Master's Thesis

The Almost Splitting theorem and some aspects of Cheeger-Colding theory

Daniel Magdalas

Supervisor:

Panagiotis Gianniotis

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Master Thesis Examination Committee

Nicholas Alikakos, Emeritus Professor, Department of Mathematics, NKUA

Panagiotis Gianniotis, Assistant Professor, Department of Mathematics, NKUA

Antonis Melas, Professor, Department of Mathematics, NKUA

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Chapter 1

Introduction

One of the major objects of study in Riemannian geometry is the relationship between curvature, structure, and the topology of a space. Topogonov proved that if a manifold $\mathcal{M}$ has positive sectional curvature $K \geq 0$ and contains a line i.e a doubly infinite geodesic, where each finite segment is minimal, then $\mathcal{M}^n = \mathbb{R}^k \times \mathcal{N}^{n-k}$, where $\mathcal{N}$ contains no lines. This theorem is called the Splitting Theorem and describes how the condition of positive curvature contradicts the condition of being "highly" non compact and that they can coexist only in the case where space contains an $\mathbb{R}^k-$ factor.

Now it's natural to ask what happens in the case of bounds on Ricci curvature. Initially an upper bound on Ricci curvature has no topological obstructions by Lokham, so we are mainly interested for lower Ricci curvature bounds. If $\text{Ric} \geq \delta > 0$, then the Bonnet-Myers theorem, which states that $\mathcal{M}^n$ has to be compact and isometric with the sphere in the rigid case holds. If $\text{Ric} \geq 0$, then Cheeger and Gromoll proved that the Splitting theorem holds and following this proof, an observation was made:

For a complete manifold $\mathcal{M}^n$ that satisfies a lower Ricci curvature bound (say $\text{Ric} \geq -(n-1)$) bound and contains a point within a minimal segment, after rescaling the metric $r^{-2}g$ (where $r << 1$), there in a small ball near the point appears to be a kind of splitting. In particular, if $\gamma : [-1, 1] \to \mathcal{M}^n$ is the geodesic then the rescaled ball $B_r(\gamma(0))$ looks as if it were a ball in some isometric product $\mathbb{R} \times \mathcal{N}$. Indeed, in the 1990s, Jeff Cheeger and Tobias Colding [ChCo1], building on this observation, proved a quantitative version of this theorem—the Almost Splitting Theorem. While the idea of this generalization comes naturally, the proof is much more challenging and requires new techniques and tools. Specifically, in the classic theorem, a distance function called the Busemann function b_+ is constructed and is also harmonic. Then, using Bochner's formula one obtains a parallel field on the manifold . However, in the generalized case, the function b_+ is not harmonic, and instead, we consider a harmonic function β_+ that closely approximates b_+. Using Bochner's formula and a special function named cutoff function, we can show that the Hessian of β_+ is small in the L^2 sense. Finally, applying a new method named segment inequality developed by Cheeger and Colding we can derive information about this function along minimal geodesics, thus proving the desired theorem. Namely, we demonstrate the theorem :

Let $\mathcal{M}^n$ be complete p, q_1, q_2 and

$$\overline{p, q_1} + \overline{p, q_2} - \overline{q_1, q_2} < \delta \ ,$$

$$\overline{p, q_1} \geq \delta^{-1} \ , \ \overline{p, q_2} \geq \delta^{-1} \ .$$

Also assume that on $B_{\delta^{-1}}(p)$

$$\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)\delta \ .$$

Given $\varepsilon > 0, \ R < \infty$, there exists $\delta(\varepsilon, R) > 0$ such that if the above bounds hold with $\delta(\varepsilon, R)$, then there exist N^{n-1} and

$$\underline{p} \in \mathbb{R} \times \mathcal{N}^{n-1}$$

such that

$$d_{GH}(B_R(p), B_R(\underline{p})) \leq \varepsilon \ .$$

Another significant result from the study of manifolds with a lower bound on Ricci curvature is that a sequence of such compact manifolds, with respect to a suitable topology—namely, the Gromov-Hausdorff topology—has a convergent subsequence. That is, sequences with a lower Ricci bound and compactness are precompact in the Gromov-Hausdorff topology. We deonte the limit above with Y and it's called a limit space . Using this theorem, we can now study how the spaces that arise as limits of such sequences behave. The behavior of the limit spaces is very important information.

For instance, if we want to show that a sequence with a lower Ricci bound does not satisfy a property, we assume the property holds and demonstrate that this reflects some " bad behaviour " in the limit, thus leading to a contradiction. Therefore, the study of the space Y that arises as a limit is of great interest on its own. Initially, we can easily observe that Y is not necessarily a manifold; its dimension may differ from the manifolds in the sequence, and we also know that it is a length space. Since Y is not necessarily a manifold, it makes sense to study its local structure . Using again Gromov's Compactness theorem, we observe that for a sequence $r_j \to 0$ the subsequential limit (Y, r_j^{-1}, y) exists and its called tangent cone of Y at a point y and it's denoted with Y_y. This cone depends on the choice of r_j.

Continuing the study of Y and its cones', we consider the "bad points" of Y as the singular set $\mathcal{S}$-the points where the cones are not Euclidean spaces.

Using the Arzelà-Ascoli theorem and Carathéodory's theorem, it is shown that there exists a measure ν(ν need not be unique in general) on the space Y which satisfies $\nu(S) = 0$.

Finally assuming that the space Y is non-collapsed (i.e. the dimension is the same as the dimension of the manifolds in the sequence), so $\mathrm{Vol}(B_1(p_i)) \geq v > 0$, it is proven that the cones over each point are metric cones $C(Z)$, where Z is a length space that satisfies $\mathrm{diam}(Z) \leq \pi$ and it is shown that the Hausdorff dimension of $\mathcal{S}$ satisfies $\dim(\mathcal{S}) \leq n - 2$, which is optimal.

The purpose of this work is to present some of the above concepts. Specifically, we demonstrate, by constructing the appropriate theory, the proof of the Almost Splitting Theorem and provide the key points for proving that the dimension of the singular part is less than or equal to $n - 2$.

Chapter 2

Riemannian geometry basics

2.1 Bochner's formula

Theorem 2.1.1. *Let* $(\mathcal{M}, g)$ *be a Riemannian manifold and* $u \in C^\infty(\mathcal{M})$. *Then,*

$$\frac{1}{2}\,\Delta|\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla(\Delta u)\rangle + \mathrm{Ric}(\nabla u, \nabla u). \tag{2.1}$$

Proof. We will prove the statement using local geodesic frames . Let $p \in \mathcal{M}$ and E_i be an orthogonal frame in a neighborhood of p such that $\nabla_{E_i} E_j(p) = 0$ for all i, j. Then at p,

$$\frac{1}{2}\,\Delta|\nabla u|^2 = \frac{1}{2}\sum_{i=1}^{n} E_i\; E_i\langle \nabla u, \nabla u\rangle$$

We can calculate inside terms of the sum on the right side above to get

$$\frac{1}{2} E_i E_i \langle \nabla u, \nabla u\rangle = E_i \langle \nabla_{E_i} \nabla u, \nabla u\rangle = E_i \mathrm{Hess}_u(E_i, \nabla u) =$$

$$= E_i \mathrm{Hess}_u(\nabla u, E_i) = E_i \langle \nabla_{\nabla u} \nabla u, E_i\rangle = \langle \nabla_{E_i} \nabla_{\nabla u} \nabla u, E_i\rangle$$

$$= \langle E_i, R(E_i, \nabla u)\nabla u\rangle + \langle E_i, \nabla_{[E_i, \nabla u]} \nabla u\rangle + \langle E_i, \nabla u \nabla_{\nabla u} \nabla_{E_i}\rangle\;.$$

Where in the last equality we used the definition of the Riemannian curvature tensor, particularly,

$$\nabla_{E_i} \nabla_{\nabla u} \nabla u = R(E_i, \nabla u)\nabla u + \nabla_{[E_i, \nabla u]} \nabla u + \nabla_{\nabla u} \nabla_{E_i} \nabla u$$

so taking inner product with E_i

$$\langle \nabla_{E_i} \nabla_{\nabla u} \nabla u, E_i \rangle = \langle E_i, R(E_i, \nabla u)\nabla u \rangle + \langle E_i, \nabla_{[E_i,\nabla u]} \nabla u \rangle + \langle E_i, \nabla u \nabla_{\nabla u} \nabla_{E_i} \rangle \ .$$

Summing over i we conclude that

$$\frac{1}{2}\ \Delta|\nabla u|^2 = \sum_{i=1}^{n} \langle E_i, R(E_i, \nabla u)\nabla u \rangle + \langle E_i, \nabla_{[E_i,\nabla u]} \nabla u \rangle + \langle E_i, \nabla u \nabla_{\nabla u} \nabla_{E_i} \rangle \ .$$

We observe that the first term is the Ricci curvature (the trace of the riemmanian curvature tensor).

For the second term we have

$$\begin{aligned} \sum_{i=1}^{n} \langle E_i, \nabla_{[E_i,\nabla u]} \nabla u \rangle &= \sum_{i=1}^{n} \mathrm{Hess}_u([E_i, \nabla u], E_i) = \sum_{i=1}^{n} \mathrm{Hess}_u(E_i), \nabla_{E_i \nabla u}) \\ &= \sum_{i=1}^{n} \langle \nabla_{E_i} \nabla u, \nabla_{E_i} \nabla u \rangle = |\mathrm{Hess}_u|^2 \end{aligned}$$

(also note that the last term is the Hilbert-Schmidt norm). Finally in the last term of the equality it is

$$\nabla u \langle \nabla_{E_i} \nabla u, E_i \rangle = \langle \nabla_{\nabla u} \nabla_{E_i} \nabla u, E_i \rangle + \langle \nabla_{E_i} \nabla u, \nabla_{\nabla u} E_i \rangle \ .$$

Summing up over i, at x it is :

$$\begin{aligned} \sum_{i=1}^{n} \langle \nabla_{\nabla u} \nabla_{E_i} \nabla u, E_i \rangle = \sum_{i=1}^{n} [\nabla u \langle \nabla_{E_i} \nabla u, E_i \rangle - \langle \nabla_{E_i} \nabla u, \nabla_{\nabla u} E_i \rangle] \\ = \nabla u(\Delta u) = \langle \nabla u, \nabla(\Delta u) \rangle \end{aligned}$$

Combining the calculations above we get Bochner's formula i.e

$$\frac{1}{2}\ \Delta|\nabla u|^2 = |\mathrm{Hess}_u|^2 + \langle \nabla u, \nabla(\Delta u) \rangle + \mathrm{Ric}(\nabla u, \nabla u)$$

and the proof is complete. □

We can use Bochner's formula to derive a characterization for Ricci curvature lower bounds. Applying the Cauchy-Schwarz inequality we obtain

$$|\mathrm{Hess}_u|^2 \geq \frac{(\Delta u)^2}{n} \ ,$$

so

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u) \rangle + \mathrm{Ric}(\nabla u, \nabla u) \ .$$

If we assume as usually that $\mathrm{Ric} \geq (n-1)k$, the last inequality yields

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \frac{(\Delta u)^2}{n} + \langle \nabla u, \nabla(\Delta u)\rangle + k(n-1)|\nabla u|^2 \ .$$

The last inequality is the aforementioned characterization for Ricci curvature lower bounds. There are many more equivalent conditions and for every new condition a whole theory appears (see [Wei] 1.6).

The equation above is quite useful for a specific class of functions, like harmonic or distance functions. For example, using the formula we can prove the existence of a map such that $\mathrm{Hess}_u = 0$ and we can conclude the classic splitting theorem using the de Rham decomposition. In particular for a sketch of the proof we have the following ideas : A geodesic line γ on a complete Riemannian manifold is a geodesic $\gamma : (-\infty, +\infty) \to \mathcal{M}$ such that

$$d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|, \ t_1, t_2 \in \mathbb{R} \ .$$

A geodesic defined on $[0, \infty)$ is called a (geodesic) ray .

Now suppose γ is a ray on $\mathcal{M}$. We can define the Busemann function as follows

$$b^+ = \lim_{t\to\infty} \left(d(x, \gamma(t)) - t\right) \ .$$

Using Rademacher's theorerm (Because b^+ is Lipschitz continuous), we can prove that b^+ is almost everywhere differentiable. Now, having a geodesic line allows us to define b^- with a similar way (by taking the limit as $t \to -\infty$).

But it is easy to see that

$$\Delta(b^+ + b^-) \leq 0$$

in a weaker sense , and applying the mean value inequality yields

$$\Delta b^+ = \Delta b^- = 0$$

and classic elliptic PDE theory guarantees that these maps are smooth. Finally, by Bochner's formula it follows that ∇b^+ is a parallel field. Now, by the de Rham decomposition with the flow generated by ∇b^+ we can construct an isometric diffeomorphism from $\mathbb{R} \times N$ onto $\mathcal{M}$, where $N = \{b^+ = 0\}$.

2.2 Laplacian and Mean curvature comparison

Comparison theorems are a useful tool when trying to understand the geometry of a manifold. Specifically, these theorems provide a convenient way to "translate" information about a lower bound on curvature into information about the distance function. These type of theorems are presented with more details in [DWei],[SY], [Zhu]

Before proceeding with the proof of the theorem, we first recall weaker types of inequalities for the Laplacian and for the rest of this work, the inequalities will be in the weak sense where is needed.

Definition 2.2.1. *If f, h are two continuous functions f, h defined on an open domain $\Omega \subseteq \mathcal{M}$, we say $\Delta f \leq h$ in the distributional sense on Ω if :*

$$\int_\Omega f \Delta \phi \leq \int_\Omega h\phi$$

for all $\phi \geq 0$ in $C_0^\infty(\Omega)$.

Definition 2.2.2. *Suppose $f \in C(\mathcal{M})$ and $q \in M$, a map f_q defined in a neighborhood of q, is an upper barrier of f at q , if $f_q \in C^2(U)$ and satisfies :*

$$f_q(q) = f(q)$$

and

$$f_q(x) \geq f(x) \;\; \forall\, x \in U$$

Definition 2.2.3. *Suppose $f \in C(\mathcal{M})$, we say $\Delta f(q) \leq c$ in the barrier sence, if for all $\varepsilon > 0$ there exists an upper barrier $f_{q,\varepsilon}$ such that*

$$\Delta f_{q,\varepsilon}(q) \leq c + \varepsilon$$

We also define a specific function called the comparison function $sn_H(r)$

$$sn_H(r) = \begin{cases} \frac{1}{\sqrt{\mathrm{H}}} \sin(\sqrt{\mathrm{H}}\, r)\ , & H > 0 \\ r\ , \ \ \mathrm{H} = 0 \\ \frac{1}{\sqrt{-\mathrm{H}}} \sinh(\sqrt{-\mathrm{H}}\, r)\ , & H < 0\ . \end{cases}$$

Considering the definitions above we prove the first theorem, through which the Laplacian comparison can be obtained.

Theorem 2.2.4 (Mean curvature comparison)**.** *Let M^n be a smooth manifold satisfying $\mathrm{Ric} \geq (n-1)H$, and $q \in M^n$ then along any minimal geodesic segment from q:*

$$m(r) \leq m_H(r)\ .$$

Proof. Let the following map ρ be a distance function.
First note that since $\mathrm{Ric} \geq (n-1)H$, by Bochner's formula we get

$$m' \leq -\frac{m^2}{n-1} - (n-1)H$$

Particularly, putting $u(x) = \rho(x)$ in the equation (2.1) yields

$$0 = |II|^2 + m' + \mathrm{Ric}(\partial_r, \partial_r)$$

and by the Cauchy- Schwarz inequality it is $|II|^2 \geq \frac{m^2}{n-1}$, so we obtain

$$m' \leq -\frac{m^2}{n-1} - (n-1)H$$

because of the fact $Hess_\rho = II$, where II is the second fundamental form of the level sets ρ, $\Delta\rho = m$ is the mean curvature and $\nabla\rho = \partial r$, the covariant derivative of the normal direction. Also the following equalities hold

$$m_H' = -\frac{m_H^2}{n-1} - (n-1)H\ .$$

Where m_H is the mean curvature of geodesic spheres in model space M_H^n, and

$$m_H = (n-1)\frac{sn_H'}{sn_H}$$

Finally we get :

$$\begin{aligned}(sn_H^2(m-m_H))' =& 2sn_H' sn_H(m-m_H) + sn_H^2(m-m_H)' \\ \leq& \frac{2}{n-1} sn_H^2 m_H(m-m_H) - \frac{1}{n-1} sn_H^2(m^2-m_H^2) \\ -& \frac{sn_H^2}{n-1}(2m_H^2 - 2mm_H + m^2 - m_H^2) \\ =& -\frac{sn_H^2}{n-1}(m-m_H)^2 \leq 0\ ,\end{aligned}$$

also

$$\lim_{r\to 0} sn_H^2(m-m_H) = 0\ .$$

So integrating from 0 to r yields

$$sn_H^2(r)(m(r) - m_H(r)) \leq 0\ ,$$

then we proved that

$$m(r) \leq m_H(r)\ .$$

□

Theorem 2.2.5. *If $\mathcal{M}^n$ is a Riemannian manifold with* $\mathrm{Ric}_{\mathcal{M}^n} \geq (n-1)\mathrm{H}$*, r is the distance function and r_p is the distance function from p, then in the distributional (or weak) and barrier sense the following inequality holds :*

$$\Delta r_p \leq (n-1)\frac{sn'_H}{sn_H}(r_p) \; .$$

It means that for every compactly supported C^∞ non negative function ϕ on $\mathcal{M}^n$ we have

$$\int_{\mathcal{M}^n} r_p \, \Delta\phi \; \leq \; \int_{\mathcal{M}^n} (n-1) \, \frac{sn'_{\mathrm{H}}}{sn_{\mathrm{H}}} \, (r_p) \, \phi$$

Proof. First for the smooth part of the distance function, the theorem follows immediately from the above (because of the $\Delta r = m$). So it remains to prove this in the distributional and barrier sense. Let $\mathrm{Cut}(p)$ be the cut locus of the point p (all points for which the geodesic segment from p stops being minimizing).

- Distributional Sense means that for every compactly supported C^∞ non negative function ϕ on $\mathcal{M}^n$ we have

  $$\int_{\mathcal{M}^n} r_p \, \Delta\phi \; \leq \; \int_{\mathcal{M}^n} (n-1) \, \frac{sn'_{\mathrm{H}}}{sn_{\mathrm{H}}} \, (r_p) \, \phi$$

  For the proof of that check the theorem 4.1 of [Ch].

- Barrier sense

  Let $p \in \mathcal{M}$ and r_p be the distance function . In the case that $q \notin Cut(p)$ we have already proven that the inequality holds. Assume that $q \in Cut(p)$ and denote the function $d(\gamma(\varepsilon), x) + \varepsilon$, where γ be a minimal geodesic with $\gamma(0) = p$ and $\gamma(l) = q$ this is a support function for r_p at the point q . First of all it's smooth in a neighborhood of q Indeed, since

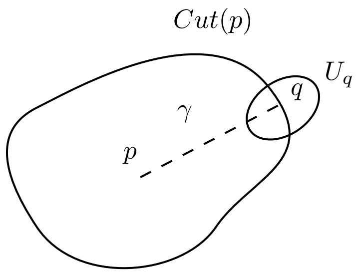
  $$d(\gamma(\varepsilon), q) + \varepsilon = r_p(q) \; ,$$

  Also,

  $$d(\gamma(\varepsilon), x) + \varepsilon \geq d(p, q)$$

  $$\begin{aligned} \Delta(d(\gamma(\varepsilon), x) + \varepsilon) \leq \Delta_H(d(\gamma(\varepsilon), x)) =& m_H(d(\gamma(\varepsilon), x)) \\ &\leq m_H(d(p, x)) + c\varepsilon \\ &= \Delta_H(r_p(x)) + c\varepsilon \; , \end{aligned}$$

  so we proved the Laplacian comparison in the barrier sense.



