-Lusztig basis.*

Proposition 6.6.1. *For $s \in S$, $w \in W$*

$$T_s C_w = \begin{cases} u^{\frac{1}{2}} C_{sw} + uC_w + \sum_{z: z \prec w, sz < z} u^{\frac{1}{2}} \mu(z,w) C_z & \textit{if } sw > w \\ -C_w & \textit{if } sw < w \end{cases}$$

Proposition 6.6.2. *For $s \in S$, $w \in W$*

$$C_s C_w = \begin{cases} C_{sw} + \sum_{z: z \prec w, sz < z} \mu(z,w) C_z & \textit{if } sw > w \\ -(u^{\frac{1}{2}} + u^{-\frac{1}{2}}) C_w & \textit{if } sw < w \end{cases}$$

Proposition 6.6.3. *1. $P_{y,w}$ has constant term 1*

2. *Suppose $y < w$*

- *If $\ell(y) = \ell(w) - 1$ then $P_{y,w} = 1$*
- *If $\ell(y) = \ell(w) - 2$ then $P_{y,w} = 1$*

Corollary 6.6.1. *Let $x < w$. If $sw < w$ but $sx > x$ for some $s \in S$, then $P_{x,y} = P_{sx,y}$*

Example 6.6.1. *Let $W = W(A_2) \cong S_3$*

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, w_o\}$$

where $w_o = s_1s_2s_1 = s_2s_1s_2$. The following table compute the polynomials $P_{y,w}(u)$:

y \ w	1	s_1	s_2	s_1s_2	s_2s_1	w_o
1	1	1	1	1	1	1
s_1	0	1	0	1	1	1
s_2	0	0	1	1	1	1
s_1s_2	0	0	0	1	0	1
s_2s_1	0	0	0	0	1	1
w_o	0	0	0	0	0	1

- *By the proposition 6.6.3, if $\ell(y) = \ell(w) - 1$ or $\ell(y) = \ell(w) - 2$ we get that $P_{y,w} = 1$.*
- *Also if $y = w$ we have that $P_{w,w} = 1$.*
- *For every time that $y \nleq w$, i.e for the polynomials that occurs below the diagonal and on some other places, we get that $P_{y,w} = 0$.*
- *So the only case that remains to calculate again is the polynomial P_{1,w_o}. By setting in the corollary 6.6.1, $s = s_1$, $x = 1$, and $y = w_o$, we get that $P_{1,w_o} = P_{s_1,w_o} = 1$.*

7 Cells in Coxeter Groups

Now that we have defined the Kazhdan-Lusztig polynomials, and construct a new basis $\{C_w\}$, indexed by the elements of W, for the generic Hecke algebra $\mathcal{H}$, we will use them to finally define the cells of a Coxeter group W, which as we will see, are separated into three kinds of cells (left,right, and two-sided). Furthermore, it will turn out (by using the theory that we have until now developed in the previous chapters) that for each type of cell, we can associate a representation of the generic Hecke algebra $\mathcal{H}$.

In particular, each kind of cell is an equivalence class of a corresponding equivalence relation on the group, and the set of cells of a given type has a partial order imposed on it. This partial order is compatible with the multiplication formulas involving the C-basis, as described in the last propositions of the previous chapter, and we can exploit this to find relatively low-dimensional representations of the Hecke algebra, which we will called it, later, (left, right, two-sides) cells representations, respectively. Also notice that, by specialising $u^{\frac{1}{2}} \mapsto 1$, we obtain a representation of W. The dimension of the representation will turn out to be the size of the cell, and hence will be small compared to $|W|$.

In this direction, first by looking at the action of the elements T_s of $\mathcal{H}$ on the Kazhdan-Lusztig basis C_w we can see how $\mathcal{H}$ acts on itself in the left (regular) representation, relative to the $C-$basis. But $\mathcal{H}$ is still very large module, so in order to construct representations of $\mathcal{H}$, is natural to look for smaller submodules of $\mathcal{H}$ (or better as we will see subquotients). At this point, the Kazhdan-Lusztig basis play important role, since lead us to a constructions of representations associated with a particular sets, as earlier we called them cells, which partition W.

7.1 The Left Cells of a Coxeter group W

Now, in order to construct the cells of a Coxeter group W, our first goal is to define the preorders $\leq_L$, $\leq_R$, and $\leq_{LR}$ on the group W and the left and right descent sets of an element $w \in W$.

Definition 7.1.1. *For any $w \in W$, we define the left and right Descent set of w, respectively, as:*

$$\mathcal{L}(w) := \{s \in S : sw < w\} \text{ and } \mathcal{R}(w) := \{s \in S : ws < w\}$$

Definition 7.1.2. *Given $z, v \in W$ we write $z \longleftrightarrow v$ if either [$z < v$ and $\mu(z,v) \neq 0$] or [$z > v$ and $\mu(v,z) \neq 0$]. i.e*

$$z \longleftrightarrow v \iff z \prec v \text{ or } v \prec z$$

Definition 7.1.3. *Let $v, y \in W$. Write $y \leq_L v$ if there is a chain of elements $v = x_0, x_1, \cdots, x_r = y$ such that for each i $x_i \longleftrightarrow x_{i+1}$ and $\mathcal{L}(x_{i+1}) \nsubseteq \mathcal{L}(x_i)$. So $x_{i+1} \leq_L x_i$.*

This isn't quite a partial ordering since it doesn't satisfy $a \leq b, b \leq a \Longrightarrow a = b$

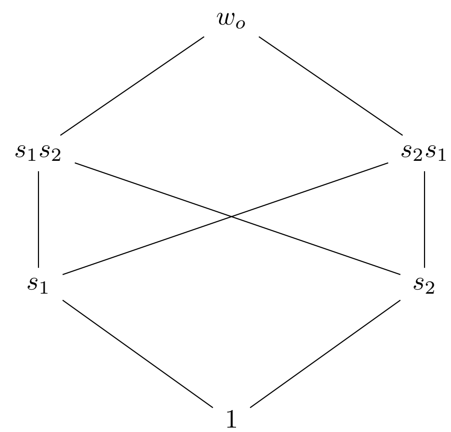
Definition 7.1.4. *We define the relation $v \sim_L y$ to mean both $y \leq_L v$ and $v \leq_L y$.The $\sim$ is an equivalence relation on W. The equivalence classes are called left cells of W.*

It's worth saying that the preorder $\leq_L$ gives a partial order on set of left cells.

Example 7.1.1. *Let $W = W(A_2) \cong S_3$, i.e*

$$W = \{1, s_1, s_2, s_1s_2, s_2s_1, w_o\}$$

where $w_o = s_1s_2s_1 = s_2s_1s_2$ the longest element of W. Now the Bruhat partial ordering gave a diagram



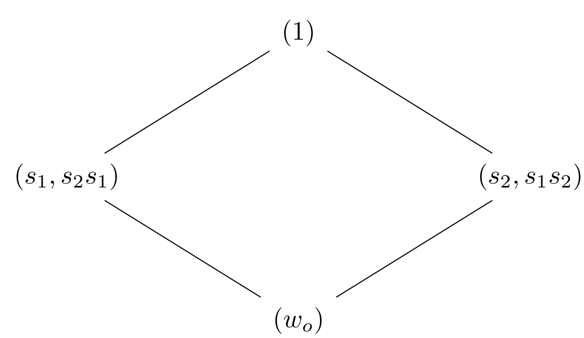
In this case $v \longleftrightarrow y \iff [v < y$ *and* $\ell(v) = \ell(y) - 1]$ *or* $[y < v$ *and* $\ell(y) = \ell(v) - 1]$*. So we get for the sets* $\mathcal{L}(w)$*, for each* $w \in W$*:*

$$\begin{aligned}
\mathcal{L}(1) &= \emptyset \\
\mathcal{L}(s_1) &= \{s_1\} \\
\mathcal{L}(s_2) &= \{s_2\} \\
\mathcal{L}(s_1s_2) &= \{s_1\} \\
\mathcal{L}(s_2s_1) &= \{s_2\} \\
\mathcal{L}(w_o) &= \{s_1, s_2\}
\end{aligned}$$

Now note that $w_o \leq_L s_1s_2$, $w_o \leq_L s_2s_1$
$s_1s_2 \leq_L s_2$, $s_2s_1 \leq_L s_1$
$s_1 \leq_L 1$, $s_2 \leq_L 1$
s_1s_2 *and* s_1 *are joined but* $\mathcal{L}(s_1s_2) = \mathcal{L}(s_1)$ *so* $s_1s_2 \not\leq_L s_1$
But we get $s_2 \leq_L s_1s_2$ *and* $s_1 \leq_L s_2s_1$
Thus the left cells of S_3 *are*

$$(1),\ (s_1, s_2s_1),\ (s_2, s_1s_2),\ (w_o)$$

and the ordering on these cells gives:



The partial order $\leq_L$ *of the left cells.*

Now, another useful way to store this information is by using the Kazhdan-Lusztig graph. So we give the following definition:

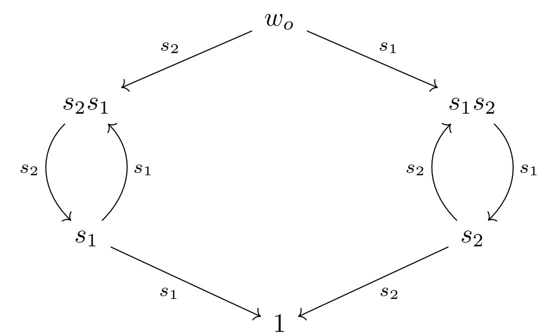
7.1.1 Kazhdan-Lusztig graph

Definition 7.1.5. *We define as Kazhdan-Lusztig graph, a biweighted directed multi-graph with:*

- *For each* $w \in W$ *we associate a vertex for the graph.*

- *And we label with an arrow $x \to w$, for an edge of the graph, if $x \leftrightarrow w$ and $\mathcal{L}(x) \nsubseteq \mathcal{L}(w)$. For such a pair (x,w) we put one arrow for each $s \in \mathcal{L}(x) - \mathcal{L}(w)$, and weight it with both s and $\mu(x,w)$ if $x \prec w$, or both s and $\mu(w,x)$ if $w \prec x$. In addition, to these arrows, we also put arrows $x \to x$ for each $s \in S$ weighted with s and 1 if $s \notin \mathcal{L}(x)$ and s and -1 if $s \in \mathcal{L}(x)$. Then the left cells corresponds to the vertices in each connected component of our graph.*

Example 7.1.2. *By applying the above definition again when the Coxeter group $W = W(A_2) \cong S_3$. We obtain another way to construct the left cells for W, from the corresponding graph. Below we present the Kazhdan-Lusztig graph for the Coxeter group $W = W(A_2) \cong S_3$, which by taking the connected components of it we lead us to the left cells of W.*



7.2 Left Cell Representations

Let Z be a left cell of W. Let I_Z be the A-submodule of $\mathcal{H}$ spanned by the C_v for every $v \in W$ such that $v \leq_L w$ for some $w \in Z$. (i.e all elements in some cell and all smaller ones)

Let $\overline{I}_Z$ be the A-submodule of $\mathcal{H}$ spanned by the C_v, $\forall v \in W$ s.t $v \leq_L w$ for some $w \in Z$ and $v \notin Z$. (i.e all elements in smaller cells but not in same one).

Lemma 7.2.1. *1. The A-module I_Z is a left ideal of $\mathcal{H}$.*

2. The A-module $\overline{I}_Z$ is a left ideal of $\mathcal{H}$.

Proof. 1. Let $w \in Z$, so $C_w \in I_Z$. Consider $C_s C_w$.
If $sw < w$ this is just a multiple of C_w
If $sw > w$ then $C_s C_w = C_{sw} + \sum_{z:\, z<w, sz<z} \mu(z,w) C_z$ and so under these circumstances we know $sw \leq_L w$ and $z \leq_L w$ when $\mu(z,w) \neq 0$. So $C_s C_w \in I_Z$.
So we get that $C_s I_Z \subseteq I_Z$ (we need those in lower cells too but the above argument holds for those too).
Thus,

$$\mathcal{H} I_Z \subseteq I_Z$$

and so I_Z is a left ideal of $\mathcal{H}$.

2. Now suppose $w \notin Z$ but $w \leq_L w'$ where $w' \in Z$. So $C_w \in \overline{I}_Z$.
Suppose $sw > w$, so $sw \leq_L w$, $z \leq_L w$ when $\mu(z,w) \neq 0$. But $sw \notin Z$, since if $sw \in Z$ then $sw \leq_L w'$ and $w' \leq_L sw$ so $sw \leq_L w \leq_L w' \leq_L sw \Longrightarrow w \sim_L w'$ (which is a contradiction).
Also $z \notin Z$ if $\mu(z,w) \neq 0$, since if $z \in Z$ then $z \leq_L w \leq_L w \leq_L w' \leq_L z$ (which is a contradiction).
So

$$\mathcal{H} \overline{I}_Z \subseteq \overline{I}_Z$$

and thus $\overline{I}_Z$ is also a left ideal of $\mathcal{H}$.

$\square$

Proposition 7.2.1. *It is clear that $\overline{I}_Z \subseteq I_Z$, so we can form the quotient left $\mathcal{H}-$ module*

$$\mathcal{M}_Z := {}^{I_Z}\!\big/\overline{I}_Z$$

$\mathcal{M}_Z$ *has a basis* $\overline{C_w}$, $w \in Z$ *(where* $\overline{C_w} = \overline{I}_Z + C_w$*)*

$$C_s\overline{C_v} = \begin{cases} \overline{C_{sv}} + \sum_{z\in Z:\ z<v, sz<z} \mu(z,v)\overline{C_z} & \text{if } sv > v \\ -(u^{\frac{1}{2}} + u^{-\frac{1}{2}})\overline{C_v} & \text{if } sv < v \end{cases}$$

$$T_s\overline{C_v} = \begin{cases} u^{\frac{1}{2}}\overline{C_s v} + u\overline{C_v} + u^{\frac{1}{2}} \sum_{z\in Z:\ z<v, sz<z} \mu(z,v)\overline{C_z} & \text{if } sv > v \\ -\overline{C_v} & \text{if } sv < v \end{cases}$$

where the term $\overline{C_{sv}}$ *in the above relations exists if* $sv \in Z$*, otherwise, if* $sv \notin Z$ *the term involving* $\overline{C_{sv}}$ *is omitted since then* $\overline{C_{sv}} \in \overline{I}_Z$ *and it gets factored out.*

Definition 7.2.1. *A left cell representation of the Hecke algebra* $\mathcal{H}$*, coming from the left cell Z, is the representation afforded by the the* $\mathcal{H}-$*module* $\mathcal{M}_Z$*.*

Example 7.2.1. *Let* $W = W(A_2) \cong S_3$*, we have seen that the left cells of W are*

$$(1),\ (s_1, s_2s_1),\ (s_2, s_1s_2),\ (w_o)$$

Then the 1-dimension representation arising from the left cell (1) *is given by:*

$$T_{s_1}\overline{C_1} = u\overline{C_1}, \quad T_{s_2}\overline{C_1} = u\overline{C_1} \tag{12}$$

The 1-dimension representation arising from the left cell (w_o) *is given by:*

$$T_{s_1}\overline{C_{w_o}} = -\overline{C_{w_o}}, \quad T_{s_2}\overline{C_{w_o}} = -\overline{C_{w_o}} \tag{13}$$

The 2-dimension representation arising from the left cell (s_1, s_2s_1) *is given by:*

$$\begin{aligned} T_{s_1}\overline{C_{s_1}} &= -\overline{C_{s_1}}, \quad T_{s_1}\overline{C_{s_2s_1}} = u^{\frac{1}{2}}\overline{C_{s_1}} + u\overline{C_{s_2s_1}} \\ T_{s_2}\overline{C_{s_1}} &= u\overline{C_{s_1}} + u^{\frac{1}{2}}\overline{C_{s_2s_1}}, \quad T_{s_2}\overline{C_{s_2s_1}} = -\overline{C_{s_2s_1}} \end{aligned} \tag{14}$$

The 2-dimension representation arising from the left cell (s_2, s_1s_2) *is given by:*

$$\begin{aligned} T_{s_1}\overline{C_{s_2}} &= u\overline{C_{s_2}} + u^{\frac{1}{2}}\overline{C_{s_1s_2}}, \quad T_{s_1}\overline{C_{s_1s_2}} = -\overline{C_{s_1s_2}} \\ T_{s_2}\overline{C_{s_2}} &= -\overline{C_{s_2}}, \quad T_{s_2}\overline{C_{s_1s_2}} = u^{\frac{1}{2}}\overline{C_{s_2}} + u\overline{C_{s_1s_2}} \end{aligned} \tag{15}$$

Then the matrix representations we get from these modules are:
(12) $T_{s_1} \mapsto (u)$, $T_{s_2} \mapsto (u)$
(13) $T_{s_1} \mapsto (-1)$, $T_{s_2} \mapsto (-1)$
(14) $T_{s_1} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{pmatrix}$, $T_{s_2} \mapsto \begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{pmatrix}$
(15) $T_{s_1} \mapsto \begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{pmatrix}$, $T_{s_2} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{pmatrix}$

The two representations of degree two are equivalent, since

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & -1 \end{pmatrix}$$

But the two 1-dimension representations aren't equivalent.