□

2.3 Volume on Riemannian manifolds

The main target of this chapter is to prove volume comparison. Lets begin by defining the volume: As it's known, we can use partitions of unity to define volume and for the form it is

$$dv = \sqrt{|det(g_{ij})|}\psi_a^{-1}dx_1...dx_n \ .$$

Let $q \in M^n$ and D_q the segment disc then $\exp_q : D_q \to M \setminus C_q$ is a diffeomorphism.

Then now, using polar coordinates and diffeomorphism $exp_q : D_q \setminus \{0\} \to M \setminus (C_q \cup \{q\})$ we set

$$E_i = (\exp_q)_*(\frac{\partial}{\partial\theta_i})$$

and

$$E_n = (\exp_q)_*(\frac{\partial}{\partial r}) \ .$$

Now $g_{nn} = 1$ and $g_{ni} = 0$ for $1 \leq i < n$ since $\exp_q$ is a radial isometry. Let $J_i(r, \theta)$ be the Jacobi field with $J_i(0) = 0$ and $J_i'(0) = \frac{\partial}{\partial\theta_i}$. Then

$$E_i(\exp_q(r, \theta)) = J_i(r, \theta) \ .$$

So we write J_i and $\frac{\partial}{\partial r}$ in terms of an orthogonal basis $\{e_k\}$, then it is $J_i = \sum_{k=1}^{n} a_{ik}e_k$ and that yields

$$\sqrt{\det(g_{ij})(r, \theta)} = |det(a_{ik})| = \left\| J_1 \wedge \ldots \wedge J_{n-1} \wedge \frac{\partial}{\partial r} \right\| \ .$$

Finally letting $\mathcal{A}(r, \theta) = \left\| J_1 \wedge \ldots \wedge J_{n-1} \wedge \frac{\partial}{\partial r} \right\|$, the volume element of $\mathcal{M}$ is

$$\text{dvol} = \mathcal{A}(r, \theta) dr d\theta_{n-1} \ .$$

Let $\mathcal{A}(r,\theta)$ be a map such that $\mathrm{dvol}_g = \mathcal{A}(r,\theta)drd\theta$ be the volume element of $\mathcal{M}$ in geodesic polar coordinates at a point q and $\mathrm{dvol}_H = \mathcal{A}_H(r,\theta)drd\theta$ is the volume element of the model space $\mathcal{M}^n_H$.

For example, in the classic model spaces $\mathbb{R}^n$, $\mathbb{S}^n$, $\mathbb{H}^n$ we can calculate

$$\mathrm{dvol} = sn_k^{n-1}(r)drd\theta_{n-1}$$

i.e

$$\mathrm{dvol} = r^{n-1}drd\theta_{n-1} \ \ (H=0)$$
$$\mathrm{dvol} = \sin^{n-1}(r)drd\theta_{n-1} \ \ (H=1)$$
$$\mathrm{dvol} = \sinh^{n-1}(r)drd\theta_{n-1} \ \ (H=-1)\ ,$$

respectively.

2.4 Bishop Volume comparison

Now we are going to prove the three lemmas we need for the proof of Volume comparison.

Lemma 2.4.1. *We define $\mathcal{A}$ as in the section above. Then it is*

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)} = m(r,\theta)\ .$$

Proof. Let γ be a unit speed geodesic with $\gamma(0)=q$, $J_i(0)=0$ and $J_i'(0)=\frac{\partial}{\partial\theta_i}$ for $i=0,\ldots,n-1$ and $J_n'(0)=\gamma'(0)$ where $\{\frac{\partial}{\partial\theta_1},\frac{\partial}{\partial\theta_2},\ldots,\gamma'(0)\}$ is an orthogonal basis of $T_q\mathcal{M}$. Note also that :

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)} = \frac{|J_1\wedge\ldots\wedge J_n|'}{|J_1\wedge\ldots\wedge J_n|}\ .$$

For any $r=r_0$ such that $\gamma|_{[0,r_0+\varepsilon)}$ is minimal, let $\{\overline{J}_i(r_0)\}$ be an orthogonal basis of $T_{\gamma(r_0)}\mathcal{M}$ with $\overline{J}_n(r_0)=\gamma'(r_0)$. We are inside the cut locus so there are no conjugate points . Therefore, $\{J_i(r_0)\}$ is also a basis of $T_{\gamma(r_0)}\mathcal{M}$. So we can write

$$\overline{J}_i(r_0) = \sum_{k=1}^n b_{ik}J_k(r_0)\ .$$

Then for all $0\le r<r_0+\varepsilon$ we define

$$\overline{J}_i(r) = \sum_{k=1}^n b_{ik}J_k(r)\ .$$

Then $\{\overline{J}_i\}$ are Jacobi fields along γ which is an orthogonal basis at $\gamma(r_0)$. We can easily calculate

$$|\overline{J_1} \wedge \ldots \wedge \overline{J_n}| = \det(b_{ij})|J_1 \wedge \ldots \wedge J_n| \ .$$

The previous equation holds for all r , thus

$$\frac{|J_1 \wedge \ldots \wedge J_n|'}{|J_1| \wedge \ldots \wedge J_n|}(r) = \frac{|\overline{J_1} \wedge \ldots \wedge \overline{J}_n|'}{|\overline{J}_1 \wedge \ldots \wedge \overline{J}_n|}(r)$$

But at $r = r_0$, $|\overline{J}_1 \wedge \ldots \wedge \overline{J}_n|(r_0) = 1$. Therefore,

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) = |\overline{J}_1 \wedge \ldots \wedge \overline{J}_n|'(r_0) = \sum_{k=1}^{n} |\overline{J}_1 \wedge \ldots \wedge \overline{J}'_k \wedge \ldots \wedge \overline{J}_n|(r_0)$$

but $\{\overline{J}_i(r_0)\}$ is an orthonormal basis of $T_{\gamma(r_0)}\mathcal{M}$, so we have

$$\overline{J}'_k(r_0) = \sum_{l=1}^{n} \langle \overline{J}'_k(r_0), \overline{J}_l(r_0)\rangle \overline{J}_l(r_0) \ .$$

If we put that in the previous equality it is

$$\sum_{k=1}^{n} |\overline{J_1} \wedge ... \wedge \sum_{l=1}^{n} \langle \overline{J'_k}(r_0), \overline{J_l}(r_0)\rangle \overline{J_l}(r_0) \wedge .. \wedge \overline{J_n}|$$

but $\{\overline{J_1}, ..., \overline{J_n}\}$ are orthonormal (for $i \neq j$ it is $\langle \overline{J_i}(r_0), \overline{J_j}(r_0)\rangle = 0$) so we get

$$\frac{\mathcal{A}'(r,\theta)}{\mathcal{A}(r,\theta)}(r_0) = \sum_{k=1}^{n} \langle \overline{J'_k}(r_0), \overline{J_k}(r_0)\rangle$$

Then finally by the fact $\overline{J'_i}(r_0) = \nabla_{\overline{J_i}(r_0)}\gamma'$ we get the mean curvature because

$$\sum_{k=1}^{n} \langle \overline{J'_k}(r_0), \overline{J_k}(r_0)\rangle = \sum_{k=1}^{n-1} \langle \nabla_{\overline{J}_k}\gamma', \overline{J}_k\rangle = m(r_0, \gamma'(0)) \ .$$

□

Corollary 2.4.2. *From the lemma above and mean curvature comparison the map*

$$r \to \frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}$$

is non-increasing along any minimal geodesic segment from q.

Proof. The proof is just a calculation of

$$\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)'$$

□

Lemma 2.4.3. *We denote the volume of a geodesic sphere of* $\mathcal{M}^n$ *by* $A(x,r)$ *and the volume of a geodesic sphere on* $\mathcal{M}^n_H$ *by* $A_H(x,r)$.

$$\textit{then } \frac{A(x,r)}{A_H(x,r)} \textit{ is non-increasing in } r$$

Proof.

$$\frac{d}{dr}\frac{A(x,r)}{A_H(x,r)} = \frac{d}{dr}\frac{\int_{\mathbb{S}^{n-1}} \mathcal{A}(r,\theta)d\theta_{n-1}}{\int_{\mathbb{S}^{n-1}} \mathcal{A}_H(r)d\theta_{n-1}} = \frac{1}{Vol(S^{n-1})}\frac{d}{dr}\int_{\mathbb{S}^{n-1}}\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)d\theta_{n-1}\ ,$$

but

$$\left(\frac{\mathcal{A}(r,\theta)}{\mathcal{A}_H(r)}\right)' \leq 0$$

Finally,

$$\frac{d}{dr}\left(\frac{A(r,\theta)}{A_H(r)}\right) \leq 0\ .$$

□

Lemma 2.4.4. *Assume* f, g *are two functions such that* $f(t)/g(t)$ *is non increasing in* t *and* $g(t) > 0$. *Then the map*

$$\frac{\int_r^R f(t)dt}{\int_r^R g(t)dt}$$

is non increasing in r *and* R .

Proof. We have

$$\frac{\partial}{\partial r}\frac{\int_r^R f(t)dt}{\int_r^R g(t)dt} = \frac{-f(r)\int_r^R g(t)dt + g(r)\int_r^R f(t)dt}{\left(\int_r^R g(t)^2dt\right)}\ .$$

Also, since $\frac{f(t)}{g(t)} \leq \frac{f(r)}{g(r)}$, we have $g(r)f(t) \leq f(r)g(t)$, so finally

$$\int_r^R g(r)f(t)dt \leq \int_r^R f(r)g(t)dt\ .$$

□

We can now prove the theorem as a corollary of the lemmas above.

Theorem 2.4.5. *If* $\mathcal{M}^n$ *is a Riemannian manifold with* $\mathrm{Ric}_{\mathcal{M}^n} \geq (n-1)\mathrm{H}$, *then the function*

$$r \mapsto \frac{\mathrm{Vol}(B(p,r))}{\mathrm{Vol}_{\mathrm{H}}(B(\underline{p},r))}$$

is non-increasing, where $\mathrm{Vol}_{\mathrm{H}}(B(\underline{p},r))$ *is the volume of the ball with radius* r *in the model space (there is no dependence of the center because of symmetry).*

Remarks

1. Under the same assumptions, $\text{Vol}(B(x,r)) \leq \text{Vol}_H B(x,r)$

The proof follows from the fact that $\lim_{r\to 0} \frac{\text{Vol}(B(x,r))}{\text{Vol}(B_H(r))} = 1$.

2. If $r \leq R$, then

$$\frac{\text{Vol}(B(x,r))}{\text{Vol}(B_H(r))} \geq \frac{\text{Vol}(B(x,R))}{\text{Vol}(B_H(R))} \ .$$

so that implies

$$\text{Vol}(B(x,R)) \leq \frac{\text{Vol}(B_H(R))}{\text{Vol}(B_H(r))} \text{Vol}(B(x,r))$$

Sometimes we let $R = 2r$ and then we can take a lower bound on the ratio $\frac{\text{Vol}(B(x,r))}{\text{Vol}(B(x,2r))}$ i.e we take

$$\text{Vol}(B(x,2r)) \leq \frac{\text{Vol}(B_H(2r))}{\text{Vol}(B_H(r))} \ \text{Vol}(B(x,r)) \ .$$

The inequality above says that if we double the radius of a ball we can have a control for the volume $\text{Vol}(B(x,2r))$ provided that $\text{Ric} \geq H(n-1)$. Informally speaking, Ricci curvature says that given the volume $\text{Vol}(B(x,r))$, then the volume $\text{Vol}(B(x,R))$, $(r < R)$ cannot be arbitrarily larger than $\text{Vol}(B(x,r))$.

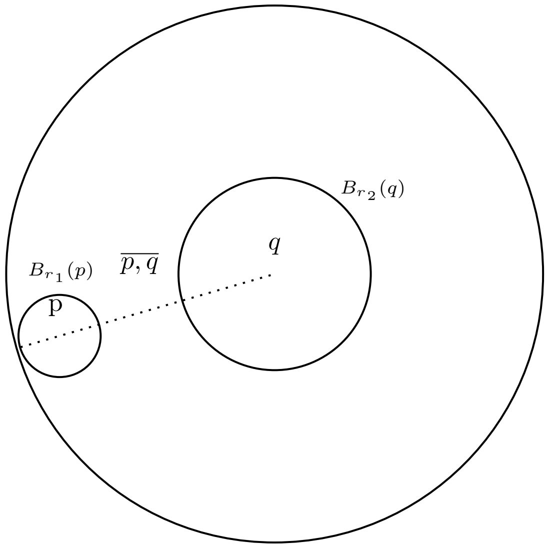
3. Topogonov proved that the equality holds if and only if $B(x,r)$ is isometric to $B_H(r)$, and also by volume comparison we can conclude the maximal diameter theorem

4. An also very usefull result that uses the theorem above it that having a Volume bound on a ball centered at $p \in M$ with radius r_1 we can find a volume bound for any other point of manifold, depending on $Vol(B(p,r_1))$ and distace $\overline{p,q}$. More specifically let r_2 be the radius of the second ball and $s = \overline{p,q}$ then it is

$$Vol(B(p,r_1)) \subseteq Vol(B(q,s+r_1))$$

so it gives

$$\frac{Vol(B(p,r_1))}{Vol(B(q,r_2))} \leq \frac{Vol(B(q,s+r_1))}{Vol(B(q,r_2))} \leq \frac{V_{-H}(r_1+s)}{V_{-H}(r_2)}$$



The next statement is useful because it allows us to achieve a bound along geodesics given an integral bound.

In the following theorem , for a non negative function g ,we denote with $\mathcal{F}_g$ the following

$$\mathcal{F}_g(y,z) = \inf_{\gamma} \int_0^l g(\gamma(s))ds$$

where the infimum is taken over all minimal geodesics γ from y to z .

Theorem 2.4.6 (Segment inequality)**.** *Let* $(\mathcal{M}^n, g)$ *be a Riemannian manifold with* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)k$ *(We can assume for simplicity that* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)$*) and let* $A_1, A_2 \subseteq B_r(p)$*, with* $r < R$*. Then*

$$\int_{A_1 \times A_2} \mathcal{F}_g(y,z) \leq c(n,R)r(\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2)) \int_{B_{2R}(p)} g \ .$$

Proof. We will use polar coordinates. Let $(y,z) \in A_1 \times A_2$. Due to the fact that the set $\mathrm{Cut}(y)$ has measure zero we can assume that for every pair of points there exists a unique minimal geodesic (particularly in $A_1 \times A_2$). For technical reasons we decompose the function $\mathcal{F}_g$ to $\mathcal{F}_g^+$ and $\mathcal{F}_g^-$.
These $\mathcal{F}_g^+$ and $\mathcal{F}_g^-$ are the following

$$\mathcal{F}_g^+ = \int_{\frac{\overline{y,z}}{2}}^{\overline{y,z}} g(\gamma_{y,z}(s))ds \ , \ \mathcal{F}_g^- = \int_0^{\frac{\overline{y,z}}{2}} g(\gamma_{y,z}(s))ds$$

Where $g_{y,z}$ is the minimal geodesic from y to z .

We fix $y \in A_1$ and consider the integral as the function $z \mapsto \mathcal{F}_g(y,z)$. We can express the integral of this function in the following way because the exponential map is a diffeomorphism almost everywhere.
We calculate the integral as

$$\int_{A_2} \mathcal{F}_g^+(y,z)dz = \int_{S_x}\int_{I_\theta} \mathcal{F}_g^+(y, \exp_x(r\theta))\mathcal{A}(r,\theta)drd\theta \ ,$$

where $I_\theta = \{t \mid \exp_x(t\theta) \in A_2\}$. Since $A_1, A_2 \subseteq B_r(p)$ and $I_\theta \subseteq [0,2r]$, the result follows from area comparison and from the definition of $\mathcal{F}_g$.
Note that so

$$\mathcal{F}_g^+(y, \exp(r\theta))\mathcal{A}(r,\theta) = \int_{\frac{r}{2}}^{r} g(\exp(t\theta))dt\mathcal{A}(r,\theta) \ .$$

But from volume comparison,

$$\mathcal{A}(r,\theta) \le \frac{\mathcal{A}_{-1}(r)}{\mathcal{A}_{-1}(t)}\mathcal{A}(t,\theta) \ , \ t < r \ ,$$

therefore

$$\mathcal{F}_g^+(y, \exp(r\theta))\mathcal{A}(r,\theta) \le c(n,R)\int_{\frac{r}{2}}^{r} g(\exp(t\theta))\mathcal{A}(t,\theta)\ dt$$

and then

$$\begin{aligned}\int_{A_2} \mathcal{F}_g^+(y,z)dz &\le c(n,R)\int_{S_y}\int_{I_\theta}\int_{\frac{r}{2}}^{r} g(\exp_y(t\theta))\mathcal{A}(t,\theta)dtdrd\theta \\ &\le \int_{S_x}\int_0^{2r}\int_0^{2R} g(\exp(t\theta))\mathcal{A}(t,\theta)dtd\theta dr \\ &\le r\ c(n,R) \int\limits_{B(p,2R)} g \ .\end{aligned}$$

$y \in A_1$ was fixed, so integrating over the set A_1 we get

$$\int_{A_1}\int_{A_2} \mathcal{F}_g^+(y,z) \le c(n,R)rVol(A_1) \int\limits_{B(p,2R)} g \ .$$

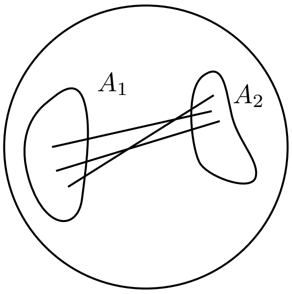
We work for $\mathcal{F}_g^-(y,z)$ in a similar way and we get

$$\int_{A_2}\int_{A_1} \mathcal{F}_g^-(y,z) \le c(n,R)rVol(A_2) \int\limits_{B(p,2R)} g \ .$$

Summing these up, we conclude that

$$\int_{A_1\times A_2} \mathcal{F}(y,z) \ \le \ c(n,R)\ r\ (\mathrm{Vol}(A_1) + \mathrm{Vol}(A_2)) \int\limits_{B(p,2R)} g \ ,$$

which completes the proof. □



Remarks. *1. We use comparison theorems as mentioned before to prove a basic inequality that generalizes Fubini's theorem.*

2. It is not necessary for the proof that we work on a ball , we can generalize the theorem on every subset W and demand segments between points x and y to be contained to W.