Note that if we replace u by q, where q is a prime power, we shall obtain representations of the ordinary Hecke algebra, $\mathcal{H}(q)$*, over* $\mathbb{Q}(q^{\frac{1}{2}})$

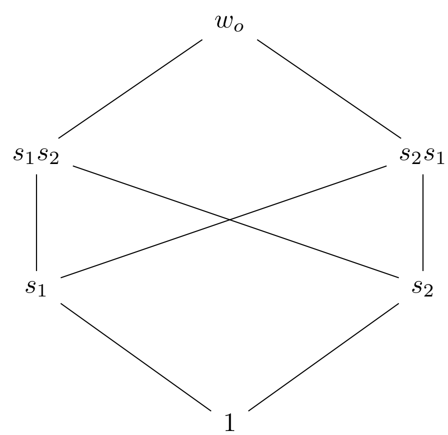
Also if we replace u by 1 we shall obtain representations of W over $\mathbb{Q}$*, since we get the group algebra.*

7.3 The Right Cells of a Coxeter group W

Definition 7.3.1. *Let $v, y \in W$ we define $y \leq_R v$ if exists a chain of elements such that $v = x_0, x_1, \cdots, x_r = y$ where $x_i \longleftrightarrow x_{i+1}$ and $\mathcal{R}(x_{i+1}) \not\subseteq \mathcal{R}(x_i)$, $\forall i$*

Definition 7.3.2. *$v \sim_R y$ if $v \leq_R y$ and $y \leq_R v$. The equivalence classes are the right cells.*

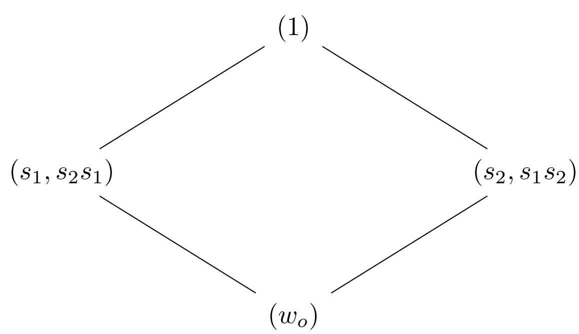
Example 7.3.1. *Let $W = W(A_2) \cong S_3$ the Bruhat partial ordering on S_3 is*



Now all Kazhdan-Lusztig polynomials are equal to 1, so only consecutive elements are joined. So right cells are

$$(1),\ (w_o),\ (s_1, s_1s_2),\ (s_2, s_2s_1)$$

and we get the ordering $\leq_R$ on W



7.4 The Two-sided Cells of a Coxeter group W

Definition 7.4.1. *We write $y \leq_{LR} v$ if there exists a sequence $v = x_0, x_1, \cdots, x_r = y$ where at each stage either $x_{i+1} \leq_L x_i$ or $x_{i+1} \leq_R x_i$ (Note this isn't the as saying either $y \leq_R$ or $y \leq_L v$)*

Definition 7.4.2. *$v \sim_{LR} y$ if $y \leq_{LR} v$ and $v \leq_{LR} y$.*
The equivalence classes are called two-sided cells.

Proposition 7.4.1.

1. *Suppose $x \leq_L y$. Then $\mathcal{R}(y) \subseteq \mathcal{R}(x)$.*
2. *Suppose $x \sim_L y$. Then $\mathcal{R}(x) = \mathcal{R}(y)$.*
3. *Suppose $x \leq_R y$. Then $\mathcal{L}(y) \subseteq \mathcal{L}(x)$.*
4. *Suppose $x \sim_R y$. Then $\mathcal{L}(x) = \mathcal{L}(y)$.*

Example 7.4.1. *The two-side cells of S_3 are* (1), $(s_1, s_2, s_1s_2, s_2s_1)$, (w_o)

Example 7.4.2. *Let $W = W(A_3) \cong S_4$ with dynkin diagram*

$$\overset{s_1}{\bullet} \;\text{———}\; \overset{s_2}{\bullet} \;\text{———}\; \overset{s_3}{\bullet}$$

$$W = < s_1, s_2, s_3 \mid {s_i}^2 = 1,\ (s_1s_2)^3 = 1 = (s_2s_3)^3,\ (s_1s_3)^2 = 1 >$$

and $|W| = 24$ In details the elements of W are:

$$\begin{gathered} 1,\ s_1,\ s_2,\ s_3,\ s_1s_2,\ s_2s_1,\ s_1s_3,\ s_2s_3,\ s_3s_2, \\ s_1s_2s_1,\ s_1s_2s_3,\ s_1s_3s_2,\ s_2s_1s_3,\ s_2s_3s_2,\ s_3s_2s_1, \\ s_1s_2s_1s_3,\ s_1s_2s_3s_2,\ s_1s_3s_2s_1,\ s_2s_1s_3s_2,\ s_2s_3s_2s_1, \\ s_1s_2s_1s_3s_2,\ s_1s_2s_3s_2s_1,\ s_2s_1s_3s_2s_1, \\ s_1s_2s_1s_3s_2s_1 \end{gathered}$$

For the Bruhat ordering, bearing in mind that $s_1s_3 = s_3s_1$ and $s_1s_2s_1 = s_2s_1s_2,\ s_2s_3s_2 = s_3s_2s_3$

w	$\mathcal{L}(w)$	$\mathcal{R}(w)$
1	$\emptyset$	$\emptyset$
s_1	s_1	s_1
s_2	s_2	s_2
s_3	s_3	s_3
s_1s_2	s_1	s_2
s_2s_1	s_2	s_1
s_1s_3	s_1, s_3	s_1, s_3
s_2s_3	s_2	s_3
s_3s_2	s_3	s_2
$s_1s_2s_1$	s_1, s_2	s_1, s_2
$s_1s_2s_3$	s_1	s_3
$s_1s_3s_2$	s_1, s_3	s_2
$s_2s_1s_3$	s_2	s_1, s_3
$s_2s_3s_2$	s_2, s_3	s_2, s_3
$s_3s_2s_1$	s_3	s_1
$s_1s_2s_1s_3$	s_1, s_2	s_1, s_3
$s_1s_2s_3s_2$	s_1, s_3	s_2, s_3
$s_1s_3s_2s_1$	s_1, s_3	s_1, s_2
$s_2s_1s_3s_2$	s_2	s_2
$s_2s_3s_2s_1$	s_2, s_3	s_1, s_3
$s_1s_2s_1s_3s_2$	s_1, s_2	s_2, s_3
$s_1s_2s_3s_2s_1$	s_1, s_3	s_1, s_3
$s_2s_1s_3s_2s_1$	s_2, s_3	s_1, s_2
$s_1s_2s_1s_3s_2s_1$	s_1, s_2, s_3	s_1, s_2, s_3

Now for the left cells of W we have :

Recall from the previous proposition that if $w \sim_L w'$ then $\mathcal{R}(w) = \mathcal{R}(w')$.

So we make the following list for the sets $\mathcal{R}(w)$:

w	(1)	$(s_1, s_2s_1, s_3s_2s_1)$	$(s_2, s_1s_2, s_3s_2, s_1s_3s_2, s_2s_1s_3s_2)$	$(s_3, s_2s_3, s_1s_2s_3)$
$\mathcal{R}(w)$	$\emptyset$	s_1	s_2	s_3

w	$(s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1)$	$(s_1s_3, s_2s_1s_3, s_1s_2s_1s_3, s_2s_3s_2s_1, s_1s_2s_3s_2s_1)$
$\mathcal{R}(w)$	s_1, s_2	s_1, s_3

w	$(s_2s_3s_2, s_1s_2s_3s_2, s_1s_2s_1s_3s_2)$	$(s_1s_2s_1s_3s_2s_1)$
$\mathcal{R}(w)$	s_2, s_3	s_1, s_2, s_3

These sets decompose into left cells.
Recall $y < w$ $\ell(w) = \ell(y) + 1 \implies y \longleftrightarrow w$. So for two elements to be in a left cell we must have $y \longleftrightarrow w$, $\mathcal{L}(y) \nsubseteq \mathcal{L}(w)$ and $\mathcal{L}(w) \nsubseteq \mathcal{L}(y)$. Thus by distinguish in the following cases we get:

(i) Let $\mathcal{R}(w) = s_1$:
- $s_1 < s_1s_2$ *so* $s_1 \longleftrightarrow s_2$. *Also* $\mathcal{L}(s_1) = s_1$, $\mathcal{L}(s_2s_1) = s_2$ *so* $s_1 \sim_L s_2s_1$.
- $s_2s_1 < s_3s_2s_1$, *so* $s_2s_1 \longleftrightarrow s_3s_2s_1$. *Also* $\mathcal{L}(s_2s_1) = s_2$, $\mathcal{L}(s_3s_2s_1) = s_3$ *so* $s_2s_1 \sim_L s_3s_2s_1$.

So in case all the elements $w \in W$ such that $\mathcal{R}(w) = s_1$, form one left cell. i.e the left cell $(s_1, s_2s_1, s_3s_2s_1)$.

(ii) Let $\mathcal{R}(w) = s_2$:
- $s_2 < s_1s_2$, $\mathcal{L}(s_2) = s_2$, $\mathcal{L}(s_1s_2) = s_1$ *so* $s_2 \sim_L s_1s_2$.
- $s_2 < s_3s_2$, $\mathcal{L}(s_2) = s_2$, $\mathcal{L}(s_3s_2) = s_3$ *so* $s_2 \sim_L s_3s_2$.
- $s_1s_2 < s_1s_3s_2$, $\mathcal{L}(s_1s_2) = s_1 \subseteq \mathcal{L}(s_1s_3s_2) = \{s_1, s_3\}$.
- $s_3s_2 < s_1s_3s_2$, $\mathcal{L}(s_3s_2) = s_3 \subseteq \mathcal{L}(s_1s_3s_2) = \{s_1, s_3\}$.
- $s_1s_3s_2 < s_2s_1s_3s_2$, $\mathcal{L}(s_1s_2s_3) = \{s_1, s_3\}$, $\mathcal{L}(s_2s_1s_3s_2) = s_2$ *so* $s_1s_3s_2 \sim_L s_2s_1s_3s_2$.

So in this case (i.e for the elements such that $\mathcal{R}(w) = s_2$) decomposes into two left cells (s_2, s_1s_2, s_3s_2) and $(s_1s_3s_2, s_2s_1s_3s_2)$

(iii) Let $\mathcal{R}(w) = s_3$:
- $s_3 < s_2s_3$, $\mathcal{L}(s_3) = s_3$, $\mathcal{L}(s_2s_3) = s_2$ *so* $s_3 \sim_L s_2s_3$.
- $s_2s_3 < s_1s_2s_3$, $\mathcal{L}(s_1s_2s_3) = s_1$ *so* $s_2s_3 \sim_L s_1s_2s_3$.

So in that case all elements are in one left cell, i.e form the left cell $(s_3, s_2s_3, s_1s_2s_3)$.

(iv) Let $\mathcal{R}(w) = \{s_1, s_2\}$:
- $s_1s_2s_1 < s_1s_3s_2s_1$, $\mathcal{L}(s_1s_2s_1) = \{s_1, s_2\}$, $\mathcal{L}(s_1s_3s_2s_1) = \{s_1, s_3\}$.
- $s_1s_3s_2s_1 < s_2s_1s_3s_2s_1$, $\mathcal{L}(s_2s_1s_3s_2s_1) = \{s_2, s_3\}$ *so* $s_1s_2s_1 \sim_L s_1s_3s_2s_1 \sim_L s_2s_1s_3s_2s_1$.

So in that case all elements are in one left cell, i.e form the left cell $(s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1)$.

(v) Let $\mathcal{R}(w) = \{s_1, s_3\}$:
- $s_1s_3 < s_2s_1s_3$, $\mathcal{L}(s_1s_3) = \{s_1, s_3\}$, $\mathcal{L}(s_2s_1s_3) = s_2$ *so* $s_1s_3 \sim_L s_2s_1s_3$.
- $s_2s_1s_3 < s_1s_2s_1s_3$, $\mathcal{L}(s_2s_1s_3) = s_2 \subseteq \mathcal{L}(s_1s_2s_1s_3) = \{s_1, s_2\}$.
- $s_2s_1s_3 < s_2s_3s_2s_1$. $\mathcal{L}(s_2s_3s_2s_1) = \{s_2, s_3\} \supseteq \mathcal{L}(s_2s_1s_3)$.
- $s_2s_3s_2s_1 < s_1s_2s_3s_2s_1$, $\mathcal{L}(s_1s_2s_3s_2s_1) = \{s_1, s_3\}$ *so* $s_2s_3s_2s_1 \sim_L s_1s_2s_3s_2s_1$.
- $s_1s_2s_1s_3 < s_1s_2s_3s_2s_1$ *so also* $s_1s_2s_1s_3 \sim_L s_1s_2s_3s_2s_1$.

So in this case (i.e for the elements such that $\mathcal{R}(w) = \{s_1, s_3\}$) decomposes into two left cells, $(s_1s_3, s_2s_1s_3)$ and $(s_1s_2s_1s_3, s_2s_3s_2s_1, s_1s_2s_3s_2s_1)$.

(vi) Let $\mathcal{R}(w) = \{s_2, s_3\}$:
- $s_2s_3s_2 < s_1s_2s_3s_2$, $\mathcal{L}(s_2s_3s_2) = \{s_2, s_3\}$, $\mathcal{L}(s_1s_2s_3s_2) = \{s_1, s_3\}$.
- $s_1s_2s_3s_2 < s_1s_2s_1s_3s_2$, $\mathcal{L}(s_1s_2s_1s_3s_2) = \{s_1, s_2\}$ so $s_2s_3s_2 \sim_L s_1s_2s_3s_2 \sim_L s_1s_2s_1s_3s_2$.
So in that case all elements are in one left cell, i.e form the left cell $(s_2s_3s_2, s_1s_2s_3s_2, s_1s_2s_1s_3s_2s_1)$.

So the left cells are

$$(1)$$
$$(s_1, s_2s_1, s_3s_2s_1)$$
$$(s_2, s_1s_2, s_3s_2)$$
$$(s_3, s_2s_3, s_1s_2s_3)$$
$$(s_1s_3s_2, s_2s_1s_3s_2)$$
$$(s_1s_3, s_2s_1s_3)$$
$$(s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1)$$
$$(s_1s_2s_1s_3, s_2s_3s_2s_1, s_1s_2s_3s_2s_1)$$
$$(s_2s_3s_2, s_1s_2s_3s_2, s_1s_2s_1s_3s_2)$$
$$(s_1s_2s_1s_3s_2s_1)$$

Note that $y \sim_L w \Longleftrightarrow y^{-1} \sim_R w^{-1}$. So we also know the Right cells:

$$(1)$$
$$(s_1, s_1s_2, s_1s_2s_3)$$
$$(s_2, s_2s_1, s_2s_3)$$
$$(s_3, s_3s_2, s_3s_2s_1)$$
$$(s_2s_1s_3, s_2s_1s_3s_2)$$
$$(s_1s_3, s_1s_3s_2)$$
$$(s_1s_2s_1, s_1s_2s_1s_3, s_1s_2s_1s_3s_2)$$
$$(s_1s_3s_2s_1, s_1s_2s_3s_2, s_1s_2s_3s_2s_1)$$
$$(s_2s_3s_2, s_2s_3s_2s_1, s_2s_1s_3s_2s_1)$$
$$(s_1s_2s_1s_3s_2s_1)$$

Then the Two-sided cells are:

$$(1)$$

$$\begin{pmatrix} s_1, & s_2s_1, & s_3s_2s_1, \\ s_2, & s_1s_2, & s_3s_2, \\ s_3, & s_2s_3, & s_1s_2s_3 \end{pmatrix}$$

$$\begin{pmatrix} s_1s_3s_2, & s_2s_1s_3s_2, \\ s_1s_3, & s_2s_1s_3 \end{pmatrix}$$

$$\begin{pmatrix} s_1s_2s_1, & s_1s_3s_2s_1, & s_2s_1s_3s_2s_1, \\ s_1s_2s_1s_3, & s_2s_3s_2s_1, & s_1s_2s_3s_2s_1, \\ s_2s_3s_2, & s_1s_2s_3s_1, & s_1s_2s_1s_3s_2 \end{pmatrix}$$

$$(s_1s_2s_1s_3s_2s_1)$$

Now we present some observations from the above :

(i) Consider those elements of order two, i.e $w^2 = 1$. Because of what we know about s_4 we know there are 10 of them. So we jave 10 elements satisfying $w^2 = 1$ in the left cells list.
Each left cell contains just one element with $w^2 = 1$, and each right cell contains just one element with $w^2 = 1$.