Using the theorem above we can convert integral bounds to pointwise bounds. Later we are going to ensure integral bounds for $|\text{Hess}_f|$ and then we will use the following. Assume that $A_1 = A_2 = B_{\frac{R}{4}}(p)$, $\underline{x} \in A_1$ and then, using the theorem for $\mathcal{F}_g(\underline{x}, .)$ ($\mathcal{F}_g(\underline{x}, .)$ which is a non negative function from $B_{\frac{R}{2}}$ to $\mathbb{R}$ since g is non-negative), we obtain

$$\int_{B(p,\frac{R}{4})\times B(p,\frac{R}{4})} \mathcal{F}_{\mathcal{F}_g(\underline{x},.)}(y,z) \leq C(n,R)\text{Vol}(B(p,R/4)) \int_{B(p,\frac{R}{2})} \mathcal{F}_g(\underline{x},.)$$

$\underline{x} \in A_1$ and then we integrate over $B_{\frac{R}{4}}(p)$ to get

$$\begin{aligned}\int_{B(p,\frac{R}{4})} \int_{B(p,\frac{R}{4})\times B(p,\frac{R}{4})} \mathcal{F}_{\mathcal{F}_g(\underline{x},.)}(y,z) \\ &\leq c(n,R)\text{Vol}(B(p,R/4)) \int_{B(p,\frac{R}{4})\times B(p,\frac{R}{4})} \mathcal{F}_g \\ &\leq c(n,R)\text{Vol}(B(p,R/4))Vol(B(p,R/2)) \int_{B(p,R)} g\ .\end{aligned}$$

The last inequality actually tells us that if the L^1-norm of a function g is small enough, there exists a point x' near $\underline{x}$ such that the integrating function is small as well i.e

$$\int_{B(p,\frac{R}{4})\times B(p,\frac{R}{4})} \mathcal{F}_{\mathcal{F}(x',.)}(y,z)dydz$$

For (y, z) *in* $B_{\frac{R}{4}}(p) \times B_{\frac{R}{4}}(p)$ there exist y', z' near y, z respectively, such that $\mathcal{F}_{\mathcal{F}(x',.)}(y', z')$ is small.

So if γ is unique minimizing geodesic from y' to z' and τ_s minimizing from x' to $\gamma(s)$ then

$$\mathcal{F}_{\mathcal{F}_g(x',\cdot)}(y',z') = \int_0^{\overline{y'z'}} \mathcal{F}_g(x',\gamma(s))ds = \int_0^{\overline{y'z'}} \int_0^{l(s)} g(\tau_s(t))dt$$

To clarify this assume we have bounds for specific functions (for instance here we will assume a bound for $|Hess_{\beta_+}|$ for a map β_+ , which will be usefull in the future). We will prove the existance of points x^*, y^*, z^* satisying

$$\mathcal{F}_{\mathcal{F}_{|Hess_{\beta_+}|}(y^*,z^*)}(x^*,\cdot) \leq \Psi$$

and

$$\overline{x,x^*}\ ,\ \overline{y,y^*}\ \overline{z,z^*}\ \leq \Psi$$

Where by Ψ we mean a positive functiondefined as 5.0.2
Consider $x, y, z \in B_R(p)$ and B_x, B_y, B_z small balls centred at these points with small radius ε , such that $B_x, B_y, B_z \subseteq B_R(p)$
First by Markov's inequality the following holds for non negative measurable functions

$$|x \in U : f \geq \alpha| \leq \frac{1}{\alpha}\int_U f$$

Where α is positive and we deonte with $|\cdot|$ the usual measure of the set

From segement inequality it is

$$\int_{B_y\times B_z} \mathcal{F}_{\mathcal{F}_{|Hess_{\beta_+}|}}(x^*,\cdot)dydz \leq C\cdot R\cdot(|B_y|+|B_z|)\cdot\int_{B(p,2R)} \mathcal{F}_{|Hess_{\beta_+}|}(x^*,w)dw \tag{2.2}$$

if we apply segment inequality again for the map $\mathcal{F}_{|Hess_{\beta_+}|}$ we get

$$\int_{B_x\times B_{2R}(p)} \mathcal{F}_{|Hess_{\beta_+}|} \leq c\ R\ (Vol(B_x) + Vol(B_{2R}(p)))\int_{B_{4R}(p)} |Hess_{\beta_+}|$$

Now we apply Markov's inequality for the measurable map

$$\int_{B_{2R}(p)} \mathcal{F}_{|Hess_{\beta_+}|}(\cdot,w)dw$$

so it is

$$\begin{aligned}|x^* \in B_x : \int_{B_{2R}(p)} \mathcal{F}_{|Hess_{\beta_+}|}(x^*,w)dw \geq \alpha| &\leq \frac{1}{\alpha}\int_{B_x\times B_{2R}(p)} \mathcal{F}_{|Hess_{\beta_+}|}(x,w)dxdw \\ &\leq \frac{1}{\alpha}\ c\ R\ (Vol(B_x) + Vol(B_{2R}(p)))\int_{B_{4R}(p)} |Hess_{\beta_+}|\end{aligned}$$

So we have to pick an α to ensure that the set

$$\{x^* \in B_x : \int_{B_{2R}(p)} \mathcal{F}_{|Hess_{\beta_+}|}(x^*, w) dw \geq \alpha\}$$

has small measure so take α satisfying :

$$\frac{1}{\alpha} c\ R(Vol(B_x) + Vol(B_{2R}(p)) \leq \frac{1}{100} Vol(B_x))$$

for instance we can take

$$\alpha = \frac{100\ c\ R\ (Vol(B_x) + Vol(B_{2R}(p)))}{Vol(B_x)} \int_{B_{4R}(p)} |Hess_{\beta_+}|$$

Then in a set of measure $\frac{99}{100} Vol(B_x)$ the following holds :

$$\mathcal{F}_{|Hess_{\beta_+}|}(x^*, w) dw \leq \frac{cR(Vol(B_x) + Vol(B_{2R}(p)))}{Vol(B_x)} \int_{B_{4R}(p)} |Hess_{\beta_+}|$$

Now we can repeat exactly the same argument for $B_y \times B_z$ so in a subset of $B_y \times B_z$ for y^*, z^* it is

$$\mathcal{F}_{\mathcal{F}_{|Hess_{\beta_+}|}(x^*,\cdot)}(y^*, z^*) \leq C\ R^2\ \frac{(Vol(B_x) + Vol(B_{2R}(p)))(Vol(B_y) + Vol(B_z))}{Vol(B_x) \cdot Vol(B_y) \cdot Vol(B_z)} \int_{B_{4R}(p)} |Hess_{\beta_+}|$$

So it suffices to complete the proof because we can use Volume comparison to disapear dependence on x,y,z , so if $\int_{B_{4R}(p)} |Hess_{\beta_+}| \leq \Psi$ we shows that

$$\mathcal{F}_{\mathcal{F}_{|Hess_{\beta_+}|}(x^*,\cdot)}(y^*, z^*) \leq \hat{C}\Psi$$

The idea is simple and says that if f is nonnegative and we have a control for $\int_A f$, then f cannot be arbitrarily large in A. For more applications of the inequality above the reader can see [Petersen1]. One application of this is Poincaré's inequality .

Chapter 3

Cutoff functions

3.1 Gradient estimate

In this chapter, we continue by presenting the classic Cheng-Yau theorem for the gradient.

Theorem 3.1.1 (Cheng-Yau estimates). *Let* $\mathrm{Ric}_{\mathcal{M}^n} \geq (n-1)H$ *on* $B_{R_2}(p)$ *,* *K smooth and $u : B_{R_2}(p) \to \mathbb{R}$ which satisfies*

$$u > 0 \text{ and } \Delta u = K(u) .$$

Then on $B_{R_1}(p)$ $(R_1 < R_2)$ *:*

$$\frac{|\nabla u|^2}{u^2} \leq \max\{2u^{-1}K(u), c(n, R_1, R_2, H) + 2u^{-1}K(u) - 2K'(u)\}.$$

Proof. First we set $v := \log u$ and then a simple calculation yields

$$|\nabla v| = \frac{|\nabla u|}{u}$$

and

$$\Delta v = -\frac{|\nabla u|^2}{u^2} + \frac{K(u)}{u} = -|\nabla v|^2 + e^{-v}K(e^v) = -|\nabla v|^2 + F(v),$$

where $F(v) = e^{-v}K(e^v) = u^{-1}K(u)$.

The function $Q = \phi|\nabla v|^2$ is smooth and attains a maximum at some interior point q of B_{R_2}. We assume that $q \notin \mathrm{Cut}(q)$.

Then at q we have by product rule

$$\nabla Q = 0 \Leftrightarrow \nabla(\phi|\nabla v|^2) = 0 \Leftrightarrow \nabla\phi|\nabla v| = -\phi\nabla|\nabla v|^2.$$

By the definition of Q we get

$$\nabla|\nabla v|^2 = \phi^{-1}\phi(\nabla|\nabla v|^2) = -\phi^{-1}\nabla\phi|\nabla v|^2 = -\phi^{-2}Q\nabla\phi\ .$$

We have $\Delta Q \leq 0$ at q as well. Then by laplacian product rule

$$\Delta Q = \Delta\phi|\nabla v|^2 + \langle\nabla\phi, \nabla|\nabla v|^2\rangle + \phi\Delta|\nabla v|^2 = (\phi^{-1}\Delta\phi - 2\phi^{-2}|\nabla\phi|^2)Q + \phi\Delta|\nabla v|^2\ .$$

We can use Bochner's formula on the last term of right hand i.e we have

$$\phi\Delta|\nabla v|^2 = 2\phi|\mathrm{Hess}_v|^2 + 2\phi\langle\nabla\Delta v, \nabla v\rangle + 2\phi\mathrm{Ric}(\nabla v, \nabla v).$$

We work with each term of the inequality above separately.
For the first one we have

$$2\phi|\mathrm{Hess}_v|^2 \geq \frac{2\phi}{n}(\Delta v)^2 = \frac{2}{n}\phi^{-1}(-Q + \phi F(v))^2.$$

For the third, from the Ricci curvature lower bound we get

$$\mathrm{Ric}(\nabla v, \nabla v) \geq \mathrm{H}|\nabla v|^2 \Leftrightarrow 2\phi\mathrm{Ric}(\nabla v, \nabla v) \geq 2\phi|\nabla v|^2\ .$$

Finally for the second term we have

$$2\phi\langle\nabla\Delta v, \nabla v\rangle = 2F'Q - 2\phi\langle\nabla|\nabla v|^2, \nabla v\rangle Q = 2F'Q + 2\phi^{-1}\langle\nabla\phi, \nabla v\rangle Q\ .$$

Using Cauchy's inequality we have for every $\alpha > 0$,

$$\langle\nabla\phi, \nabla v\rangle \geq (-\frac{1}{2\alpha\phi}|\nabla\phi|^2 - \frac{\alpha}{2}\phi|\nabla v|^2)$$

so

$$\phi^{-1}Q\langle\nabla\phi, \nabla v\rangle \geq -\frac{1}{2\alpha}\phi^{-1}|\nabla\phi|^2 - \frac{\alpha}{2}\phi Q^2$$

so finally it is

$$2\phi^{-1}Q\langle\nabla\phi, \nabla v\rangle \geq -\frac{1}{\alpha}\phi^{-2}|\nabla\phi|^2 Q - \alpha\phi^{-1}Q^2$$

$$\geq 2F'Q - 2\phi^{-1}|\nabla\phi||\nabla v|Q \geq 2F'Q - \alpha^{-1}\phi^{-2}|\nabla v|^2 Q - \alpha\phi^{-1}Q^2$$

and picking $\alpha = \frac{1}{4n}$, we have

$$2\phi\langle\nabla\Delta v, \nabla v\rangle \geq 2F'Q - 4n\phi^{-2}|\nabla\phi|^2 Q - \frac{1}{4n}\phi^{-1}Q^2 \ .$$

Adding all these above we finally get the following inequality ($\Delta Q \leq 0$)

$$(-\Delta\phi+(2+4n)\phi^{-1}|\nabla\phi|^2-2(n-1)H\phi-2F'\phi)Q \geq \frac{2}{n}(-Q+\phi F)^2-\frac{1}{4n}Q^2 \quad (3.1)$$

Now if

$$Q \leq 2\phi F$$

then

$$|\nabla v|^2 \leq 2F$$

because $\phi \leq 1$

If not , then $Q > 2\phi F$ so in that case

$$-Q + \phi F \leq -\frac{Q}{2} \leq 0$$

and

$$\frac{2}{n}(-Q + \phi F)^2 - \frac{1}{4n}Q^2 \geq \frac{1}{4n}Q^2$$

Now from (3.1) we get

$$4n(-\Delta\phi + (2 + 4n)\phi^{-1}|\nabla\phi|^2 - 2(n - 1)H\phi - 2F'\phi) \geq Q \ .$$

Finally it is the time to define the wanted map ϕ. We define as $\phi = f(r_q)$, where $f : [0, R_2] \to [0, 1]$ is a function satisfying the following :

$$f|_{[0,R_1]} \equiv 1, \ suppf \subseteq [0, R_2)$$

$$-cR_1^{-1}f^{\frac{1}{2}} \leq f' \leq 0$$

$$|f''| \leq cR_1^{-2}$$

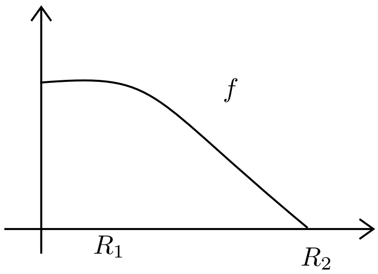
Thus

$$\phi = f(r_q) \ , \ |\nabla\phi| = |f'| \ , \ \Delta\phi(r_q) = f'\Delta r_q + f'' \quad (3.2)$$

For example a map like the following does the trick .

$$f(x) = \frac{\psi(R_2 - x)}{\psi(R_2 - x) + \psi(x - R_1)}$$

$$\psi(x) = \begin{cases} e^{-\frac{1}{x}}, x > 0 \\ 0 \ , \ x < 0 \end{cases}$$



because from the construction of f and the facts of equation (3.2) It suffices to bound the term $-\Delta\phi$

More precisely it is

$$\Delta\phi(r_q) \geq f'\Delta_H r_p - cR^{-2}$$

so because of the fact $f' \leq 0$ we want a lower bound for the Δr_q and that completes the proof.

Note that

$$\Delta r_H = (n-1)\sqrt{-H}coth(\sqrt{-H}r)$$

but on $[R_1, R_2]$ its

$$\Delta r_q \leq \Delta r_H \leq (n-1)\sqrt{-H}\coth(\sqrt{-H}R_1)\ .$$

So finally we proved

$$-\Delta\phi \leq c(n, R_1, R_2, H)\ .$$

Now for the case that $q \in \mathrm{Cut}(p)$, the inequality holds in the barrier sense. We use the barrier (as defined in 2.2)$r_{q,\varepsilon}(x)$ for $r_q(x)$ and let $\varepsilon \to 0$ gives the same i.e from the previous equations

The following theorem is typically encountered in the following form. $\Delta u = \lambda$ on the ball of radius R_2 and we can conclude that in a ball of a radius $R_1 < R_2$ the following holds .

$$|\nabla u| \leq c(n, R_1, R_2, H, \lambda) \sup_{B_{R_2}} u$$

□

In the next chapter we are going to use Yau's gradient estimate to prove the existence of cutoff functions.

3.2 Cutoff functions

The maximum principle is of course known to hold for subharmonic functions and elliptic operators. In this Chapter, we present and prove a quantitative version of the principle .

Before proving the quantitative theorems we need to construct some comparison functions.

We write the metric of the model space $\mathcal{M}^n_H$ as $g = dr^2 + \underline{k}^2 g_{\mathbb{S}^{n-1}}$ (where $\underline{k} = sn_k$ as previous defined).

For any $n \geq 3$, we construct the radial function :

$$\underline{G}(r) = \frac{1}{(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})} \int_r^\infty \underline{k}^{1-n}(s)ds \ ,$$

which satisfies

$$\underline{\Delta G} = 0 \ \ , \ \underline{G}(0) = \infty, \ \ \underline{G'} < 0$$

(Note that G is the Green's function with singularity at 0 and $\underline{G}(\overline{x, \underline{p}}))$ has singularity at $\underline{p} \in \mathcal{M}^n_H$. The smooth function

$$\underline{U}(r) = \int_0^r \underline{k}^{-(n-1)}(s)\Big(\int_0^s \underline{k}^{n-1}(u)\ du\Big)\ ds$$

satisifies

$$\underline{\Delta}\ \underline{U} = 1\ , \underline{U}(0) = 0,\ \underline{U'} \geq 0,\ |\nabla \underline{U}(r)| = \frac{\mathrm{Vol}(B_r(\underline{p}))}{\mathrm{Vol}(\partial B_r(\underline{p}))}\ .$$

Note for example that if H $\equiv 0$ then $\mathcal{M}^n$ is $\mathbb{R}^n$ and by simple calculations we can conclude that

$$\underline{U}(r) = \frac{r^2}{2n} \text{ and } \underline{G}(r) = \frac{1}{(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})} r^{2-n}\ .$$

Given $R > 0$, put $\underline{G}_R = \underline{G} - \underline{G}(R)$ and $\underline{U} - \underline{U}(R)$ and set $c = -\frac{\underline{U'}(R)}{\underline{G'}(R)}$.

We have

$$\underline{G''} \geq 0\ ,\ \lim_{r\to 0} \underline{G'}(R) = -\infty \ \text{ and } \ \underline{U''} \geq 0,\ \underline{U}'(0) = 0$$

Finally, we define the function $\underline{L}_R = c\ \underline{G}_R + \underline{U}_R$ which satisfies

$$\underline{\Delta}\ \underline{L}_R = 1\ ,\ \underline{L}'_R \leq 0 \ \text{ on } (0, R],\ \ \underline{L}_R(R) = 0\ .$$

Suppose f is a function satisfying $\Delta f \geq \delta > 0$, which is stronger than simply saying that f is subharmonic. According to Hopf's principle f attains a minimum at some boundary point of Ω.

Since these special functions are available we can easily obtain a stronger bound than the one in Hopf's classical principle i.e

Theorem 3.2.1. *Let* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)H$ *and* $f : \overline{\Omega} \to \mathbb{R}$ *be a continuous function. If* f *satisfies*

$$\Delta f \geq \delta > 0 \ ,$$

then for all $x \in \Omega$*,*

$$f(x) \leq \max_{\partial\Omega}(f - \delta \ \underline{U}(\rho_x)) \ . \tag{3.3}$$

Similarly, if $\Delta f \leq -\delta < 0$*, then for all* $x \in \Omega$

$$f(x) \geq \min_{\partial\Omega}(f + \delta \ \underline{U}(\rho_x)) \ .$$

Proof. Note that from the definition of $\underline{U}$, $\underline{U}(0) = 0$ holds, so we can apply the maximum principle to $f - \delta \ \underline{U}(\rho_x)$ because

$$\Delta(f - \delta\Delta\underline{U}(\rho_x)) = \Delta f - \delta\Delta\underline{\rho_x} = \Delta f - \delta \geq 0 \ ,$$

which shows it's subharmonic and then

$$f - \delta\Delta\underline{U}(\rho_x) \leq \max_{\partial\Omega}(f - \delta\Delta\underline{U}(\rho_x))$$

i.e .

$$f(x) \leq \max_{\partial\Omega}(f - \delta\Delta\underline{U}(\rho_x))$$

□

For example, if we take $n = 3$, then $\underline{U}(r) = \frac{r^2}{6}$ and a subharmonic function f which satisfies $\Delta f \geq 1$ we can take a much better bound i.e.

$$f(x) \leq \max_{\partial\Omega}(f - \frac{\rho_x^2}{6}) \leq \max_{\partial\Omega} f$$

Note that for the Laplacian of $\underline{U}$ obviously Laplacian comparison used .