(ii) If L is a left cell and R is a right cell with L,R in the same two-sided cell then $|L \cap R| = 1$. (i.e consists only of $\{w\}$ where $w^2 = 1$).

(iii) The number of elements in each two-sided cell is a square.

(iv) The sum of the squares of the irreducible representations of W = $|W|$.

(v) Each of the two-sided cells gives rise to an irreducible representation.

All the above observations hold for symmetric groups in general, (but not for all Coxeter groups in general).

7.5 The Relationship between Cell Representations and the Classic Representation Theory of S_n

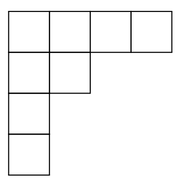
7.5.1 The Classical Representation Theory of S_n

For this section, we specialize to $W = W(A_{n-1}) \cong S_n$ and consider left cells, right cells and two-sided cells in S_n.
We will demonstrate the connection between the cells and the Young tableau. In this direction we quote the following definitions:

Definition 7.5.1 (Partition)**.** *Given a positive integer n, we define a partition λ of n to be a set of positive integers $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$ and $n = \lambda_1 + \cdots + \lambda_k$. We write for a partition λ of n, $\lambda \vdash n$.*

Definition 7.5.2 (Young Diagram)**.** *Each partition is associated with a Young diagram, a diagram of left justified boxes with k rows and λ_i boxes in each row, with the 1-st row being the one on the top and the k-th row being on the bottom.*

For example, a partition of 8 can be$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (4, 2, 1, 1)$ with the Young diagram



To any partition λ of n and its associated Young diagram, we say that a standard Young tableau of λ is a bijection between the boxes of the Young diagram and the natural numbers $\{1, \cdots, n\}$ such that each column of boxes increases from top to bottom and each row increases from left to right. So we have the following definition:

Definition 7.5.3 (Young Tableau)**.** *A $\lambda-$tableau is a $\lambda-$diagram filling it with the numbers $1, \cdots, n$ in the squares, such that each number appears once.*
A standard $\lambda-$tableau obtained by filling in the numbers in the increasing order along the rows and down the columns.

A standard tableau of the partition considered before is

1	3	5	8
2	4		
6			
7			

Definition 7.5.4. *We let SYT_λ to be the set of standard tableaux T of the partition $\lambda \vdash n$, also we let $SYT = \bigcup_{\lambda \vdash n} SYT_\lambda$, and a given tableau $T \in SYT_\lambda$, we define the descent set D(T) to be the set of $s_i \in \{s_1, \cdots, s_n - 1\}$ such that $i + 1$ appears in a strictly lower ro of T.*

Now, given a symmetric group S_n, the classical theory gives us that the irreducible representations can be indexed by the conjugacy classes of S_n. But these conjugacy classes can be indexed by partitions $\lambda \vdash n$: a conjugacy class is characterized by the size of its cycles. For example, the conjugacy class containing the trivial element has cycle sizes (1,1,...,1), and can be identified with the partition of n by $(1, 1, \cdots, 1)$. Therefore, by the representation theory of a finite group, the number of the inequivalent irreducible representations, over $\mathbb{C}$, is equal to the number of partitions of n. So from the above is clear, that we have 1-1 correspondence between the irreducible representations and the partitions of n, and hence there exists 1-1 correspondence of Young diagrams with irreducible representations.

Also notice that, the dimension of the irreducible representation ρ_λ of the symmetric group S_n corresponding to the the partition λ of n, have dimension $|SYT_\lambda|$, i.e is equal to the number of different standard Young tableaux that can be obtained from the diagram of the representation. This number can be calculated by the hook length formula. More precisely, the hook length of a certain box in Young diagram of shape λ, denoted by d_λ is the number of boxes that are in the same row to the right of it plus the boxes in the same column below it, plus one for the box itself. By the hook length formula we obtain that the dimension of the irreducible representation are given by

$$d_\lambda = dim\ \rho_\lambda = \frac{n!}{\text{product of all hooks lengths of boxes in the Young diagram}}$$

One way to transmute the irreducible representations is of what we called Specht modules. In particular, we have the following Theorem, from the combinatorial point of view in representation theory of S_n:

Theorem 7.5.1. *The subspace $V_\lambda := \mathbb{C}[S_n]c_\lambda$ of $\mathbb{C}[S_n]$ is an irreducible representation of S_n under left multiplication. Every irreducible representation of S_n is isomorphic to V_λ for a unique λ. For the definition of c_λ. We can define two subgroups of S_n corresponding to SYT_λ:*

1. *The row subgroup P_λ: the subgroup which maps every element of 1,...,n into an element standing in the same row in SYT_λ.*

2. *The column subgroup Q_λ: the subgroup which maps every element of 1,...,n into an element standing in the same column in SYT_λ.*

Then we define the Young projectors

$$a_\lambda := \frac{1}{|P_\lambda|} \sum_{g \in P_\lambda} g$$

and

$$b_\lambda := \frac{1}{|Q_\lambda|} \sum_{g \in Q_\lambda} sgn(g) g$$

and finally we let $c_\lambda = a_\lambda b_\lambda$

7.5.2 The Robinson-Schensted Correspondence

We now state the Robinson-Schensted Correspondence, along with some miraculous facts about it. Since the proofs of these theorems are largely combinatorial, we will leave them out due to space considerations.

Earlier we represented each $w \in S_n$ with reduced word expressions $w = s_1 \cdots s_r$. Another way to represent permutations, however, is by simply writing where each letter is sent. For example, we write $w = x_1 \cdots x_n$, where $w(i) = x_i$.

Now we associate to each $w \in S_n$ a pair (A(w), B(w)) of tableaux, where both A(w) and B(w) are tableaux of the same partition. To do this, let $w = x_1 \cdots x_n$, and let us construct (A(w), B(w)) recursively. Supposing that the (i-1)-th step has already been completed, the i-th step goes as follows:

1. Consider x_i and the A that has been constructed so far.

2. Compare x_i with the elements of the first row, from left to right.

3. If x_i is greater than all the elements of the row, create a box at the end of the row, and put x_i into it.

4. If not, then let the first box that x_i is less than have p in it. Put x_i in the box that p was in, and start this process over again considering p and now going to the second row.
5. Continue this process until there are no rows left.
6. In B, place a new box with i in it in the location that a new box was created in A.

Let us do an example: let $w = 43125 \in S_5$. The process goes as follows, with A on the left and B on the right.

$$\begin{array}{|c|}\hline 4 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline 3 \\ \hline 4 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$$

$$\begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline\end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$$

Example 7.5.1. $W \cong S_3$

w	$A(w)$	$B(w)$
$1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{array}{ccc} 1 & 2 & 3 \end{array}$	$\begin{array}{ccc} 1 & 2 & 3 \end{array}$
$s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{array}{cc} 1 & 3 \\ 2 \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 \end{array}$
$s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{array}{cc} 1 & 2 \\ 3 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 \end{array}$
$s_1 s_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{array}{cc} 1 & 2 \\ 3 \end{array}$	$\begin{array}{cc} 1 & 3 \\ 2 \end{array}$
$s_2 s_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{array}{cc} 1 & 3 \\ 2 \end{array}$	$\begin{array}{cc} 1 & 2 \\ 3 \end{array}$
$w_o = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$	$\begin{array}{c} 1 \\ 2 \\ 3 \end{array}$

All standard λ-tableaux appear. Some appear more than once.
Then we get, for a given $A(w)$ the sets

$$(1),\ (s_1, s_2 s_1),\ (s_2, s_1 s_2),\ (w_o)$$

and for given $B(w)$ the sets

$$(1),\ (s_1, s_1 s_2),\ (s_2, s_2 s_1),\ (w_o)$$

Also notice that $B(w) = A(w^{-1})$ and that the sets for the same $A(w)$ gives the left cells of S_3 and similarly for the same $B(w)$ we get the right cells of S_3. Furthermore, we have the following theorem.

Theorem 7.5.2. *For $W = W(A_{n-1}) \cong S_n$*

(i) w, w' *lie in the same left cell if and only if* $A(w) = A(w')$.

(ii) w, w' *lie in the same right cell if and only if* $B(w) = B(w')$.

(iii) w, w' *lie in the same two-sided cell if and only if* $A(w)$ *and* $A(w')$ *have the same shape.*

(iv) The maps $A : S_n \longrightarrow SYT$ *and* $B : S_n \longrightarrow STY$ *are surjective.*

(v) The map $S_n \longrightarrow \bigsqcup_{\lambda \vdash n}(SYT_\lambda \times SYT_\lambda)$ *defined by* $w \mapsto (A(w), B(w))$ *is bijective. (Note* $A(w)$ *and* $B(w)$ *always have the same shape).*

(vi) There is a bijection between 2-sided cells of S_n *and partitions of* n*. The 2-sided cell corresponding to* λ *has* ${d_\lambda}^2$ *elements,* $d_\lambda = |SYT_\lambda|$

(vii) There is a bijection between left cells of S_n *and standard tableaux. The number of elements in each left cell contained in the 2-sided cell corresponding to* λ *is* d_λ*. Similarly for the right cells.*

(viii) If L, R are a left cell and a right cell contained in the same 2-sided cell then $|L \cap R| = 1$

(ix) Each left cell contains just one element with $w^2 = 1$
Each right cell contains just one element with $w^2 = 1$

(x) Each left cell representation o S_n *is irreducible.*
Two left cell representations are equivalent if and only if the left cells lie in the same two-sided cell.

7.6 The Example of the Left Cell representations of S_4

Left cell representations of S_4

As we already have seen, there are 5 2-sided cells :

$$(1)$$

$$\begin{pmatrix} s_1, & s_2s_1, & s_3s_2s_1, \\ s_2, & s_1s_2, & s_3s_2, \\ s_3, & s_2s_3, & s_1s_2s_3 \end{pmatrix}$$

$$\begin{pmatrix} s_1s_3s_2, & s_2s_1s_3s_2, \\ s_1s_3, & s_2s_1s_3 \end{pmatrix}$$

$$\begin{pmatrix} s_1s_2s_1, & s_1s_3s_2s_1, & s_2s_1s_3s_2s_1, \\ s_1s_2s_1s_3, & s_2s_3s_2s_1, & s_1s_2s_3s_2s_1, \\ s_2s_3s_2, & s_1s_2s_3s_1, & s_1s_2s_1s_3s_2 \end{pmatrix}$$

$$(s_1s_2s_1s_3s_2s_1)$$

and the number of left cells in each 2-sided cell is respectively: (1,3,2,3,1)

Now take one left cell in 2-sided cell, e.g :

$$\begin{gathered} (1) \\ (s_1, s_2s_1, s_3s_2s_1) \\ (s_1s_3s_2, s_2s_1s_3s_2) \\ (s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1) \\ (w_o) \end{gathered}$$

Each left cell from above gives an irreducible representation of $\mathcal{H}$, and also with specialization by $u \mapsto 1$,we get representations of W.

So

- For the left cell (1):
$$T_{s_1} \mapsto (u),\ T_{s_2} \mapsto (u),\ T_{s_3} \mapsto (u)$$
This gives rise to the trivial representation of W by specialization $u \mapsto 1$.

- For the left cell $(s_1, s_2s_1, s_3s_2s_1)$:
$$T_{s_1} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} & 0 \\ 0 & u & 0 \\ 0 & 0 & u \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} u & 0 & 0 \\ u^{\frac{1}{2}} & -1 & u^{\frac{1}{2}} \\ 0 & 0 & u \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} u & 0 & 0 \\ 0 & u & 0 \\ 0 & u^{\frac{1}{2}} & -1 \end{pmatrix}$$

- For the left cell $(s_1s_3s_2, s_2s_1s_3s_2)$:
$$T_{s_1} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} u & 0 \\ u^{\frac{1}{2}} & 1 \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} \\ 0 & u \end{pmatrix}$$

- For the left cell $(s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1)$:
$$T_{s_1} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & u^{\frac{1}{2}} \\ 0 & 0 & u \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} -1 & u^{\frac{1}{2}} & 0 \\ 0 & u & 0 \\ 0 & u^{\frac{1}{2}} & -1 \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} u & 0 & 0 \\ u^{\frac{1}{2}} & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

- For the left cell (w_o):
$$T_{s_1} \mapsto (-1),\ T_{s_2} \mapsto (-1),\ T_{s_3} \mapsto (-1)$$

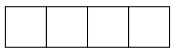
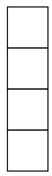
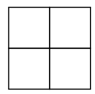
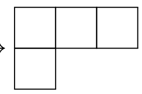
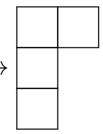
This is a complete set of irreducible nonequivalent representations.

Now we will use the classical representation theory of S_4 in order to construct the irreducible representations by characters, but retain the association with partitions from the combinatorial approach into the representations of S_4. After that, by specialising $u \mapsto 1$ we will get irreducible representations of S_4, but by using the methods of the left cells and the corresponding left representations that occurs. Finally, from the Robinson-Schensted Correspondence we will see that the irreducible representations, which have been constructed by the two different methods described above, are in reality the same. So

- **From the classical theory:** We have that the symmetric group S_4 has 5 conjugacy classes, $\{1\}$, $C(s_1s_3)$, $C(s_1)$, $C(s_1s_2s_3)$, $C(s_1s_2)$.Thus there are five irreducible representations. Three of these are common to every symmetric group : the trivial representation 1, the sign representation sgn, both of dimension 1 and the standard representation std of dimension 3 (obtained by the action of S_4 on the 3-dimensional subspace of vectors whose sum of coordinates in the basis is zero). Thus, as the sum of the dimensions squared must be the order of the group, we get the other two representations, let ρ and σ, which will be of dimension 2 and 3, respectively. And as the sum of all characters must be the character of the regular representation, we have the following character table:

	$\{1\}$	$C(s_1s_3)$	$C(s_1)$	$C(s_1s_2s_3)$	$C(s_1s_2)$
trivial	1	1	1	1	1
sign	1	1	-1	-1	1
ρ	2	2	0	0	-1
standard	3	-1	1	-1	0
σ	3	-1	-1	1	0

Now, in previous section we have seen the connection between the irreducible representations to the partitions and further to the Young diagrams. So we have that:

$1 \leftrightarrow$  sgn $\leftrightarrow$  $\rho \leftrightarrow$  std $\leftrightarrow$  $\sigma \leftrightarrow$ 

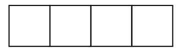
- **Left Cell Representations:** By specialising $u \mapsto 1$ into the previous relations, we get the left cell representations of S_4. More precisely, we obtain the following maps:
  - For the left cell (1):
  $$T_{s_1} \mapsto (1),\ T_{s_2} \mapsto (1),\ T_{s_3} \mapsto (1)$$
  This gives rise to the trivial representation of W.
  - For the left cell $(s_1, s_2s_1, s_3s_2s_1)$:
  $$T_{s_1} \mapsto \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$
  By simple calculations of the action of T into the elements of every conjugacy class of S_4, we get that the character of this cell has
  $$\chi(\{1\}) = 3,\ \chi(C(s_1s_3)) = -1,\ \chi(C(s_1)) = 1,\ \chi(C(s_1s_2s_3)) = -1,\ \chi(C(s_1s_2)) = 0$$
  Show by the character table of the classical theory, we conclude that this representation corresponds to standard representation.
  - For the left cell $(s_1s_3s_2, s_2s_1s_3s_2)$:
  $$T_{s_1} \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$$
  Again by calculations we get the character of this cell to be:
  $$\chi(\{1\}) = 2,\ \chi(C(s_1s_3)) = 2,\ \chi(C(s_1)) = 0,\ \chi(C(s_1s_2s_3)) = 0,\ \chi(C(s_1s_2)) = -1$$
  Show by the character table of the classical theory, we conclude that this representation corresponds to ρ.
  - For the left cell $(s_1s_2s_1, s_1s_3s_2s_1, s_2s_1s_3s_2s_1)$:
  $$T_{s_1} \mapsto \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},\ T_{s_2} \mapsto \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix},\ T_{s_3} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$
  Similarly to the previous cases we get that the character of this cell has
  $$\chi(\{1\}) = 3,\ \chi(C(s_1s_3)) = -1,\ \chi(C(s_1)) = -1,\ \chi(C(s_1s_2s_3)) = 1,\ \chi(C(s_1s_2)) = 0$$
  Show by the character table of the classical theory, we conclude that this representation corresponds to σ.