Finally we present the quantitative version. We state the case in which $\Delta f \leq \delta$. Put

$$A_{R_1,R_2}(p) = B_{R_2}(p) \backslash \overline{B_{R_1}(p)} \ .$$

Theorem 3.2.2. *Let* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)H$*. Let* $f : \overline{\Omega} \to \mathbf{R}$*, where* $\Omega \subseteq A_{R_1,R_2}(p)$*. If* f *satisfies*

$$\Delta f \leq \delta \ (\delta \geq 0)$$

then for all $x \in \Omega$, $t \geq 0$

$$f(x) \geq (\delta \underline{L}_{R_2} + t\ \underline{G}_{R_2})(R) + \max_{\partial\Omega}(f - (\delta \underline{L}_{R_2}) + t\underline{G}_{R_2}(\rho_p)) \ .$$

Proof. We apply the minimum principle to the function $f - \delta \underline{L}_{R_2} - t\underline{G}_{R_2}$, because

$$\Delta(f - \delta \underline{L}_{R_2} - t\underline{G}_{R_2}) = \Delta(f - \delta \underline{L}_{R_2}) = \Delta f - \Delta \delta \underline{L}_{R_2} \leq 0$$

(that follows from the fact $\Delta f \leq \delta$ and the definition of $\underline{L}_{R_2}$) □

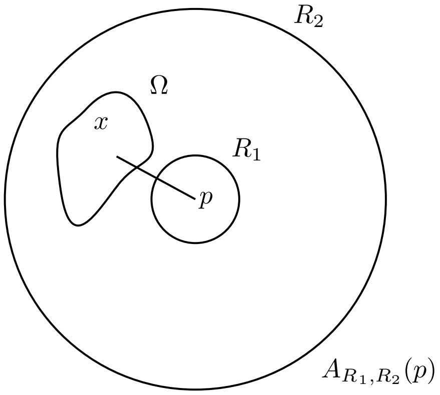


Figure 3.1: $x \in \Omega \subseteq A_{R_1,R_1}(p)$

Lemma 3.2.3. *Let* $\mathrm{Ric}_{\mathcal{M}^n} \geq -(n-1)$ *and let* $f : \overline{A_{R_1,R_2}(p)} \to \mathbb{R}$ *satisfy*

$$\Delta f \leq \delta$$

and

$$f|_{\partial B_{R_p}} \geq 0 \ .$$

Then for all $t \geq 0$, *either*

$$\min_{\partial B_{R_2}(p)} f < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1) \ ,$$

or for all x *in the interior of the Annulus* $A_{R_1,R_2}(p)$,

$$f(x) \geq (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \ \ , \ \ (\overline{x,p} = R) \ .$$

Proof. First note that the function $f - (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})$ satisfies

$$\Delta(f - \delta \underline{L}_{R_2} - t\underline{G}_{R_2}) = \Delta f - \Delta(\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) = \Delta f - \delta \Delta \underline{L}_{R_2} = \Delta f - \delta \leq 0 \ ,$$

because by definition $\Delta \underline{L}_{R_2} = 1$ and $\Delta \underline{G}_{R_2} = 0$ inside the annulus. Assume also that the first case doesn't hold. This means that for every $y \in \partial B_{R_1}(p)$,

$$f(y) \geq (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1)$$

and also by hypothesis $f|_{\partial B_{R_p}} \geq 0$ holds as well. Then using the minimum principle for the superharmonic function $f - (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})$ (which is positive on the boundary $\partial A_{R_1,R_2}(p)$) we get

$$f(x) - \delta \underline{L}_{R_2}(R) - t\underline{G}_{R_2}(R) \geq 0$$

i.e

$$f(x) \geq (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \ \ , \ x \in A_{R_1,R_2}(p) \ .$$

□

Theorem 3.2.4. *If* Ric $\geq -(n-1)H$ *and* $f : \underline{B_{R_2}(p)} \to \mathbb{R}$ *satisfies*

$$\Delta f \leq \delta \ \ f|_{B_{R_2}(p)} \geq 0 \text{ on } \overline{A_{R_1,R_2}}(p) \text{ and } \mathrm{Lip}(f) \leq c \quad \text{on } B_{R_1}(p) \ .$$

If for some $x \in A_{R_1,R_2}(p)$,

$$f(x) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R) \ \ , \ \ (R = \overline{x,p}), \tag{3.4}$$

then

$$f(p) < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1) + cR_1 \ . \tag{3.5}$$

Proof. Using lemma 3.2.3 we get $\min_{\partial B_{R_1}(p)} f < (\delta \underline{L}_{R_2} + t\underline{G}_{R_2})(R_1)$. We need to prove that $f(p) - cR_1 < \min_{\partial B_{R_1}(p)} f$. First note that by the Lipschitz property of f, it is

$$|f(p) - f(x)| \leq cR_1$$

for every $x \in \partial B_{R_1}(p)$, so

$$f(x) > f(p) - cR_1$$

Taking the minimum of f over $\partial B_{R_1}(p)$,

$$\min_{\partial B_{R_1}(p)} f > f(p) - cR_1 \ .$$

□

We will use the previous results and Yau-Cheng estimates for the gradient to construct a very useful type of functions called cutoff functions. Generally, cutoff functions are like bump functions known from differential geometry. The difference is that for cutoff functions we also assume that gradient and Laplacian are bounded by constant depending on n, R_1, R_2 so the next theorem undoubtedly presents great interest on its own, but it is useful for the results that will follow . The hypothesis $Ric \geq -H(n-1)$ has been replaced by $Ric \geq -(n-1)$ for simplicity.

Theorem 3.2.5 (Cutoff functions)**.** *Let* $\text{Ric} > -(n-1)$, *and* R_1 , R_2 *with* $R_1 < R_2$. *There exists a function* $\phi : \mathcal{M}^n \to [0,1]$ *such that*

$$\begin{aligned} \phi|_{B_{R_1}}(p) &\equiv 1 \\ \text{supp}\phi &\subset B_{R_2}(p) \\ |\nabla\phi| &\leq c(n, R_1, R_2) \\ |\Delta\phi| &\leq C(n, R_1, R_2) \end{aligned}$$

Proof. Let $f : \overline{B_{R_2}} \to \mathbb{R}$ such that

$$\Delta f = 1 \text{ and } f|_{\partial(B_{R_2} \setminus B_{R_1})} = L_{R_2} \; .^1$$

Set $\phi = \psi(f)$, where the smooth function ψ will be specified below. We extend ϕ to all of $\mathcal{M}^n$ by putting $\phi(x) = 1$ for $x \in B_{R_1}(p)$ and $\phi(x) = 0$ for x outside the ball of radius R_2. Using the chain rule

$$\Delta\phi(x) = \psi''(f(x))|\nabla f(x)|^2 + \psi'(f(x))\Delta f(x) \; .$$

Since our goal is to find a bound for $\Delta\phi$, it is enough to show that ψ'' is bounded for the values near the boundary $\partial B_{R_2}(p)$, because for all points “away” from the boundary Cheng-Yau gradient estimates ensure the bound

$$|\Delta\phi| \leq |\psi''(f(x))||\nabla f(x)|^2 + |\psi'(f(x))| \; .$$

We now turn to the quantitative maximum principle to get

$$f(x) \leq \max_{\partial B(p,R_1)} (f - \underline{U}(\rho_x)) = \underline{L}_{R_2}(R_1) - \underline{U}(R - R_1) \; .$$

[1] $f|_{\partial B_{R_2}} \equiv 0$ because $L_{R_2}(R_2) = 0$, with L_R defined as below 3.2

Take a, b such that

$$\underline{L}_{R_2}(R_1) > b > a > \underline{L}_{R_2}(R_1) - \underline{U}(R_2 - R_1) \ .$$

We define ψ just requiring that

$$\psi(s) = 1 \ , s \geq b \ \text{ and } \ \psi(s) = 0, \ s \leq \alpha$$

By monotonicity we can choose η_1, η_2 to satisfy

$$b = \underline{L}_{R_2}(R_1 + \eta_1)$$

and

$$a = \underline{L}_{R_2}(R_1) - \underline{U}(R_2 - \eta_2) \ ,$$

where

$$\underline{L_{R_2}}(R_1) > b > a > \underline{L_{R_2}}(R_1) - \underline{U}(R_2 - R_1) \ .$$

First, by (3.4) we get

$$f(x) \geq \underline{L}_{R_2}(R) \geq 0 \ .$$

Now, from the fact that $\underline{L}_{R_2}(r)$ is decreasing we can say

$$f(x) \geq \underline{L}_{R_2}(R) \geq \underline{L}_{R_2}(R + \eta_1) \ ,$$

so for all $R_1 \leq R \leq R_1 + \eta_1$,

$$f(x) \geq b$$

holds. In a similar way we can conclude that

$$f(x) \leq \alpha$$

holds for all $R_2 - \eta_2 \leq R \leq R_2$.

Analytically, because of (3.3) we have

$$f(x) \leq \max_{\partial(B_{R_2}(p) \setminus \overline{B_{R_1}(p)})} f - \underline{U}(\overline{x, p}) \ ,$$

but the maximum occurs on the inner boundary $\partial B_{R_1}(p)$ (because $f - \underline{U} \leq 0$ and $f(x) \geq 0$), so

$$f(x) \leq \underline{L}_{R_2}(R_1) - \underline{U}(R - R_1)$$

$\underline{U}_{R_2}(r)$ is increasing, thus $f(x) \leq \alpha$. So for points in the annulus $R_2 - \eta_2 \leq R_2$, ψ vanishes and $\Delta\phi(x) = 0$. So to verify that $\phi(x) = \psi(f(x))$ is the desired one, we need to check that it vanishes outside of B_{R_2} and is identically equal to one inside B_{R_1}. Inside the annulus we can use the Yau estimate to find the bound for $\Delta\phi$, because we already saw that

$$|\Delta\phi| \leq |\psi''||\nabla f|^2 + |\psi| \ .$$

Clearly we can construct ψ to be bounded so it suffices to bound the term $\psi''|\nabla f|^2$. But this is also bounded by Yau's estimate in a ball with radius smaller that R_2.

Finally, to ensure the bound we have to select the map ψ to vanish in that "problematic" region, as we already did above. □

Chapter 4

Gromov-Hausdorff convergence

In this section we present some basic notions about convergence, We will provide suitable notions to allow convergence between manifolds and more generally, between metric spaces.
First we define the Hausdorff distance between two subsets A, B of a metric space X :

Definition 4.0.1. *Let (X, d) be a metric space and $A, B \subseteq X$. Then, the Hausdorff distance of A and B is defined as:*

$$d_H(A, B) = \inf\{d(a, b) \mid a \in A, b \in B\} \ .$$

Equivalently, we can define the Hausdorff distance of A and B as

$$d_H(A, B) = \inf\{r > 0 \mid A \subset U_r(B), B \subset U_r(A)\} \ ,$$

where with $U_r(S)$ we denote the r-neighborhood of a set S i.e the set of points x such that $\operatorname{dist}(x, S) < r$ *or* $\bigcup_{x \in S} B_r(x)$.

This is a nice tool to measure distances of subsets on a metric space but we need to modify this to obtain a distance sense between two arbitrary metric spaces.
We need a distance notion such that, if two subsets of a metric space are close

enough (in the Hausdorff sense), they must be close as arbitrary metric spaces as well. We also need the distance between isometric spaces to be zero. The Gromov-Hausdorff distance satisfies both of these requirements. We begin by defining it.

Definition 4.0.2. *Let (X, d_X) , (Y, d_Y) be arbitrary metric spaces. The Hausdorff-Gromov distance between X and Y is defined as follows : We say that $d_{GH}(X, Y) < r$ for some $r > 0$ if and only if there exists a metric space Z and subspaces X' and Y' of Z, isometric to X and Y, respectively, such that $d_H(X', Y') < r$.*

Remarks.

1. *Spaces X' and Y' are regarded with the restriction of metric of the space Z . If, for example $X = \mathbb{S}^1$ is unit cycle of $\mathbb{R}^2$, we can't choose $X = X' = \mathbb{S}^1$ and $Z = \mathbb{R}^2$, because X and X' are not isometric (we can take a shortcut in X')*

2. *It's trivial to verify that if $d_{GH}(X, Y) = 0$ then X is isometric to Y. The inverse is not generally true. We need to add a compactness hypothesis on Z and then, d_{GH} is a metric in the class of compact metric spaces.*

3. *Actually the definition above deals with huge classes of metric spaces (all spaces Z containing X and Y isometrically), so it's possible to reduce this huge class to disjoint unions of X and Y i.e*

$$d_{GH}(X, Y) = \inf\{d_H(X, Y) \ : \ d \text{ admissible metric on } X \sqcup Y\} .$$

It is evident from the definitions above that computing the Hausdorff-Gromov distance may be difficult and painful, so we are going to need a more useful tool to estimate the Gromov-Hausdorff distance between two metric spaces. The tool we are going to use is called a correspondence between spaces X and Y , which roughly means that for every point of X exist at least one corresponding point in Y, and vice versa. The simplest example of a correspondence is the obvious one, any surjective map $f : X \to Y$ defines a correspondence between X and Y.

Definition 4.0.3. *Let X, Y be two metric spaces. A correspondence between X and Y is a set $\mathcal{R} \subset X \times Y$ satisfying the following condition:*

For every $x \in X$ exists at least one $y \in Y$ such that $(x, y) \in \mathcal{R}$ and similarly for every $y \in Y$, there exists an $x \in X$ such that $(x, y) \in \mathcal{R}$

Let $\mathcal{R}$ *be a correspondence between metric spaces* X *and* Y*. The distorsion of* $\mathcal{R}$ *is defined by*

$$\text{dis}\mathcal{R} = \sup\{|d_X(x,x') - d_Y(y,y')| : (x,y),(x',y') \in \mathcal{R}\} ,$$

where d_X *and* d_Y *are the obvious metrics.*

Theorem 4.0.4. *Let* X, Y *be metric spaces. Then*

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\mathcal{R}} \text{dis}(\mathcal{R}) .$$

Proof. Let X, Y be metric spaces such that $d_{GH}(X,Y) < r$. We may assume that X, Y are subspaces of some metric space Z (we can always do that because X and Y are compact) and $d_H(X,Y) < r$ in Z . We define

$$\mathcal{R} = \{(x,y) : x \in X, y \in Y, d(x,y) < r\}$$

where d is the metric of Z, and observe that $\mathcal{R}$ is correspondence between X and Y because $d_H(X,Y) < r$. From the triangle inequality

$$|d(x,x') - d(y,y')| \leq d(x,y) + d(x',y') < 2r ,$$

then taking the supremum over all pairs $(x,y),(x',y') \in \mathcal{R}$ we get

$$\text{dis}\mathcal{R} < 2r .$$

For the opposite inequality

$$d_{GH}(X,Y) \leq \frac{1}{2}\text{dis}\mathcal{R}$$

for any correspondence $\mathcal{R}$, let $\text{dis}\mathcal{R} = 2r$. It suffices to show that there is a semi-metric d on the disjoint union $X \sqcup Y$ such that $d|_{X\times X} = d_X$ and $d|_{Y\times Y} = d_Y$ and $d_H(X,Y) \leq r$ (d_X, d_Y are the metrics of X and Y respectively). To achieve this we define the following metric

$$d(x,y) = \inf\{d_X(x,x') + r + d_Y(y,y') : (x',y') \in \mathcal{R}\} .$$

□

Definition 4.0.5. *A map* $f : X \to Y$ *is called an* $\varepsilon-$*isometry if* $\text{dis}(f) < \varepsilon$ *and the set* $f(X)$ *is an* ε*- net on* Y*.*

Note that in the definition we don't require the function be continuous.

We are now ready to refer and prove the following corollary which connects the correspondence with the GH distance :

Corollary 4.0.6. *Let X and Y be compact metric spaces. Then :*

1. *If $d_{GH}(X,Y) < \varepsilon$, then $\exists$ 2ε-isometry $f : X \to Y$.*

2. *If $\exists$ $\varepsilon-$ isometry $f : X \to Y$, then $d_{GH}(X,Y) < 2\varepsilon$.*

Proof. 1. Let $\mathcal{R}$ be a correspondence between X and Y with dis$\mathcal{R} < 2\varepsilon$. For every $x \in X$, we choose a point $f(x) \in Y$ such that $(x, f(x)) \in \mathcal{R}$. This defines a map $f : X \to Y$ which satisfies $disf < 2\varepsilon$ and it's sufficient to prove that $f(X)$ is 2ε- net in Y. For a $y \in Y$, we consider an $x \in X$ such that $(x, y) \in \mathcal{R}$. Note that both y and $f(x)$ are in correspondence with x , one has $|d(x, x) - d(y, f(x)| \le dis(\mathcal{R})$ which it means

$$d(y, f(x)) \le d(x, x) + \text{dis}\mathcal{R} < 2\varepsilon$$

so $d(y, f(x)) < 2\varepsilon$ and hence $d(y, f(X)) < 2\varepsilon$.

2. Let f be an ε-isometry .
Define $\mathcal{R} \subset X \times Y$ by $\mathcal{R} = \{(x, y) \in X \times Y : d(y, f(x)) \le \varepsilon\}$
Then $\mathcal{R}$ is a correspondence because $f(X)$ is an ε- net in Y. If $(x, y) \in \mathcal{R}$ and $(x', y') \in \mathcal{R}$, one has

$$\begin{aligned} |d(y, y') - d(x, x')| &\le |d(f(x), f(x')) - d(x, x')| + d(y, f(x)) + d(y', f(x')) \\ &\le \text{dis}f + 2\varepsilon \le 3\varepsilon \ . \end{aligned}$$

Then, from the previous theorem it's $d_{GH}(X,Y) \le \frac{3}{2}r < 2r$ and we are done.

□

4.1 Precompactness

Gromov-Hausdorff convergence is similar to uniform convergence, in the sense that it's not necessary for a sequence of functions to converge uniformly in the whole space, given that it converges uniformly on the compact subsets, since

the sequence may not be bounded .
Similarly, Gromov-Hausdorff convergence may fail in the non-compact case, even if every member of the sequence is compact. We can however extend this notation and it's still useful.

As a motivation we can consider the classic sphere example. Consider a sequence of spheres of radii increasing to infinity. There is no reason for this sequence to converge in the Gromov-Hausdorff sense, but if we work locally on a set of fixed diameter as the radii tend to infinity this sequence of subsets should converge to the Euclidean plane as seen in the next picture .

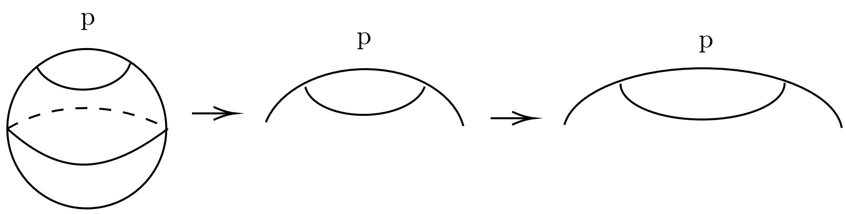


Figure 4.1: Spheres with increasing radii

Using this observation, we will give an extended notion of convergence for non compact sets.

Definition 4.1.1. *A pointed metric space is a pair* (X, p) *consisting of a metric space* X *and a point* $p \in X$.