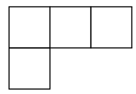
– For the left cell (w_o):

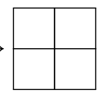
$$T_{s_1} \mapsto (-1),\ T_{s_2} \mapsto (-1),\ T_{s_3} \mapsto (-1)$$

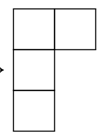
This gives rise to the sign representation of W.

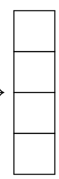
Now by taking a representative of each cell, we will find its tableau shape under the Robinson-Schensted Correspondence. Recall the notation we have introduced previous, $A(w) = A(x_1 \cdots x_n)$, where $w = x_1 \cdots x_n$ with $w(i) = x_i$. So we get:

$A(1) = A(1234) \leftrightarrow$ 

$A(s_1) = A(2134) \leftrightarrow$ 

$A(s_1 s_3 s_2) = A(2413) \leftrightarrow$ 

$A(s_1 s_2 s_1) = A(3214) \leftrightarrow$ 

$A(s_1 s_2 s_1 s_3 s_2 s_1) = A(4321) \leftrightarrow$ 

so indeed we have the same associations of representations to Young diagrams, as desired.

8 Tits Deformation Theorem

Theorem 8.0.1. *The Hecke algebra over $\mathbb{C}$ is isomorphic to the group algebra over $\mathbb{C}$ of the Weyl group. Specifically, let G be a finite group with BN-pair, and let (W,S) be the Weyl group of G. Let $\mathcal{H}$ be the Hecke algebra $H(G, B, 1_B)$. Then there exists an isomorphism of $\mathbb{C}$-algebras :*

$$\mathcal{H} \cong \mathbb{C}W$$

We consider homomorphisms : $\mathbb{C}[u] \longrightarrow \mathbb{C}$ such that $f(u) \mapsto f(t)$ if $u \mapsto t$. These homomorphisms are called specializations.

Proposition 8.0.1. *Let $\mathcal{E}_{\mathbb{C}}(u)$ be the generic algebra of a finite Coxeter System (W, S), over $\mathbb{C}[u]$.*

(i) Assume that we have in the way that described above a specialization $f : \mathbb{C}[u] \longrightarrow \mathbb{C}$ such that $u \mapsto 1$. Then the generic algebra specialized to group algebra over $\mathbb{C}$, i.e

$$\mathcal{E}_{\mathbb{C}}(1) \cong \mathbb{C}W$$

as $\mathbb{C}$-algebras.

(ii) Assume that W is the Weyl group of a finite group G with BN-pair, with parameter q. We consider the specialization $f' : \mathbb{C}[u] \longrightarrow \mathbb{C}$ such that $u \mapsto q$. Then the generic algebra $\mathcal{E}_{\mathbb{C}}(q)$ specialized to the Hecke algebra $H(G, B, 1_B)$, i.e

$$\mathcal{E}_{\mathbb{C}}(q) \cong H(G, B, 1_B)$$

as $\mathbb{C}$-algebras.

Lemma 8.0.1. *The $\mathcal{E}_{\mathbb{C}}(u)$ is semisimple when u specializes to q, and also when specializes to 1 and for all but a finite number of values of u.*

Proof. Let $\mathcal{E}$ be a finite dimensional algebra over a field and $a \in \mathcal{E}$. Consider the map $p(a) : \mathcal{E} \longrightarrow \mathcal{E}$ such that $x \mapsto xa$. We also consider the map $T : \mathcal{E} \times \mathcal{E} \longrightarrow$ field such that $(a, b) \mapsto T(a, b)$, where

$$T(a, b) = trace(p(a), p(b))$$

T called the trace form of the algebra $\mathcal{E}$, it is a symmetric bilinear form and we say that

$$T \text{ is non-degenerate if and only if } T(a, b) = 0,\ \forall b \in \mathcal{E},\ \text{implies } a = 0$$

The discriminant of the form with respect to a given basis is the determinant of the matrix of the form with respect to this basis of $\mathcal{E}$. So if $e_1, \cdots, e_n$ is a basis of $\mathcal{E}$ then, discriminant $= det(T(e_i, e_j))$ It can be shown that

$$T \text{ is non-degenerate if and only if the discriminant is different from zero}$$

So when we specialize $u \mapsto 1$, from the above the generic alg □

Recall that a finite dimensional semisimple algebra over an algebraically closed field is a direct sum of a complete matrix algebras of a certain degrees over the field.

Definition 8.0.1.

- *We call a finite dimensional associative algebra $\mathcal{S}$ separable, if it is semisimple, when the based field is extended to it's algebraic closure.*
- *The degrees of the resulting matrix algebras called numerical invariants of $\mathcal{S}$.*

Proposition 8.0.2. *Let B be an associative finite dimensional simple algebra over an algebraically closed field L. So B is the direct sum of complete matrix algebras of certain degrees over L, say $d_1, \cdots, d_r$. Let $b_1, \cdots, b_n$ be a basis for B over L and $x_1, \cdots, x_n$ independent indeterminates over L. Consider $B_{L(x_1,\cdots,x_n)}$ and let $b \in B_{L(x_1,\cdots,x_n)}$ be the element*

$$b = \sum_i x_i b_i$$

where $B_{L(x_1,\cdots,x_n)} = \bigoplus L(x_1,\cdots,x_n)b_i$. *We call* b *a general element of* B. *Let also* $B_{L(x_1,\cdots,x_n)} \longrightarrow B_{L(x_1,\cdots,x_n)}$ *be the transformation such that* $z \mapsto bz$ *and* $P(t)$ *the characteristic polynomial of this transformation with coefficients in* $L(x_1,\cdots,x_n)$.
Let

$$P(t) = \prod P_i(t)^{p_i}$$

the factorization of $P(t)$ *into distinct monic irreducible polynomials over* $L(x_1,\cdots,x_n)$. *Then*

- *(i) The multiplicities* $\{p_i\}$ *are the numerical invariants of* B.
- *(ii)* $p_i = degP_i(t)$, *for all* i.
- *(iii) If* $P(t) = \prod Q_j(t)^{q_j}$ *be another factorization of* $P(t)$ *over* $L(x_1,\cdots,x_n)$ *such that* $q_j = degQ_j$, *for all* j. *Then the polynomials* $Q_j(t)$ *are distinct and coincide with the polynomials* $P_i(t)$. *So* q_j *are the numerical invariants of* B.

Proof. First, the statements of the proposition are independent of the choice of the basis and the set of indeterminates used to define a general element. So we let E^k_{ij} be the basis of elementary matrices for B where $k = 1,\cdots,r$, $i,j = 1,2,\cdots,d_k$. By denoting the indeterminates $\{x_{ij}\}$, the general element $b \in B$ is represented by the matrix

$$b = \sum_{i',j',k'} x^{k'}_{i'j'} E^{k'}_{i'j'}$$

. So when $E^k_{ij} \mapsto bE^k_{ij}$ we have that

$$bE^k_{ij} = \sum_{i',j',k'} x^{k'}_{i'j'} E^{k'}_{i'j'} E^k_{ij} = \sum_{i'} x^k_{i'i} E^k_{i'i} E^k_{ij} = \sum_{i'} x^k_{i'i} E^k_{i'j}$$

Thus the characteristic polynomial with respect to this basis is

$$\prod_{k=1}^{r} \left(det(tI - x^{(k)})\right)^{d_k}$$

where $x^{(k)} = [x^k_{ij}]$.

The two factorization of the $P(t)$ are the same. For $det(tI - x^{(k)})$ is irreducible, by specializing

$$x^{(k)} \mapsto \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ c_1 & c_2 & \cdots & c_{d_k} \end{bmatrix}$$

Then $det(tI - x^{(k)})$ specializes to $t^{d_k} - c_{d_k} t^{d_k - 1} - \cdots - c_2 t - c_1$. By proper choose of the $c_1,\cdots,c_{d_k}$ we obtain a specialization for the characteristic polynomial, that is given by $t^{d_k} - c_1$. But this polynomial is irreducible by Einstein's criterion, so it follows that the original polynomial is also irreducible. Moreover if $k \neq k'$ then $det(tI - x^{(k)}) \neq det(tI - x^{(k')})$ because their coefficients involve different indeterminates.

So the d_i's coincide with the p_i's and d_k =deg of $det(tI - x^{(k)})$. So $p_i = degP_i(t)$. Finally, let

$$P(t) = \prod P_i(t)^{p_i} = \prod Q_j(t)^{q_j}$$

where the first factorization is as given above, and where $q_j = detQ_j(t)$, $\forall j$. If some Q_j is equal to some Q_k, for $k \neq j$, then for the Q_j we would have a contradiction to (ii). Now, from the fact that the polynomials P_i are distinct irreducible polynomials, since they involve different indeterminates, we have that the factors $P_i(t)$ shows up exactly p_i times in the polynomial $P(t)$. So if $Q_j(t) \neq P_i(t)$, we would have that the polynomial $P_i(t)$ would occur more than p_i times in $P(t)$ which is a contradiction. Hence the polynomials $\{Q_j\}$ are a permutation of the $\{P_i\}$. □

Also later we will need the following property that is true for the separable algebras:
Two separable algebras over a field K are isomorphic if and only if in a extension $\overline{K}$ of the field, they have the same numerical invariants.

Lemma 8.0.2. *Let $R^\star$ be the integral closure of R in $\overline{F}$ and $x_1, \cdots, x_n$ indeterminates over $\overline{F}$. Then $R^\star[x_1, \cdots, x_n]$ is the integral closure of $R[x_1, \cdots, x_n]$ in $\overline{F}[x_1, \cdots, x_n]$.*

Lemma 8.0.3. *Let $R^\star$ as previous. Then (by using Zorn's Lemma) any homomorphism $f : R \longrightarrow K$, where K field, can be extended to a homomorphism $f^\star : R^\star \longrightarrow K^\star$, where $K^\star$ is an algebraic closure of K.*

Theorem 8.0.2 (Tits Deformation Theorem)**.** *Let R be an integral domain, F it's field of fractions, and $f : R \longrightarrow K$ be a homomorphism of R into a field K. Let $\mathcal{S}$ be a finite dimensional associative R-algebra, and let $\mathcal{S}_F$ and $\mathcal{S}_K$ be the resulting specialized algebras over F and K, respectively. If both $\mathcal{S}_F$ and $\mathcal{S}_K$ are separable, then they have the same numerical invariants.*

Sketch of the proof. Let $\{a_i\}$ be a basis for $\mathcal{S}$ over R, hence also for $\mathcal{S}_F$ over F. Let $\{x_i\}$ be independent indeterminates over $\overline{F}$ and also over $\overline{K}$. The given homomorphism $f : R \longrightarrow K$ by the above Lemma can be extended to a homomorphism $f^\star : R^\star \longrightarrow \overline{K}$ and then naturally to a homomorphism, also denoted by $f^\star$, from $R^\star[x_1, \cdots, x_n]$ into $\overline{K}[x_1, \cdots, x_n]$. Now if $a = \sum_i x_i a_i$ is a general element of the algebra $\mathcal{S}_{\overline{F}(x_1,\cdots,x_n)}$, we consider the map $z \mapsto az$ of $\mathcal{S}_{\overline{F}(x_1,\cdots,x_n)}$ into itself. Let $P(t) = \prod P_i(t)^{p_i}$ be the characteristic polynomial of the transformation above, where the $P_i(t)$ are its irreducible factors over $\overline{F}[x_1, \cdots, x_n]$. Then, by the fact that $P(t)$ is monic, its roots are integral over $R[x_1, \cdots, x_n]$. Then the coefficients of the factors $P_i(t)$, lie in the field generated by the roots of $P_i(t)$. In particular each root of $P_i(t)$ is also a root of $P(t)$, and the roots of $P(t)$ are integral over $R[x_1, \cdots, x_n]$. Thus the coefficients of each $P_i(t)$ are integral over $R[x_1, \cdots, x_n]$ hence belong to $R^\star[x_1, \cdots, x_n]$ by the previous Lemma. Now we can apply the homomorphism $f^\star$ to $P(t)$. We specializes $P(t)$ to the characteristic polynomial of the map $z \mapsto f(a)z$ of $\mathcal{S}_{\overline{K}(x_1,\cdots,x_n)}$ into itself. Then we obtain

$$f^\star(P(t)) = \prod f^\star(P_i(t))^{p_i}$$

over $\overline{K}[x_1, \cdots, x_n]$, i.e each polynomial $f^\star(P_i(t))$ are over $\overline{K}[x_1, \cdots, x_n]$. Now for each i, we have that $p_i = degP_i(t)$ so by (i) of the above proposition are the numerical invariants of $\mathcal{S}_F$ and by (ii) from the same proposition, we also have $p_i = deg f^\star(P_i(t))$, for every i. So finally, by the (iii) of the proposition, we obtain that the multiplicities p_i are also the numerical invariants of $\mathcal{S}_K$. So the theorem is proved. □

As an application of the above we have the main result that we stated in the beginning of this paragraph. Particularly we have the below theorem:

Theorem 8.0.3. *Let G be a finite group with BN-pair, and let (W,S) be the Weyl group of G. Let $\mathcal{H}$ be the Hecke algebra $H(G, B, 1_B)$. Then there exists an $\mathbb{C}$-algebras isomorphism:*

$$\mathcal{H} \cong \mathbb{C}W$$

Proof. Let $R = \mathbb{C}[u]$, so $F = \mathbb{C}[u]$. We take $\mathcal{S}$ to be the generic algebra over $\mathbb{C}[u]$. Then by the specialization $f : u \mapsto q$ we take $\mathcal{S}_{\mathbb{C}(q)} \cong \mathcal{H}$, and by the specialization $f : u \mapsto 1$ we take $\mathcal{S}_{\mathbb{C}[1]} \cong \mathbb{C}W$. Both algebras $\mathcal{H}$ and $\mathbb{C}W$ are separable, so from the above theorem they have the same numerical invariants. Now, since $\mathbb{C}$ is algebraically closed we get that $\mathcal{H} \cong \mathbb{C}W$. □

Corollary 8.0.1. *Let G be a finite group with BN-pair and with $|B : n_i B {n_i}^{-1} \cap B| = q$ for every i. Then there is a 1-1 correspondence between irreducible components of 1_B^G(over $\mathbb{C}$) and irreducible characters Φ of W. The multiplicity of a component in 1_B^G is the degree $\Phi(1)$ of Φ.*

Example 8.0.1. *Let $G = GL_n(q)$ and $W = S_n$. The irreducible characters of S_n correspond to partitions $\lambda \vdash n$. Let Φ_λ be an irreducible character of S_n. Then*

$$\Phi_\lambda(1) = \textit{ Number of standard Young tableaux of } \lambda$$

By the discussion we have already done in the section of the classical approach into the representation theory of S_n we get that

$$\Phi_\lambda(1) = dim\rho_\lambda = \frac{n!}{\textit{Product of all hooks lengths of boxes in Young diagram}}$$

e.g for the symmetric group on 3 letters, i.e for S_3, we have the following correspondence between the SYT_λ with hook length entries and the degrees of irreducible representations :

(i) the SYT_λ with hook length entries $\begin{array}{|c|c|c|}\hline 3 & 2 & 1 \\ \hline\end{array}$ *gives rise to irreducible representation of degree 1.*

(ii) the SYT_λ with hook length entries $\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$ *gives rise to irreducible representation of degree 2.*

(iii) the SYT_λ with hook length entries $\begin{array}{|c|}\hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}$ *gives rise to irreducible representation of degree 1.*

Now back to $GL_n(q)$.
Let χ_λ be the component of 1_B^G corresponding to λ. χ_λ appears with multiplicity $\Phi_\lambda(1) = \frac{n!}{\prod h_\lambda(i,j)}$, where $h_\lambda(i,j)$ be the hook length for each box (i,j). So we get that

$$\chi_\lambda(1) = q^{\lambda_2+2\lambda_3+3\lambda_4+\cdots} \cdot \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q^{h_1}-1)(q^{h_2}-1)\cdots(q^{h_n}-1)}$$

As previous let see for example $GL_3(q)$. Then we have:

1. *For $\lambda-$tableau* $\begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$ *we get $\chi_\lambda(1) = 1$, since in this case $\chi_\lambda(1) = q^0 \cdot \frac{(q-1)(q^2-1)(q^3-1)}{(q^3-1)(q^2-1)(q-1)}$*
2. *For $\lambda-$tableau* $\begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ *we get $\chi_\lambda(1) = q(q+1)$, since in this case $\chi_\lambda(1) = q^1 \cdot \frac{(q-1)(q^2-1)(q^3-1)}{(q^3-1)(q^2-1)(q-1)}$*
3. *For $\lambda-$tableau* $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$ *we get $\chi_\lambda(1) = q^3$, since in this case $\chi_\lambda(1) = q^{1+2} \cdot \frac{(q-1)(q^2-1)(q^3-1)}{(q^3-1)(q^2-1)(q-1)}$*

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