A sequence of pointed metric spaces converges in the Hausdorff-Gromov sense to a pointed metric space (X, p) if the following holds :
For every $r > 0$ and $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, there exists a map $f : B_r(p) \to X$ such that the following hold :

1. $f(p_n) = p$
2. $\text{dis} f < \varepsilon$
3. the ε-neighborhood of the set $f(B_r(p_n))$ contains the ball $B_{r-\varepsilon}(p)$.

In that case we write $(X_n, p_n) \to (X, p)$.

Remarks. *Note that many annoying or weird things may happen in this kind of convergence.*

1. *A sequence of Riemannian manifolds may fail to converge to a Riemannian manifold.*

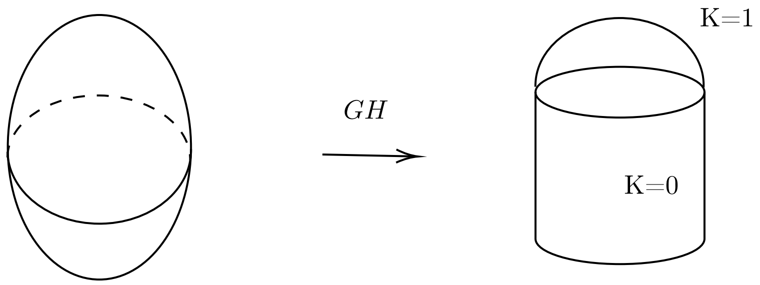


Figure 4.2: Example of GH limit of R.m not being a R.m

2. *Let* $\{X_n\}_{n=1}^{\infty}$ *be a sequence of length spaces , X a complete metric space , and* $X_n \xrightarrow{GH} X$. *Then X is a length space. (for the proof see Burago)*

3. *Consider* $\mathbb{S}^1 \times \mathbb{S}^1(\frac{1}{k}) \to \mathbb{S}^1$. *This is an example of a sequence of two dimensional manifolds with the limit being a one dimensional Riemannian manifold . This phenomenon is called "collapsing".*

4. *Under the Hausdorff-Gromov convergence the topology may change. In the previous example the first fundamental group of the sequence is :* $\pi_1(\mathcal{M}_i^n) = \mathbb{Z} \oplus \mathbb{Z}$ *, for every i. But the GH limit has* $\pi_1(\mathbb{S}^1) = \mathbb{Z}$

5. *Every compact length space can be obtained as a Gromov–Hausdorff limit of finite graphs .*

As we mentioned above if X and Y are isometric they have zero GH distance but if $d_{GH}(X,Y) = 0$ does not generally imply that X and Y are isometric.
To construct a metric space with the Gromov-Hausdorff topology we have to ensure that this equivalence holds. In order to achieve that, we restrict ourselves to compact metric spaces. Then the class of compact metric spaces modulo isometries is a metric space. We denote it by $\mathcal{M}$.
As a metric space, it's very useful to look for precompact subsets of $\mathcal{M}$ in the metric topology to ensure the existence of a convergent subsequence (in the GH topology). Since GH is a weak topology we expect many compact sets. Indeed many classes of metric spaces are precompact in the GH topology.

Theorem 4.1.2. *Let* $\mathcal{X}$ *be a class of compact metric spaces which satisfies:*

- *There is a constant D such that* $\operatorname{diam} X \leq D$ *for all* $X \in \mathcal{X}$.

- *$\forall\, \varepsilon > 0$, there exists a natural $N = N(\varepsilon)$ such that for every $X \in \mathcal{X}$, there exists an ε-net in X with no more than N points.*

Then the class $\mathcal{X}$ is precompact in the Gromov - Hausdorff topology.

Proof. We will use a diagonal argument to construct the metric . Let $\{X_i\}$ be the sequence. We know that for every ε there exists a $N(\varepsilon)$ such that there is an ε- dense set with at most $N(\varepsilon)$ elements.
First note that in a every space X_i for every j we can find an $\frac{1}{j}$−net $\Gamma_{i,j}$, then passing to a subsequence we can assume that every $\Gamma_{i,j}$ has the same number of points N_j (independent of i).
Now set $\Gamma_i = \bigcup_j \Gamma_{j,i}$, then Γ_i is a countable dense in each X_i and we can write it as $\{x_1^i, \ldots, x_j^i\}$. Now for every couple of natural numbers we can define

$$d_\infty(x_k, x_l) = \lim_{i\to\infty} d(x_k^i, x_l^i)\ .$$

We can assume the existence of this limit passing to a subsequece since $d_i(x_k^i, x_l^i)$ is bounded by D for every i , by a diagonal argument.
Finally, d_∞ can be easily proven to be a semi- metric. We consider the quotient of X over the equivalence relation on X : xR_dy if and only if $d(x, y) = 0$ (so that $\frac{X}{d}$ is well defined). The metric space we're looking for is the completion of X/R_d with respect to d_∞. □

Usually a class of compact metric spaces satisfying these is called uniformly totally bounded. The terminology comes from the fact that if for any $\varepsilon > 0$, any ε-net (actually any ε- separated subset) on a metric space (X, d) is finite, then X is totally bounded.
An alternative but similar proof can be found in Burago's book [BBI] on metric geometry .
As an application we get the next theorem (a precompact class of Riemannian manifolds). This theorem is very useful in the theory of manifolds with Ricci curvature lower bounds and is often called **Gromov's Compactness Theorem** .

Theorem 4.1.3.
For $n \geq 2$, $k \in \mathbb{R}$ and $d > 0$, the following classes are precompact in Gromov-Hausdorff topology.

1. *The class of closed Riemannian manifolds with* $\mathrm{Ric} \geq (n-1)H$ *and* $\mathrm{diam} < d$.

2. *The class of pointed complete Riemannian n-manifolds with* $\mathrm{Ric} \geq (n-1)H$.

Proof. For the proof we use Gromov's idea of packing . We will use the theorem above.
Consider a sequence of manifolds as above and let $\{x_j\}_{j=0}^N$ be a maximal subset of a manifold $\mathcal{M}_i^n$ with the property : $\{x_j\}$ being ε dense subset and the balls $B_{\frac{\varepsilon}{2}}(x_j)$ are disjoint. Then, it suffices to prove that the number of these $\{x_j\}$ are bounded by a constant independent of i. First note that

$$\mathrm{Vol}(\mathcal{M}) = \mathrm{Vol}(B_d(x_j)) \geq \mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j)) \ ,$$

so

$$\mathrm{Vol}(\mathcal{M}) \geq \sum_j \mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j)) \ ,$$

hence we have

$$\sum_j \frac{\mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j))}{\mathrm{Vol}(\mathcal{M})} \leq 1 \ .$$

Now we use the Bishop-Gromov inequality

$$\frac{\mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j))}{\mathrm{Vol}_{-H}(B_{\frac{\varepsilon}{2}}(0))} \geq \frac{\mathrm{Vol}(\mathcal{M})}{\mathrm{Vol}_{-H}(B_d(0))} \ ,$$

where $B_d(0)$ denotes the ball of radius d centered at 0 in the model space of constant curvature $-H$ and Vol_{-H} is the volume respectively. The last inequality is

$$\frac{\mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j))}{\mathrm{Vol}(\mathcal{M})} \geq \frac{\mathrm{Vol}_{-H}(B_{\frac{\varepsilon}{2}}(0))}{\mathrm{Vol}_{-H}(B_d(0))} \ .$$

We sum over j,

$$1 \geq \sum_j \frac{\mathrm{Vol}(B_{\frac{\varepsilon}{2}}(x_j))}{\mathrm{Vol}(\mathcal{M})} \geq N \ \frac{\mathrm{Vol}_{-H}(B_{\frac{\varepsilon}{2}}(0))}{\mathrm{Vol}_{-H}(B_d(0))}.$$

In the previous inequality the quantity

$$\frac{\mathrm{Vol}_{-H}(B_{\frac{\varepsilon}{2}}(0))}{\mathrm{Vol}_{-H}(B_d(0))}$$

is independent of i, so if we denote it with $c(n,d,\varepsilon)$ we proved

$$N \leq \frac{1}{c(n,d,\varepsilon)} \ .$$

□

Remarks. 1. *There are other examples of precompact classes of Riemannian manifolds. For example, the class of manifolds with bounded volume and injectivity radius and bounded diameter and (sectional) curvature.*

2. *The theorem above guarantees the existence of a tangent cone for manifolds with a lower Ricci curvature bound. Tangent cones are very useful when trying to understand the structure of the limit space and will be defined and studied later.*

3. *As we have already seen , the limit space of the previous theorem need not be a Riemannian manifold in general , but it is always a length space .*

Chapter 5

Almost Splitting theorem

Definition 5.0.1. *We denote with* $\Psi = \Psi(e_1, e_1, ..., e_n|c_1, ..., c_k)$ *some nonnegative function such that for any fixed* $c_1, ..., c_k$*,*

$$\lim_{e_1,..,e_n \to 0} \Psi = 0$$

and also denote with E the map

$$E(x) = \overline{x, q^+} + \overline{x, q^-} - \overline{q^+, q^-} \ .$$

The quantity E is sometimes called excess function .
In the classic splitting theorem we prove that the sum of Busemann functions vanishes, here we do the analogous for the almost splitting which is to prove $b_+ + b_- = E(x) - E(p)$. Note that Ψ may change from line to line in the following proofs.

Theorem 5.0.2 (Abresch-Gromoll)**.** *If*

$$\begin{aligned} \text{Ric} &\geq -(n-1)\delta \\ \overline{p, q_\pm} &\geq L \ (> 2R + 1) \\ E(p) &\leq \epsilon \ . \end{aligned}$$

Then for $\Psi(\delta, L^{-1}, \epsilon|n, R)$*,*

$$E \leq \Psi \ on \ B_R(p)$$

Proof. We will use theorem 3.2.4 and for every x in the Ball we take a bound .

By Laplacian comparison exist $\Psi_1 = \Psi(\delta, L^{-1}|n, R)$ such that $\Delta E \leq \Psi_1$ on $B_R(p)$ because of the laplacian comparison for the map r_q
Put $\overline{x,p}$ fix $0 < \eta < R$ to be specified below . We can choose ϵ to satisfy

$$\epsilon \leq \Psi_1 \underline{L}_{R+1}(R) \leq \Psi \underline{L}_{R+1}(\eta)$$

and note

$$\Delta E \leq \Psi_1$$

$$E(p) \leq \varepsilon \leq \Psi_1 \underline{L}_{R+1}(\eta)$$

then by (3.5) (Lip$E \leq 2$)

$$E(x) \leq \Psi_1 \underline{L}_{R+1}(\eta) + 2\eta$$

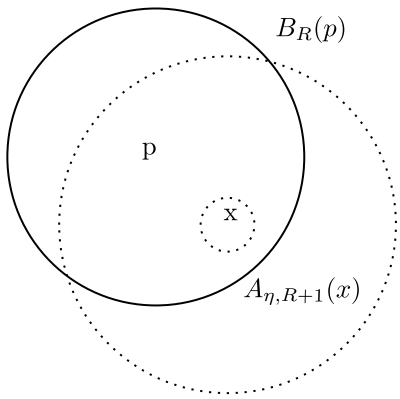


Figure 5.1: Applying previous theorem for the Annulus

In picture we can see that we use the theorem for $\delta = \Psi_1$, $t = 0$ the point p "plays the role " of x . In other words we proved that

$$E(x) \leq \Psi_1 \underline{L}_{R+1}(\eta) + 2\eta$$

for x such that $\eta \leq r < R$ but this inequality also holds for $r \leq \eta$:

$$E(x) - E(p) \leq 2r$$

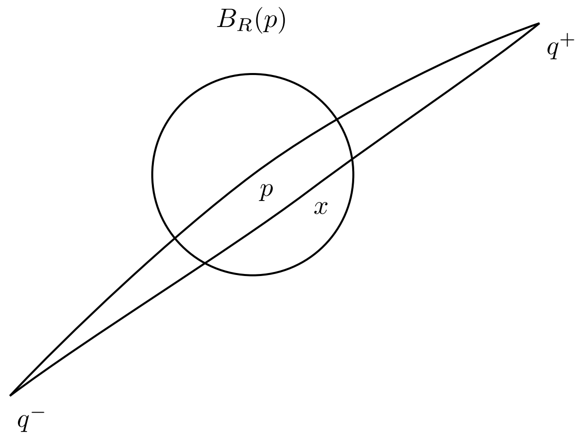
and since $E(p) \leq \varepsilon \leq \Psi_1 \underline{L}_{R+1}(\eta)$ we get the wanted inequality . Now we chose η to satisfy :

$$\Psi_1 \underline{L}_{R+1}(\eta) = 2\eta$$

then $E \leq \Psi$ follows .

□

Geometrically the above theorem says that by applying these results to E , Abresch-Gromoll obtained an estimate for thin triangles .



Let $\gamma_\pm$ denote minimal geodesics from $q_\pm$ to p and $b_\pm$ is the Busemann function. We find a harmonic function $\beta_\pm$ which is equal to the Busemann function on the boundary i.e. $b_\pm|_{\partial B_R(p)} = \beta_\pm|_{\partial B_R(p)}$. The existence of the harmonic function $\beta_\pm$ is guaranteed by classic PDE theory as we can ensure the boundary $\partial B(p, R)$ is smooth. The boundary may not be smooth in general, but a non smooth set can be approximated by smooth boundary, this follows from Whitney's approximation theorem.

Since our goal to generalize the splitting theorem in a quantitative way , our assumptions will be weaker versions of theorem , The first assumption we will make is that $Ric \geq -(n-1)\delta$ which is clear and the next assumptions are a way to describe the notion of "almost a geodesic line " whose are : There are q^+, q^- such that

$$\overline{p, q^+} \geq L, \overline{p, q^-} \geq L$$

and

$$E(p) \leq \varepsilon$$

where

$$E(x) = \overline{x, p} - \overline{x, q^+} - \overline{x, q^-}$$

As in the proof of classical splitting theorem , here an harmonic function is needed , since this generalized busseman function is not harmonic we will show that an harmonic β_+ which coincide with b^+ in the boundary of a Ball , is near to b^+ in some sense , It will be achieved using the next lemmas.

Lemma 5.0.3. *Let*

$$\mathrm{Ric} \geq -(n-1)\delta$$

$$\overline{p, q^{\pm}} \geq L$$

$$E(p) \leq \varepsilon$$

.

Then

$$|b_+ - \beta_+| \leq \Psi \ .$$

Proof. We will use the Abresch-Gromoll theorem and maximum principles , so these maps b_+ and β_+ are uniformly close .
First observe that $\Delta(b_+ - \beta_+) = \Delta b_+$, (beacuse β_+ is harmonic) and because of Laplacian comparison theorem we get in weak sense the following

$$\Delta b_+ \leq \Psi$$

that together with quantitative maximum principle implies that

$$b_{\pm} - \beta_{\pm} \geq -\Psi$$

Note also that

$$E(x) - E(p) = \overline{x, q^+} + \overline{x, q^-} - \overline{p, q^+} - \overline{p, q^-} = b_+(x) + b_-(x)$$

combined with Abresch-Gromoll $b_+(x) + b_-(x) \leq \Psi$. Thus,

$$b_+ + b_- \leq \Psi$$

so

$$\beta_+ - \beta_- = (\beta_+ - b_+) + (\beta_- - b_-) + (b_+ - b_-) \leq \Psi$$

Now for the lower bound it is

$$b_+(x) + b_-(x) = E(x) - E(p) \geq -\varepsilon$$

but on the boundary $\partial B_R(p)$ two maps coincide ie $b_+ = \beta_+$ then by the maximum principle

$$\beta_+ + \beta_- \geq -\varepsilon$$

Combining the above we get:

$$-\Psi \leq b_+ - \beta_+ = (b_+ + b_-) - (\beta_+ - \beta_-) - (b_- - \beta_-) \leq 2\Psi + \varepsilon$$

so finally

$$-\Psi \leq b_+ - \beta_+ \leq 2\Psi + \varepsilon$$

and the last implies that

$$|b_+ - \beta_+| \leq \Psi \tag{5.1}$$

□

We now present the gradients of these map are L^2 - close using the theorem above .

Lemma 5.0.4. *Let as before*

$$\mathrm{Ric} \geq -(n-1)\delta$$

$$\overline{p, q^{\pm}} \geq L$$

$$E(p) \leq \varepsilon.$$

Then

$$\fint_{B_R} |\nabla\beta_+ - \nabla b_+|^2 \ \leq \Psi \ . \tag{5.2}$$

Proof. First by integration by part we get

$$\fint_{B_R(p)} |\nabla b_+ - \nabla\beta_+|^2 = -\fint_{B_R(p)} (b_+ - \beta_+)\Delta b_+$$

because $b_+ = \beta_+$ on the boundary $\partial B_R(p)$ so obviously it is

$$\fint_{B_R(p)} |\nabla b_+ - \nabla\beta_+|^2 \leq \fint_{B_R(p)} |b_+ - \beta_+| \, |\Delta b_+|$$

now we already proved that

$$|b_+ - \beta_+| \leq \Psi$$

so the last one gives

$$\fint_{B_R(p)} |\nabla b_+ - \nabla\beta_+|^2 \leq \Psi \fint_{B_R(p)} |\Delta b_+|$$

Now we use the following (more general lemma of [C] , lemma 1.8)
With the assumptions of the theorem for $q \notin B_R(p)$ and $f : \mathbb{R}_+ \to \mathbb{R}$ for which $f' \geq 0$, then

$$\fint_{B_R(p)} |\Delta(f \circ r_q)| \leq 2 \max_{B_R(p)} \Delta_{-\delta}(f \circ r_q(p)) + \max_{\partial B_R(p)} f'(r_q(p)) \frac{\mathrm{Vol}(\partial \mathrm{B_R(p)})}{\mathrm{Vol}(\mathrm{B_R(p)})}$$

Now we present the proof of the above inequality .
for a function g we set

$$g_+(p) = \max\{g(p), 0\} \ \ and \ \ g_-(p) = \min\{g(p), 0\}$$

so

$$g = g_+ - g_- \ \text{ and } \ \int_{B_R} g = \int_{B_R} g_+ - \int_{B_R} g_-$$

Hence

$$\int_{B_R} |g| = -\int_{B_R} g + 2\int_{B_R} g_+ \leq \left| \int_{B_R} g \right| + 2\mathrm{Vol(B_R)} \max_{\mathrm{B_R}} \mathrm{g}_+$$

We apply for $\Delta(f \circ r_q)$ so by Laplace comparison and Stoke's theorem

$$\begin{aligned}\int_{B_R(p)} |\Delta(f \circ r_q)| \leq & \left| \int_{B_R(p)} \Delta(f \circ r_q) \right| + 2 \ \mathrm{Vol}(B_R(p)) \max_{B_R(p)} \Delta_{-\delta}(f \circ r_q) \\ &= \left| \int_{\partial B_R(p)} *d(f \circ r_q) \right| + 2\mathrm{Vol}(B_R(p)) \max_{B_R(p)} \Delta_{-\delta}(f \circ r_q) \\ &\leq \mathrm{Vol}(\partial B_R(p)) \max_{\partial B_R(p)} (f' \circ r_q) + 2\mathrm{Vol}(B_R(p)) \max_{B_R(p)} \Delta_{-\delta}(f \circ r_q)\end{aligned}$$

Now we can use this in the specific case $f = x$ and we get

$$\int_{B_R(p)} |\Delta b_+| \leq \mathrm{Vol}(\partial B_R(p)) + 2\mathrm{Vol}(B_R(p))\Psi$$

so it is

$$\fint_{B_R(p)} |\Delta b_+| \leq \frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol}(B_R(p))} + \frac{\mathrm{Vol}(B_R(p))}{\mathrm{Vol}(B_R(p))}\Psi$$

So we proved that

$$\fint_{B_R(p)} |\nabla b_+ - \nabla \beta_+|^2 \leq \left(2\Psi + \frac{\mathrm{Vol}(\partial B_R(p))}{\mathrm{Vol} B_R(p)} \right) \Psi$$

Finally we use Volume comparison so then

$$\fint_{B_R(p)} |\nabla b_+ - \nabla \beta_+|^2 \leq \left(2\Psi + \frac{\mathrm{Vol}(\partial B_R(\underline{p}))}{\mathrm{Vol} B_R(\underline{p})} \right) \Psi$$

$\square$

Remarks. *There are many ways to get the above inequality . Since the cut locus of* $q^{\pm}$ *has zero measure* $|\Delta b_+|$ *is well defined in the this context. On the one hand we can use for positive and negative part the fundamental inequality* $a - b \leq |a+b| + 2a$ *or alternatively since in sense of distributions* Δr *is a singed measure whose absolutely continuous part has density the smooth function,* Δr *and the singular is a measure supported on* C_q*. In [C] there is presented that for the total mass* $||\Delta r||$ *the following holds .*

$$|\Delta r| \leq 2 \int_{B_R^+(p)} \Delta r - 2 \int_{\partial B_R^-(p)} \langle \nabla r, N \rangle$$

where $B_R^+(p) \subseteq B_R(p) \setminus (q \cup C_q)$ *is the set on which* $\Delta r > 0$ *and* $\partial B_R^- \subseteq \partial B_R(p) \setminus (q \cup C_q)$ *the set on which* $\langle \nabla r, N \rangle < 0$ *and this implies the fact that*

$$||\Delta r|| \leq Vol(\partial B_R(p)) + 2\Psi Vol(B_R(p))$$

and by the last one we can easily get

$$\fint_{B_R(p)} |\Delta b_+| \leq 2\Psi + \frac{Vol(\partial B_R(p))}{Vol(B_R(p))}$$

Lemma 5.0.5. *Let as usual*

$$\mathrm{Ric} \geq -(n-1)\delta$$

$$\overline{p, q^{\pm}} \geq L$$

$$E(p) \leq \varepsilon$$

. Then,

$$\fint_{B_{\frac{R}{2}}(p)} |\mathrm{Hess}_{\beta_+}|^2 \leq \Psi \ .$$

Proof. First we note that β_+ is harmonic, thus by Bochner's formula we have

$$\frac{1}{2}\Delta |\nabla \beta_+|^2 = |\mathrm{Hess}_{\beta_+}| + \mathrm{Ric}(\nabla \beta_+, \nabla \beta_+) \ .$$

We note also that $\mathrm{Ric} \geq -(n-1)\delta$, which means

$$\mathrm{Ric}(\nabla \beta_+, \nabla \beta_+) \geq -(n-1)\delta |\nabla \beta_+|^2$$

We multiply both sides by a cutoff function ϕ we already constructed satisfying

$\phi|_{B_{R/2}(p)} \equiv 1$ (5.1.2)and $|\Delta\phi| \leq c(n,R,H)$ and we integrate over the ball $B_R(p)$

$$\begin{aligned}\fint_{B_R(p)} \phi|\mathrm{Hess}_{\beta_+}|^2 &\leq \fint_{B_R(p)} \frac{1}{2}\phi\Delta(|\nabla\beta_+|^2-1)+\delta(n-1)|\nabla\beta_+|^2\\ &\leq \fint_{B_R(p)} \frac{1}{2}\Delta\phi\ (|\nabla\beta_+|^2-1)+\delta(n-1)|\nabla\beta_+|^2\\ &\leq c(n)\fint_{B_R(p)} (||\nabla\beta_+|^2-1|)+\delta(n-1)|\nabla\beta_+|^2\\ &\leq c(n)\fint_{B_R(p)} (||\nabla\beta_+|^2-|\nabla b_+|^2|)+\delta(n-1)|\nabla\beta_+|^2\\ &\leq \Psi\ .\end{aligned}$$

For the first term we use the lemma above and the fact that in the boundary $\partial B_R(p)$ it is $\beta_+ = b_+$. Finally, for the last term we used Green's identity and the previous proposition with the fact that $\delta \to 0$ □

Since tha function ϕ is identically equal to 1 on the $B_{\frac{R}{2}}(p)$ we conclude

$$\fint_{B_{\frac{R}{2}}(p)} |\mathrm{Hess}_{\beta_+}|^2 \leq \Psi$$

5.1 Pythagorean Theorem

Finally we are going to prove a key result which is known as the quantitative version of the Pythagorean Theorem , then from that we can derive the final Almost Splitting Theorem .

Theorem 5.1.1 (Pyhthagorean theorem). *Assume*

$$\mathrm{Ric} \geq -(n-1)\delta$$

$$\overline{p, q^{\pm}} \geq L$$

$$E(p) \leq \varepsilon$$

Let $x, z, w \in B_{\frac{R}{8}}(p)$, with $x \in b_+^{-1}(\alpha)$ and z a point on $b_+^{-1}(\alpha)$ closest to w ,then

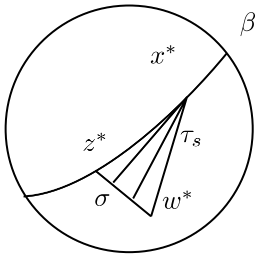
$$|\overline{x,z}^2 + \overline{z,w}^2 - \overline{x,w}^2| \leq \Psi\ . \tag{5.3}$$

Proof. Thanks to segment inequality we can find x^*, z^*, w^* such that

$$\overline{x,x^*}, \overline{z,z^*}, \overline{w,w^*} \leq \Psi$$

for which exist segment connecting z^* with w^* and if $\sigma : [0,e] \to \mathcal{M}^n$ is minimizing from z^*, w^* then there is $U \subseteq [0,e]$ (a.e equal) such that for all $s \in U$, the minimal geodesic $\tau_s : [0,l(s)] \to M^n$ joining x^* to $\sigma(s)$ is unique and the following holds

$$\int_U \int_0^{l(s)} |\mathrm{Hess}_{\beta_+}(\tau_s(t))| \mathrm{dtds} \leq \Psi$$



First using the Hessian estimate $\forall\, t \in [0,l(s)]$ we have

$$\left|\langle \nabla\beta_+(\tau_s(t)), \tau_s'(t))\ \rangle - \langle \nabla\beta_+(\tau_s(l(s))), \tau_s'(l(s))\rangle\right| \leq \int_0^{l(s)} \left|\mathrm{Hess}_{\beta_+}(\tau_s(u))\right| du \ \leq \Psi \tag{5.4}$$

Particularly,

$$\begin{aligned} \left|\ \langle \nabla\beta_+(\tau_s(t)), \tau_s'(t)\rangle - \langle \nabla\beta_+(\tau_s)(l(s))), \tau_s'(l(s))\rangle\ \right| &= \left| \int_t^{l(s)} \frac{d}{du} \langle \nabla\beta_+(\tau_s(u)), \tau_s'(u)\rangle du \right| \\ &\leq \left| \int_t^{l(s)} \mathrm{Hess}_{\beta_+}(\tau_s(u), \tau_s(u)) \right| \\ &\leq \left| \int_0^{l(s)} \mathrm{Hess}_{\beta_+}(\tau_s(u)) du \right| \leq \Psi \end{aligned}$$

Now note that we can apply the segment inequality for the function
$\left|\ |\nabla\beta_+(\sigma(s))|^2 - 1\ \right|$ i.e especially we can conclude that

$$\int_0^e ||\nabla\beta_+(\sigma(s))|^2 - 1| ds \leq \Psi \tag{5.5}$$

Also it is

$$\beta_+(w^*) - \beta_+(z^*) = \int_0^e \langle \nabla\beta_+(\sigma(s)), \sigma'(s)\rangle ds$$

and for $\gamma : [0,s] \to M^n$ geodesic connecting a point w to a point z on level set $\beta_+^{-1}(\alpha)$ closest to w then

$$|\beta_+(\gamma(s)) - \beta_+(\gamma(0)) - \overline{w,z}| \leq \Psi \tag{5.6}$$

but the last equation is

$$\beta_+(\gamma(s)) - \beta_+(\gamma(0)) = \overline{w,z} \pm \Psi$$

and we can write the last as

$$|\int_0^e (\langle \nabla\beta_+(\sigma(s)), \sigma'(s)\rangle - 1)ds| \leq \Psi \tag{5.7}$$

By (5.5) and (5.7) we get

$$\begin{aligned}\int_0^e |\langle \nabla\beta_+(\sigma(s)) - \sigma'(s)\rangle|^2 \, ds &= \int_0^e |\nabla\beta_+(\sigma(s))|^2 - 2\langle \nabla\beta_+(\sigma(s)), \sigma'(s)\rangle + 1 \, ds \\ &= \int_0^e |\nabla\beta_+(\sigma(s))|^2 - 1 - 2(\langle \nabla\beta_+(\sigma(s)), \sigma'(s)\rangle - 1) \, ds \leq \Psi \, .\end{aligned}$$

Now we are finally ready to complete the proof

$$\begin{aligned}\frac{1}{2}\overline{z,w}^2 = \frac{1}{2}\overline{z^*,w^*}^2 \pm \Psi &= \int_0^e s ds \pm \Psi \\ &= \int_0^e \beta_+(\sigma(s)) - \beta_+(\sigma(0))ds \pm \Psi \\ &= \int_U \beta_+(\tau_s(l(s))) - \beta_+(\tau_s(0))ds \pm \Psi \\ &= \int_U \int_0^{l(s)} \langle \nabla\beta_+(\tau_s(t)), \tau_s'(t)\rangle dtds \pm \Psi \\ &\overset{(5.4)}{=} \int_U \int_0^{l(s)} \langle \nabla\beta_+(\tau_s(l(s))), \tau_s'(l(s))\rangle dtds \pm \Psi \\ &= \int_U l(s)\langle \nabla\beta_+(\sigma(s)), \tau_s'(l(s))\rangle ds + \Psi \\ &= \int_U l(s)\langle \sigma(s)', \tau_s'(l(s))\rangle ds \pm \Psi\end{aligned}$$

By the classic variation formula of arc length it is $\forall \ s \in U$:

$$l'(s) = \langle \sigma'(s), \tau_s'(l(s))\rangle$$

and then we get :

$$\frac{1}{2}\overline{z,w}^2 = \int_U l(s)l'(s)ds \pm \Psi = \frac{1}{2}l^2(e) - l^2(0) \pm \Psi = \frac{1}{2}\overline{x,w}^2 - \frac{1}{2}\overline{x,z}^2 \pm \Psi$$

□

It's just an application of the lemma above that $B_{\frac{R}{8}R}(p)$ is $\Psi-$Gromov-Hausdorff close to a subset $B_{\frac{R}{8}}(p) \subset \mathbb{R} \times \beta_+^{-1}(0)$, where $\beta^{-1}(0)$ is equipped with the subset metric

Theorem 5.1.2 (Almost splitting theorem)**.**
Assume

$$\mathrm{Ric} \geq -(n-1)\delta$$

$$\overline{p, q^{\pm}} \geq L$$

$$E(p) \leq \varepsilon \ .$$

Then there is a length space X and a ball $B_R(0,x) \subseteq \mathbb{R} \times X$ such that

$$d_{GH}(B_{\frac{R}{8}}(p), B_{\frac{R}{8}}(0,x)) \leq \Psi = \Psi(\delta, L^{-1}, \varepsilon | n, R) \ .$$

Proof. The proof is an application of the quantitative version of the Pythagorean Theorem. We have to define a function f that is Ψ - GH approximation between the ball and a subset of the product .
Without loss of generality we may assume that $\beta_+(p) = 0$ and let $X = \beta_+^{-1}(0)$.
Now consider the Hausdorff Gromov approximation $f : B(0, \frac{R}{8}) \to \mathbb{R} \times X$ by

$$f : w \to (w', \beta_+(w)) \ ,$$

with w' being closest point to w in X .
We have to prove that

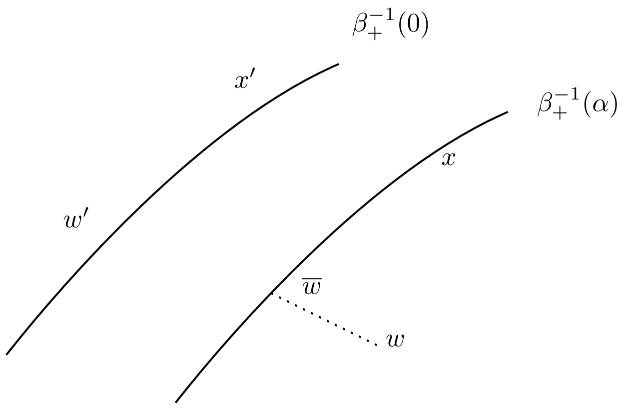
$$\overline{f(x), f(w)}^2 = \overline{x, w}^2 \pm \Psi$$

Denote $\alpha = \beta_+(x)$ and $b = \beta_+(w)$ also with x' closest point to x in $\beta^{-1}(0)$ and with w' closest point to w in $\beta^{-1}(0)$

So we have to show

$$\overline{x, w}^2 = |\beta_+(x) - \beta_+(w)|^2 + \overline{x', w'}^2 \pm \Psi$$

i.e the map f does not distorts too much.



First we denote with $\overline{w}$ closest point to w on $\beta_+^{-1}(\alpha)$ We use pythagoras theorem 5.3 for the triangle $\{x, w, \overline{w}\}$ then we get

$$\overline{x,w}^2 = \overline{x,\overline{w}}^2 + \overline{w,\overline{w}} \pm \Psi$$

so by (5.6) this is

$$\overline{x,w}^2 = \overline{x,\overline{w}}^2 + |\beta_+(w) - \beta_+(\overline{w})|^2 \pm \Psi$$

Since $\beta_+(\overline{w}) = \beta_+(x)$ the last one is

$$\overline{x,w}^2 = \overline{x,\overline{w}}^2 + |\beta_+(w) - \beta_+(x)|^2 \pm \Psi$$

To complete the proof we need to show that

$$\overline{x,\overline{w}}^2 = \overline{x',w'}^2 \pm \Psi$$

Apply pythagorean theorem theorem (5.3) for the triangle $\{x, x', w'\}$ and we get

$$\overline{x,w'}^2 = \overline{x',w'}^2 + \overline{x,x'}^2 \pm \Psi$$

And also for the triangle $\{x, w', y\}$ so it is

$$\overline{x,w'}^2 + \overline{x,y}^2 + \overline{y,w'}^2 \pm \Psi$$

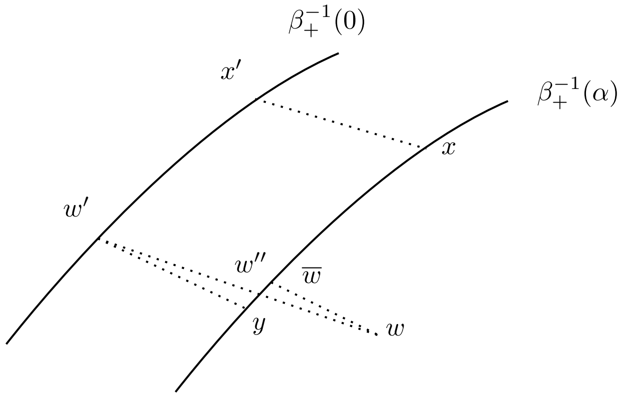
So combing these we get

$$\overline{x,y}^2 = \overline{x',w'}^2 \pm \Psi$$

because by 5.6

$$\overline{x,x'} = \overline{y,w'} = \alpha \pm \Psi$$

Then now its enough to show that points y and $\overline{w}$ are close . We prove that using the point w'', which is the point that geodesic connecting w and w' intersects $\beta_+^{-1}(\alpha)$. We use again Pythagorean theorem, for triangles $\{w, \overline{w}, w''\}$ and $\{y, w', w''\}$ then we can conclude the wanted equation .



Specifically note that by (5.6) we get

$$\overline{w, \overline{w}} = \overline{w, w''} = \beta - \alpha \pm \Psi$$

and also using the same equation

$$\overline{y, w'} = \overline{w', w''} = \alpha \pm \Psi$$

then using these and the fact that

$$\overline{w', w''}^2 = \overline{y, w''}^2 + \overline{y, w'}^2 \pm \Psi$$

and the equation 5.3 for the $\{w, w'', \overline{w}\}$

$$\overline{w, w''}^2 = \overline{w, \overline{w}} + \overline{w'', \overline{w}} \pm \Psi$$

suffices to complete the proof .

□

Remarks. *Note that we assumed that the geodesic connecting points w, w' intersects the level set $\beta_+^{-1}(\alpha)$, if points x and w lie in opposite sides of the level set $\beta^{-1}(0)$ we can similarly prove the theorem using same ideas .*

So we proved that $B_{\frac{R}{8}}(p)$ is GH close to the whole ball of product $\mathbb{R} \times \beta_+^{-1}(0)$ By Gromov's compactness and the fact that GH limit of length spaces is length space the space X can be taken to be length space. The fact above follows from the observation that for a fixed $R > 0$ the sequence of balls $B_R(p)$ with lower Ricci curvature bound $Ric \geq -(n-1)\delta$ satisfies Gromov's theorem conditions so it has to converge up to a subsequence. But since these balls are length spaces the limit has to be length space as well, so we can take X as a length space .

Before the applications of the theorem above , some useful remarks are that X may not be smooth , and although $d_{GH}(B_{\frac{R}{8}}(p), B_{\frac{R}{8}}(0,x)) \leq \Psi$, the ball $B_{\frac{R}{8}}(p)$ may not have the topology of the product , we give a sketch of the classic Anderson's counterexample in the appendix.

5.2 Applications of Almost Splitting

The next presented theorem is one of the most useful because it says that splitting theorem holds in a more general case .

Theorem 5.2.1 (Splitting Theorem for limit spaces). *Let $\mathcal{M}_i^n \xrightarrow{d_{GH}} Y$ satisfy* $\mathrm{Ric}_{\mathcal{M}_i^n} \geq -(n-1)\delta_i$ *($\delta_i \to 0$). We also assume that Y contains a line. Then Y splits isometricaly as a product , $Y = \mathbb{R} \times X$ for some length space X .*

Proof. To prove this we wil show that for every $R > 0$ there exist a ball $B_R(0,x)$ (for a length space X) , that is isometric to a subspace of Y. So let $R > 0$ and let $p \in Y$, from the line existance we can chose $q^{\pm}$ to satisfy $E(p) = 0$ and $\overline{p, q^{\pm}}$ can be chosen howmuch large we want. Also from the definition of the convergence exist ε_i - isometries $f_i : M_i^n \to Y$ and $h_i : Y \to M_i^n$ so from the fact that h_i is an ε_i- isometry it is

$$|d_{M_i}(h_i(p), h_i(q^{\pm})) - d(p, q^{\pm})| \leq \varepsilon_i$$

so

$$d_{M_i}(h_i(p), h_i(q^{\pm})) \geq \overline{p, q^{\pm}} - \varepsilon_i$$

so from the last inequality we can take $L > 2R + 1$ with $d_{M_i}(h_i(p), h_i(q^{\pm})) \geq L$ (for all i sufficiently large) and now by the fact that $\delta_i \to 0$ we can apply almost splitting theorem in a ball centered at $h_i(p)$. So it is

$$d_{GH}(B_{\frac{R}{4}}(h_i(p)), B_{\frac{R}{4}}(0,x)) \leq \Psi_i(\delta_i, L^{-1}, 3\varepsilon_i | n, R)$$

Then if g_i is the Ψ_i- GH approximation from almost splitting theorem, taking the composition $f_i \circ g_i$ shows that there is a $(\Psi_i + \varepsilon_i)$ - approximation between the ball $B_{\frac{R}{4}}(0,x)$ and $f_i(g_i(B_{\frac{R}{4}}(0,x)))$, a subset of Y. Finally we can take the limits $i \to \infty$ and $L \to \infty$ so there is a length space X as we wanted and since it holds for every R, it suffices to complete the proofs The above proof is detailed presented in [MJ]

□

Corollary 5.2.2. *The last Theorem is very usefull when we have to study tangent cone . Note that a tangent cone of a manifold with bounded Ricci curvature is such space because rescaling the metric gives the* $Ric \geq -(n-1)\delta_i$ *,* $\delta_i \to 0$ *, so it is a way to apply splitting theorem in tangent cones too.*

Corollary 5.2.3. *Let* x_i *be the standard coordinate functions on* $\mathbb{R}^n$*. Let* $B_L(0) \subseteq \mathbb{R}^n$*. Using the previous theorem we get the following.*

Let

$$Ric \geq -(n-1)\delta$$

$$d_{GH}(B_L(p), B_L(0)) \leq \delta$$

. Then there exist harmonic functions $\beta_1^+, \ldots, \beta_n^+$ *such that*

$$\fint_{B_R(p)} \sum_i ||\nabla\beta_i^+|^2 - 1| + \sum_{i \neq j} |\langle\nabla\beta_i^+, \nabla\beta_j^+\rangle| + \sum_i |\mathrm{Hess}_{\beta_i^+}|^2 \leq \Psi(\delta|\mathrm{n}, \mathrm{R})$$

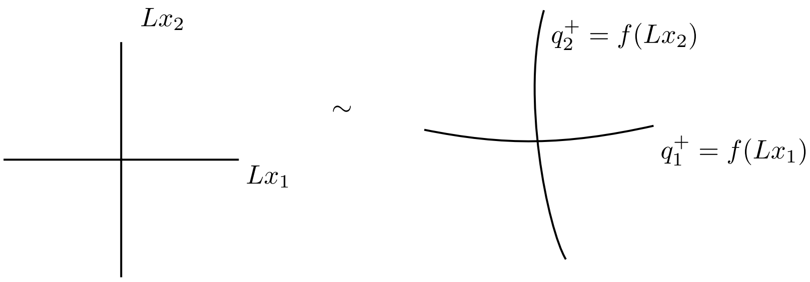
Proof. Since $d_{GH}(B_L(0), B_L(p)) \leq \delta$ we have a δ - GH approximation denoted with f so if x_i is the coordinate basis in $\mathbb{R}^n$ then we define points q_i as follows

$$f(Lx_i) = q_i^+$$

and the maps b_i^+ as

$$b_i^+(\cdot) = \overline{\cdot, q_i^+} - \overline{p, q_i^+}$$

for every i , like the proof of the almost splitting theorem we can construct an harmonic β_i^+ which coincide with b_i^+ on the boundary of the ball , So in a ball of radius R (R « L) we have bounds for $|Hess_{\beta_i^+}|^2$ and $||\nabla\beta_i^+|^2 - 1|$



So we had already proved that for every i , terms

$$\fint_{B_R(p)} ||\nabla \beta_i^+|^2 - 1| \text{ and } \fint_{\mathrm{B_R(p)}} |\mathrm{Hess}_{\beta_{\mathrm{i}}^+}|^2$$

are bounded by $\Psi(\delta)$,
so is suffices to show same for $\fint_{B_R(p)} |\langle \nabla \beta_i^+, \nabla \beta_j^+ \rangle|$ for $(i \neq j)$ Now since

$$\langle \nabla \beta_i^+, \nabla \beta_j^+ \rangle = \langle \nabla b_i^+ - \nabla \beta_i^+, \nabla b_j^+ \rangle + \langle \nabla \beta_i^+, \nabla b_j^+ - \nabla \beta_j^+ \rangle + \langle \nabla \beta_i^+, \nabla \beta_j^+ \rangle$$

but using Green's identity and lemma 2.9 of [C] we get :

$$\fint_{B_R(p)} |\langle \nabla \beta_i^+, \nabla \beta_j^+ \rangle| \leq \fint_{B_R(p)} |\langle \nabla b_i^+, \nabla b_j^+ \rangle| \leq \Psi(\delta|n, R)$$

□

Chapter 6

Structure of Limit Spaces

We could say that theorems presented here are obviously other applications of the Almost rigidity, but it is worth dedicating a separate chapter. The splitting theorem, as we have already seen, holds for singular spaces and not only for Riemannian manifolds. Using this, we will derive results about the structure of these spaces.

We consider a sequence of manifolds M_i^n with lower Ricci curvature bounds

$$Ric_{M_i^n} \geq -(n-1)$$

Then by Gromov's compactness theorem this sequence has a subsequence which converges with the pointed sense . so

$$(M_i^n, p_i) \to (Y, \overline{y})$$

One of the important issues of our theory is to understand how limit space Y looks like, ie how irregular it could be. Of course its known that its not in general a Riemannian manifold , also its known that it should be a length space ,as limit of length spaces . To study these spaces we need to study their infinitesimal behavior , so intuitively to zoom in around a point. To achieve this consider a sequence $r_l \to 0$. Then the limit $(Y, r_l^{-1}d)$ exist (passing to a subsequence) and its called tangents cone of Y at point y. We denote these spaces with Y_y

Remarks. *1) The tangent space Y_y in a point $y \in Y$ has not to be unique it depends on the choice of sequence r_j*

2) The existance of the tangent cone is guaranteed again by the Gromov's compactness theorem . Especially we can view $(Y, r_i^{-1}d, y)$ as a limit space (always we mean by passing to a subsequence) of $(M_j^n, r_i^{-2}g_j, p_j)$ as $j \to \infty$

So for every i we can find a subsequence $k(j)$ such that $(M_{k(j)}^n, r_i^{-2}g_{k(j)}, p_k(j))$ converges , then finally we can view the $(Y_y, y_\infty, d_\infty)$ as the following limit

$$(M_{k(j)}^n, r_i^{-1}d_{k(j)}, p_{k(j)}) \to (Y_y, d_\infty, y_\infty)$$

where the sequence of manifolds satisfies

$$\liminf_{k(j)\to\infty} Ric_{M_{k(j)}^n} \geq 0$$

3) An example of how limit space may look is the following, if the limit space Y is a Riemannian manifold then tangent cones at every point are isometric to $\mathbb{R}^n$ where n is the dimension of manfiold .Due to the above, we say that tangent cones are a generalization of tangent spaces
4)An other example is a cone $(\mathbb{R}^2, dr^2 + \alpha r^2 d\theta^2)$ which shows that in every point except the vertex the tangent cone are $\mathbb{R}^2$ and at the vertex of the cone the tangent cone is the cone itself .
5) Let Y_y be a tangent cone at a point y of the limit space Y , then if $z \in Y_y$ we can consider the tangent cone of Y_y over z and we denote with $(Y_y)_z$. Since it is a cone over the space Y it can be realized as limit of a sequence $\{(Y, r_j^{-1}d, y_j)\}$ and hence as we already seen as a limit of a sequence $\{(M_i^n, q_i)\}$

After we defined the limit spaces and explained what infinitesimal behavior is, a question arises: what can we say about their dimension? We can not be sure what the Hausdorff dimension is, it may collapse (i.e., the dimension of the limit is less than the dimension of the sequence elements). For instance, take the limit space of cylinders with decreasing radius , the dimension may be less than the dimension of manifolds (cylinders), despite the fact that curvature remains bounded.

We will limit our study to the case where this does not happen. Then, the collapsing phenomenon can be avoided by adding the following condition, called the 'non-collapsing condition. So from now we will Assume

$$Vol(B_1(p_i)) \geq v > 0$$

but we will repeat it wherever necessary to emphasize it.

After this introduction , we give some notions that help us understand these spaces.

Definition 6.0.1. *A point $y \in Y$, is called k- regular , if for some k every tangent cone at y is isometric to $\mathbb{R}^k$. We denote the set of k-regular points with $\mathcal{R}_k$*

An easy example to check regular points with different k is a rectifiable set .

Definition 6.0.2. *We denote the union of these sets $\mathcal{R}_k$ with $\mathcal{R}$ and we call regular set $\mathcal{R} = \bigcup_k \mathcal{R}_k$*

Definition 6.0.3. *A point $y \in Y$, is called singular , if it is not regular. We denote the set of singular points with $\mathcal{S}$*

Definition 6.0.4. *A point , $y \in Y$ is called k-weakly Euclidean, if **some** tangent cone at y splits off a factor , $\mathbb{R}^k$, isometrically . We denote the set above with $\mathcal{WE}_k$*

Definition 6.0.5. *A point , $y \in Y$ is called k-degenerate if it is not $(k+1)$-weakly Euclidean We denote the set above with $\mathcal{D}_k$*

Let $\mathcal{D}_k$ as above then note that

$$\mathcal{D}_0 \subset \mathcal{D}_1 \subset ... \subset \mathcal{D}_n = Y$$

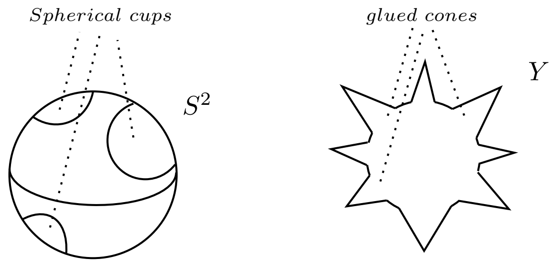
Put $D_k \setminus \mathcal{R} = \mathcal{S}_k \subset S$ then

$$S = \bigcup_k \mathcal{S}_k$$

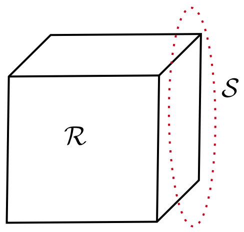
And it is

$$Y^n = \mathcal{WE}_0 \supset \mathcal{WE}_1 \supset ...\mathcal{WE}_n = \mathcal{R}_n$$

We turn our attention to the singular set of Y. It is evident that the cone of dimension 2 is an example of a non collapsed limit space with a singularity. Using that we can construct an example of a limit space with much bigger set of singularities. For example from sphere we can remove a collection (which may be countable) of caps and replace them with tops of cones so that the new constructed surface be C^1 and ensure the curvature remains bounded to construct a limit space with infinitesingular points, as you can see in the picture below



Although singular set may be dense as above, always has zero measure . The notion of the measure here is as follows: if there is no collapsing, then the limit space Y^n inherits a measure from the sequence, which is a multiple of the Hausdorff measure. In the case of collapsing, this measure is not unique, and by choosing different sequences, we can define different measures. This construction is done in details in the first chapter of [ChCo2], using volume comparison , Arzela-Ascoli theorem the construction of an outer measure.
Finally the following is one more example of a limit space, this time the surface of the cube. It is clear that edges of the cube are regular and remaining points are singular



6.1 Hausdorff Measure

Before we state and prove two central results, we will provide some definitions related to the Hausdorff measure. For more details see [EG]

Let (X, d) be a metric space and $A \subseteq X$
Denote with ω_m the volume of m dimensional ball with radius 1 , in $\mathbb{R}^n$ i.e

$$\omega_m = \frac{\pi^{\frac{m}{2}}}{\Gamma(\frac{m}{2}+1)}$$

where Γ is the usual Gamma function . For each $\delta > 0$ we define the size δ

approximation to $\mathcal{H}^m$ as follows

$$\mathcal{H}^m_\delta(A) = \omega_m 2^{-m} \inf\{\sum_{j=0}^{\infty} diam(C_j)^m : A \subset \bigcup_j C_j, diam(C_j) < \delta\}$$

for $A \subseteq X$, and $\mathcal{H}^m_\delta(\emptyset) = 0$ sometimes the quantity above is called Hausdorff content . It is obvious that it is a decreasing function of δ so this guarantees the existence of the limit $\delta \downarrow 0$ (although it may be infinity) , then

$$\lim_{\delta \downarrow 0} \mathcal{H}^m_\delta(A) = sup_\delta \mathcal{H}^m_\delta(A)$$

so finally this limit is an outer measure so we define for every $m \geq 0$ the Hausdorff (outer) measure as

$$\mathcal{H}^m(A) = \lim_{\delta \downarrow 0} \mathcal{H}^m_\delta(A)$$

Note that $\mathcal{H}^m_\delta$ is also an outer measure ,but we prefer to work with $\mathcal{H}^m$ because generally $\mathcal{H}^m_\delta$ allows many sets to not be measurable , for instance the closed interval $[0, 1]$ is not measureable respectively to unlimited content
Also given an outer measure, it is entirely straightforward to gain a measure using the well-known Carathéodory method , and the limit in the definition is always finite if the space is seperable .

Hausdorff dimension

Its clear that there is a unique value α , with $0 \leq a \leq \infty$ such that $\mathcal{H}^l(A) = 0$ for $l < a$ and $\mathcal{H}^l(A) = \infty$ for $l > a$, so we define the Hausdorff dimension of a subset $A \subseteq X$ with exactly that value and we denote with $\dim(A)$. Hausdorff dimension need not be an integer (for example fractals has not integer Hausdorff dimension.)
We call a point x of A as an l- density point of A if

$$2^{-l} \leq \limsup_{r \to 0} \frac{\mathcal{H}^l_\infty(A \cap B_r(x))}{\omega_l r^l}$$

Density points If $D_l(A)$ is the set of density points of A then $\mathcal{H}^l$- every point of A is density point i.e

$$\mathcal{H}^l(A \setminus D_l(A)) = 0$$

We present a proof of [EG]
Consider the following set

$$B(\mathcal{H}^k_\delta, t, \varepsilon) = \{x \in A : \mathcal{H}^k_\delta(A \cap S) \leq t\omega_k 2^{-k} diam(S)^k, \text{for all S} : x \in S, diam(S) < \varepsilon\}$$

so obviously if S is any set in A , with $diamS < \varepsilon$ we have

$$\mathcal{H}^k_\delta(S \cap B(\mathcal{H}^k_\delta), t, \varepsilon) \le t\omega_k 2^{-k} diam(S)^k$$

If we take a cover with sets like these we get :

$$\mathcal{H}^k_\delta(B(\mathcal{H}^k_\delta, t, \varepsilon)) \le t\omega_k 2^{-k} \sum_j (diam(S_j))^k$$

by taking infinimum it is :

$$\mathcal{H}^k_\delta(B(\mathcal{H}^k_\delta, t, \varepsilon)) \le t\mathcal{H}^k_\varepsilon(B(\mathcal{H}^k_\delta, t, \varepsilon))$$

Particularly for $\delta < 1$ we can write the last inequality as

$$\mathcal{H}^k_\delta B(\mathcal{H}^k_\delta, 1-\delta, \delta) \le (1-\delta)\mathcal{H}^k_\delta(B(H^k_\delta, 1-\delta, \delta))$$

which actually tells us that

$$\mathcal{H}^k_\delta B(\mathcal{H}^k_\delta, 1-\delta, \delta) = 0$$

Hence it is $\mathcal{H}^k_\delta = 0$ then $\mathcal{H}^k = 0$ so

$$\mathcal{H}^k(B(\mathcal{H}^k_\delta, 1-\delta, \delta)) = 0$$

Now consider

$$C = \{x \in A : \inf_n \sup\{\frac{\mathcal{H}^k_\infty(A \cap S)}{\omega_k 2^{-k} diam(S)^k} : x \in S, diam(S) < \frac{1}{n}\} < 1$$

so if $x \in C$, there some n such that

$$\sup\{\frac{\mathcal{H}^k_\infty(A \cap S)}{\omega_k 2^{-k} diam(S)^k} : x \in S, diam(S) < \frac{1}{n}\} < 1 - \frac{1}{n}$$

To finish the proof we just note that the set

$$D = \{x \in A : \limsup \frac{\mathcal{H}^k_\infty(A \cap B(x,r))}{\omega_k r^k} < 2^{-k}\}$$

is contained in C so,

$$\mathcal{H}_k(D) = 0$$

and the lemma is proved To see this ($D \subseteq C$) we just need to check

$$2^{-k} \sup\{\frac{\mathcal{H}^k_\infty(A \cap S)}{\omega_k 2^{-k} diam(S)^k} : x \in S, diamS < \delta\} \ge \sup\{\frac{\mathcal{H}^k_\infty(A \cap B(x,\rho))}{\omega_k \rho^k} : \rho < \delta\}$$

and the last one is true because, if S has $diam(S) < d$ and $x \in S$ then

$$\frac{\mathcal{H}^k_\infty(A \cap S)}{\omega_k 2^{-k} diam(S)^k} \le 2^k \ \frac{\mathcal{H}^k_\infty(A \cap B(x,r))}{\omega_k d^k}$$

6.2 $\dim \mathcal{S}_{\mathrm{k}} \leq \mathrm{k}$

Theorem 6.2.1. *Let Y be a limit space of a sequence of pointed n-manifolds satisfying:*

$$\mathrm{Ric}_{\mathcal{M}_i^n} \geq -(n-1)$$

and

$$\mathrm{Vol}(B_1(m_i)) \geq v > 0 \ .$$

Then

$$\dim \mathcal{S}_k \leq k \ .$$

Before the proof of the theorem above we present the following :

Theorem 6.2.2. *Let (Y, y) be the limit space of a sequence $\{(\mathcal{M}_i^n, p_i)\}$ with the property*

$$\mathrm{Ric}_{\mathcal{M}_i^n} \geq -(n-1)$$

and

$$\mathrm{Vol}(B_1(p_i)) \geq v > 0$$

Then for all $\underline{y} \in Y$, every tangent cone at $\underline{y}$ is a metric cone , $C(X)$, on a length (and obviously a metric) space , with $\mathrm{diam}(X) \leq \pi$

The proof of the Theorem 6.2.2 could be found in [ChCo1] and by notion of metric cone we mean the completion of the metric space $Z \times (0, +\infty)$ with the metric

$$d((z_1, r_1), (z_2, r_2)) = \begin{cases} r_1^2 + r_2^2 - 2r_1r_2cos(\overline{z_1, z_2}), \text{ if } \overline{\mathrm{z_1, z_2}} \leq \pi \\ r_1 + r_2 \ , \ \overline{z_1, z_2} > \pi \end{cases}$$

Now, we move on to the proof of the main theorem.

Proof. We will prove using the theory of density points above and by contradiction with blowup arguments. We already proved that if $\mathcal{H}^k(A) > 0$ then k - almost every point of A is a $k-$ density point.

First note that the set $\mathcal{S}_k$ is the set of points of the limit space for which there is no tangent cone with $\mathbb{R}^{k+1}$ factor ie

$$\mathcal{S}_k = \{y \in Y : \text{no tangent cone } Y_y \text{ has } \mathbb{R}^{k+1} \text{ as isometric factor } \}$$

so clearly by definitions $\mathcal{S}_k$ is the following union .

$$\mathcal{S}_k = \bigcup_i \mathcal{S}_{k,i}$$

where

$$\mathcal{S}_{k,i} = \{y \in Y : d_{GH}(B_r(y), B_r(0, x^*)) \geq \frac{r}{i} : \forall r \in (0, i^{-1})\}$$

Where 0 denotes the origin of $(k+1)$ euclidean space and x^* the vertex of $C(X)$, for some metric space X .

Assume that $dim(S_k) > k$ so there is a k' such that $\dim(\mathcal{S}_\mathrm{k}) \geq \mathrm{k}'(> \mathrm{k})$, then $\mathcal{H}^{k'}(\mathcal{S}_k) > 0$ then $\mathcal{H}^{k'}(\bigcup_i \mathcal{S}_{k,i}) > 0$ and that means there exist an i such that

$$\mathcal{H}^{k'}(\mathcal{S}_{k,i}) > 0$$

because of the positive measure we proved that almost every point of $\mathcal{S}_{k,i}$ has to be k' density point . So $\mathcal{H}^{k'}$ almost everywhere it is $y \in D_{k'}(\mathcal{S}_{k,i})$, then by definition of k' density points there is a sequence $r_l \downarrow 0$ with

$$2^{-k'} \leq \lim_{l \to \infty} \frac{\mathcal{H}^{k'}_\infty(\mathcal{S}_{k,i} \cap B_{r_l}(y))}{\omega_{k'} r_l^{k'}}$$

Also its clear that

$$(\mathcal{S}_{k,i} \cap \overline{B}_{r_l}(y), r_l^{-1} d) \to (\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}, d_\infty)$$

As a consequence , the compact set $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}$ has positive k' - dimensional measure. If not, we cover $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_\infty)}$ by open sets (finite by compactness) , $\{B_{r_j}(w_{j,\infty})\}$ with

$$\sum_j (r_j)^{k'} \leq 2^{-(k'+1)} \omega_{k'} r^{k'}$$

for sufficiently large l, its contradiction with the definition of r_l more details for this could be found in [ChCo2]. It is because on the one hand by the choice of sequence r_l it is

$$\lim_{l \to \infty} \frac{\mathcal{H}^{k'}_\infty(B_{r_l}(y) \cap \mathcal{S}_{k,i})}{r_l^{k'} \omega_{k'}} \geq 2^{-k'}$$

but on the other hand again for enough large l we can cover $B_{r_l}(y) \cap \mathcal{S}_{k,i}$ with the same number of balls with slightly larger radius. For instance take $\{B(\hat{w}_{j,\infty}, r_j + \varepsilon)\}$ and since $\sum_j (r_j)^{k'} \leq 2^{-(k'+1)} \omega_{k'} r^{k'}$ for large l we get a contradiction. (Since we have a cover with balls $B(w_j, r_j)$ we can easy see that for enough big l we have that the Gromov-Hausdorff approximation does not distors much , so we can take a union of bit larger radius balls to cover $B_{r_l}(y) \cap \mathcal{S}_{k,i})$.

We can repeat this argument so there exist a infinite sequence of tangent cones , Y_{y_j}, with base points $y_{j,\infty}$ such that $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_{j,\infty})}$ has positive k' dimensional Hausdorff measure for all j and such that y_{j+1} is an arbitrarily chosen point of destiny of Y_{y_j} (of the intersection $\mathcal{S}_{k,i}(Y_y) \cap \overline{B_1(y_{j,\infty})}$)
Now note that $Y_{y_j} = R^{k_j} \times C(X_j)$ then $k' \geq k_j$ (otherwise $\mathcal{S}_k(Y_{j_j})$ would be empty), also we can choose a $y_{j+1} \notin \mathbb{R}^{k_j} \times \{x_j^*\}$. because of the fact $\mathcal{H}^{k'}(S_{k,i}(Y_{y_j}) \cap \overline{B_1(y_{\infty,j})}) > 0$ (note that $\mathcal{H}^{k'}(\mathbb{R}^{k_j} \times \{x_j^*\}) = 0$ because $k' > k_j$)

In particular the ray from $(0, x_{j+1}^*)$ through y_{j+1} is not contained in $\mathbb{R}^{k_j} \times \{x_j^*\}$ then by spitting theorem we get $k_{j+1} \geq k_j + 1$. For example chose j such that $j > k$ and there is a contradiction with the fact that $k_j \leq k$
so we proved that for the limit space the $\dim \mathcal{S}_{\mathrm{k}} \leq \mathrm{k}$ holds .

□

Remarks. *1) Let X be a complete length space .We say $x \in X$ is a pole , if for all $\underline{x} \neq x =$ there is a ray $\gamma : [0, \infty) \to X$ with $\gamma(0) = x$ and $\gamma(t) = \underline{x}$ for some $t > 0$. As usual , γ is called a ray if each finite segment of γ is minimal .*
2) In [ChCo2] the above theorem is proven without the assumption of non collapsing we can replace that by more general fact space Y^m beeing polar , ie if for all $y \in Y^m$ the base point of every iterated tangent cone is a pole .

Now we will give a sketch of the proof for dim $\mathcal{S} \leq n - 2$

6.3 $\dim \mathcal{S} \leq n - 2$

Theorem 6.3.1. *Let Y be a limit space of a sequence of pointed n-manifolds satisfying:*

$$Ric_{M_i^n} \geq -(n-1)$$

and

$$Vol(B_1(m_i)) \geq v > 0$$

Then for the singular set of Y the following Hausdorff dimension bound holds

$$\dim \mathcal{S} \leq \mathrm{n} - 2$$

Proof. In [ChCo2] authors proved this argument by contradiction by constructing a map with degree 0 and 1 at the same time.
It is obvious that

$$\mathcal{S}_{n-2} \subseteq \mathcal{S}_{n-1}$$

we will show that actually $\mathcal{S}_{n-1} \backslash \mathcal{S}_{n-2} = \emptyset$ and this suffices for the proof because if $\mathcal{S} = \mathcal{S}_{n-2}$ by previous theorem it is

$$\dim \mathcal{S} = \dim \mathcal{S}_{\mathrm{n}-2} \leq \mathrm{n} - 2$$

We argue by contradiction , assume there is a $y \in Y$ such that $Y_y = \mathbb{R}^{n-1} \times C(Z)$, for some tangent cone . Since Y_y is not isometric with $\mathbb{R}^n$ (then it should be $S_{n-1} \setminus S_{n-2} \neq \emptyset$), so it follows that Y_y is isometric with $\mathbb{H}^n$.

The fact $Y_y \simeq H^n$, ensures by GH pointed convergence that there is a sequence of rescaled balls $\{B_1(p_i)\}$ such

$$B_1(p_i) \to B_1(0) \cap H^n$$

Then there are f_i GH- approximations with $dis(f_i) = \varepsilon_i \to 0$.

Now having this sequence we start sketching the proof by several steps.
The first step is proving that these map can be considered to be continuous if we ensure that i is sufficiently large.
Step 1 For sufficient large i, there is a **continuous** ε -GH approximation

$$f : B_1(p_i) \to B_1(0) \cap \mathbb{H}^n$$

where $B_1(p_i) \subseteq M_i^n$ and $\mathrm{B}_1(0) \subseteq \mathbb{R}^{\mathrm{n}}$. That follows from a more general proposition which could be found in [SI].

Since $\mathbb{H}^n$ is locally contractible , then we can construct a sequence of ε_k-GH approximations $i_k : X_k \to X$, with all i_k continuous . (Here of course $X_k = B_1(p_k)$ and $X = B_1(0) \cap \mathbb{H}^n$) For every k there is a ε_k- net S_k

We define the maps $i_k : X_k \to R^n$ as follows :

$$i_k(x) = \frac{\sum_{y \in S_k} g_k(d(x,y)) f_k(y)}{\sum_{y \in S_k} g_k(d(x,y))}$$

where $g_k : [0, \infty) \to \mathbb{R}$ is defined as :

$$\begin{cases} g_k(x) = -\frac{1}{2\varepsilon_k}x + 1 \ , \ 0 \le x < 2\varepsilon_k \\ 0 \quad , \ x \ \ge 2\varepsilon_k \end{cases}$$

Clearly these i_k are continuous, now we have to prove they are close to f_k and from that we can conclude the wanted and the fact that $dis(i_k) \to 0$

Let $x \in X_k$ arbitrary and $y \in S_k$ with $d(x, y) \le 2\varepsilon_k$, (otherwise if $d(x, y) > 2\varepsilon_k$ it should be $i_k(y) = 0$, by definition of g_k , so it must be

$$|f_k(x) - f_k(y))| \le 3\varepsilon_k$$

it because

$$|f_k(x) - f_k(y)| - d(x, y) \le \sup_{x,y}\{|d(x, y) - |f_k(x) - f_k(y)||\}$$

then

$$|f_k(x) - f_k(y)| \le dis(f_k) + 2\varepsilon_k$$

from the fact above we get

$$|i_k(x) - f_k(x)| \le \sup\{|y' - f_k(x)| : y' \in U(f_k(x), 3\varepsilon_k)\}$$

where with $U(f_k(x), 3\varepsilon_k)$ we denote the the $3\varepsilon_k$ neighborhood of $f_k(x)$.
that is because

$$|i_k(x)-f_k(x)| = \left|\frac{\sum_{y\in S_k} g_k(d(x, y))f_k(y)}{\sum_{y\in S_k} g_k(d(x, y))} - f_k(x)\right| \le \frac{\sum_{y\in S_k} g_k(d(x, y))|f_k(x) - f_k(y)|}{\sum_{y\in S_k} g_k(d(x, y))}$$

Now the fact $\varepsilon_k \to 0$ implies

$$\sup_{x\in X_k} |i_k(x) - f_k(x)| \xrightarrow{k\to\infty} 0$$

so we just need to verify that

$$dis(i_k) \to 0$$

which holds by the triangle inequality.
In particular

$$||i_k(x)-i_k(x')|-d(x, x')| = ||i_k(x)-i_k(x')|-|f_k(x)-f_k(x')|+|f_k(x)-f_k(x')|-d(x, x')||$$

So we get

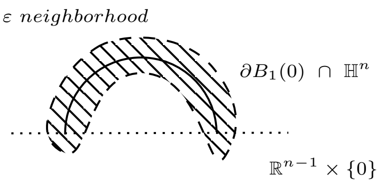
$$||i_k(x) - i_k(x')| - d(x,x')| \leq dis(f_k) + ||i_k(x) - i_k(x')| - |f_k(x) - f_k(x')||$$
$$\leq dis(f_k) + |i_k(x) - f_k(x)| + |i_k(x') - f_k(x')|$$

which goes to zero as $k \to \infty$

Step 2

We can without loss of generality assume that 1 is a regular value of the distance function, so the boundary $\partial B_1(p_i)$ is smooth. This can be achived using the Whitney's approximation lemma . Since f is an ε- GH approximation $f(\partial B_1(p_i))$ is contained in an ε neighborhood of $\partial B_1(0) \cap \mathbb{H}^n$, using radial projections we can assume without loss of generality , that

$$f(\partial B_1(p_1)) \subseteq \partial B_1(0) \cap \mathbb{H}^n$$



It is $q \in \partial B_1(p_i)$ then $\overline{p_i, q} = 1$ so from the fact that f is an ε GH- approximation, it follows that

$$\overline{f(p_i), f(q)} \leq \varepsilon + \overline{p_i, q}$$

then indeed by that and the inequality with $-\varepsilon$, $f(\partial B_1(p_i))$ is contained in an ε neighborhood of $\partial B_1(0) \cap \mathbb{H}^n$

Step 3 From compactness of $\partial B_{\frac{1}{2}}(p_i)$ there exist a point $q \in \partial B_{\frac{1}{2}}(p_i)$ such that $f(q)$ has the maximal distance form $\mathbb{R}^{n-1} \times \{0\}$ Using again radial projection on a bigger ball we can assume that :

$$f(B_{\frac{1}{4}}(q)) \subseteq B_{\frac{1}{4}}(f(q))$$

and

$$f(A_{\frac{1}{8},\frac{1}{4}}(q)) \subseteq A_{\frac{1}{8},\frac{1}{4}}(f(q))$$

Step 4

We want to construct an inverse map with properties similar with them of f, we need a map h which is continuous and an ε-GH approximation . The existence of this function is guaranteed whenever the space is locally contractible because of [Petersen2], also see [SI], but the local contractiblity follows from 7 and [ChCo2] Appendix by Reifengerg's method . So there is a map with the properties as below

$$h : B_{\frac{1}{4}}(f(q)) \to B_{\frac{1}{4}}(q)$$

and again using radial projection on $B_{\frac{2}{5}}(f(q))$ we can assume without loss of generality

$$h(B_{\frac{1}{4}}(f(q))) \subseteq B_{\frac{1}{4}}(q)$$

and

$$h(A_{\frac{1}{8},\frac{1}{4}}(f(q))) \subseteq A_{\frac{1}{8},\frac{1}{4}}(q)$$

so for $z \in B_{\frac{1}{4}(f(p))}$ and $\Psi(\varepsilon|n)$ as above it is

$$\overline{f \circ h(z), z} \leq \Psi(\varepsilon|n)$$

but according to (6.2) of [ChCo2] this is enough to get that the map of pairs

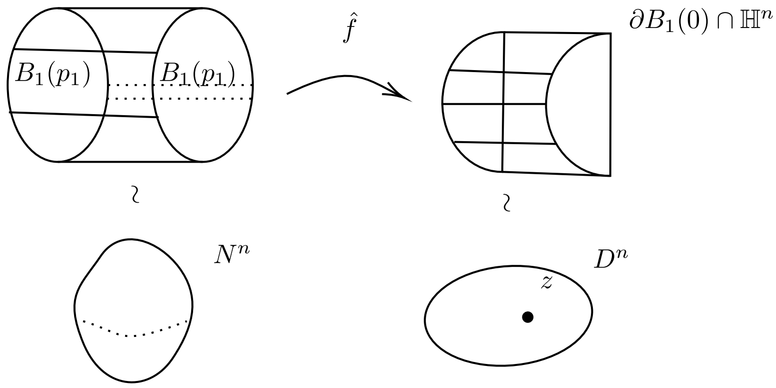
$$f \circ h : (B_{\frac{1}{4}}(f(q)), A_{\frac{1}{8},\frac{1}{4}}(f(q))) \to (B_{\frac{1}{4}}(f(q)), A_{\frac{1}{8},\frac{1}{4}}(f(q)))$$

has mod 2 degree , $\deg(f \circ h) = 1$, so if we let $\tilde{f}$ the map induced from $\hat{f}$ by restriction, we have $deg(\tilde{f}) = 1$ where

$$\tilde{f} : (B_{\frac{1}{4}}(q), A_{\frac{1}{8},\frac{1}{4}}(q)) \to (B_{\frac{1}{4}}(f(q)), A_{\frac{1}{8},\frac{1}{4}}(f(q)))$$

Step 5

The last step will give the contradiction. First recall the notion of doubling. Let M be a smooth manifold with boundary. The double of M is the manifold $M \bigcup_{id} M$, where id is the identity map of ∂M, where by that we mean the space we obtain from $M \sqcup M$ by identifying each boundary point in one copy of M with the same boundary point in the other. Now with doubling the ball $B_1(p_i)$ along the boundary we get a closed manifold N^n, similarly by doubling $B_1(0) \cap \mathbb{H}^n$ along $\partial B_1(0) \cap \mathbb{H}^n$ we get a manifold with boundary D^n, the manifold D^n is a ball topologically. Finally since we have a continuous map $f : B_1(p_i) \to B_1(0) \cap \mathbb{H}^n$, then there is the induced map $\hat{f} : N^n \to D^n$ So it is a classical algebraic topology result that this map has to be zero degree i.e. $\deg(\hat{f}) = 0$ but also in the previous step we proved that taking a point in a small neighborhood of $f(q)$, for example $z \in B_{\frac{1}{16}}(f(q))$ that fact that mod 2 degree is also equal to 1, gives the contradiction.



□

Remarks

Note that the bound $n-2$ in the inequality $dim(\mathcal{S}) \leq n-2$ is optimal, for instance in the example of ice cream cone we already presented there is a singular point and the dimension of the space is 2

Chapter 7

Appendix

The next theorem we present proved by Tobias Colding in [C].

Theorem 7.0.1. *Let*

$$Ric_{M^n} \geq -(n-1)\delta$$

$$d_{GH}(B_R(p), B_R(0)) \leq \delta$$

then for $\Psi(\delta|n)$

$$Vol(B_R(p)) \geq (1-\Psi)Vol(B_R(0))$$

Proof. The proof also could be found in [Ch]. It uses again mod 2 degree theory and the key point is showing for a splitting map the mod 2 degree is 1 □

We now present Anderson's counterexample, it is the theorem 0.3 of [An] which shows that almost splitting does not necessarily imply a topological split.

First Consider the manifold $M = S^1(\delta) \times T^3$ where T^3 is the flat 3-torus. The δ is a small parameter, $\delta = \frac{2}{R}$

The first step in construction is to remove the domain $S(\delta) \times B$ where B is a geodesic ball of radius 1 in T^3.

Consider the product $(\mathbb{R}^2 \times S^2, g_s)$ where g_s is the Schwarzchild metric i.e the metric on the product $\mathbb{R}^2 \times S^2$

$$g = \frac{1}{1-r^{-1}}dr^2 + 4(1-r)d\theta^2 + r^2 g_0$$

where g_0 is the canonical metric of curvature 1, on S^2

Next we denote with Ω_R be the domain in $\mathbb{R}^2 \times S^2$ given by $r^{-1}[0, \frac{1}{2}R]$. Similarly, let $F_R = r^{-1}[\frac{1}{4}R, \frac{1}{2}R]$ and we also define the metric $g_R = R^{-2}g_s$ Note that for R sufficiently large the metric g_R is a small perturbation of the flat product metric on $S^1(\frac{2}{R}) \times A(\frac{1}{4}, \frac{1}{2})$, where the $A(\frac{1}{4}, \frac{1}{2})$ is the usual annulus of $\mathbb{R}^3$,to verify this we calculate, consider $\frac{R}{4} \leq r \leq \frac{R}{2}$ it is

$$g_R = \frac{1}{R^2}\frac{1}{1-r^{-1}}dr^2 + 4(1-r)d\theta^2 + r^2 g_0$$

so if we write $\frac{r}{R} = s$, for $s \in [\frac{1}{4}, \frac{1}{2}]$ it becomes

$$g_R = \frac{1}{1-r^{-1}}ds^2 + \frac{4(1-r)}{R^2}d\theta^2 + s^2 g_0$$

also the flat metric of the $S^1(\frac{2}{R}) \times A(\frac{1}{4}, \frac{1}{2})$ is

$$\hat{g} = \frac{4}{R^2}d\theta^2 + ds^2 + s^2 g_0$$

So we can obviously see that these metrics are near for sufficiently large R. Finally now we can define a smooth metric on $S^1(\frac{2}{R}) \times A(\frac{1}{4}, 1)$ which agrees with g_R on F_R and its the usual flat in $S^1(\frac{2}{R}) \times A(\frac{3}{4}, 1)$. Further this metric is (C^∞) ε-close to the flat metric on $S^1(\frac{2}{R}) \times A(\frac{1}{4}, 1)$. So this actually can guarantee that in the manifold which is obtained with surgery

$$M_1 = S^1 \times (T^3 \setminus B^3) \cup (S^2 \times \mathbb{R}^2)$$

has a smooth family of metric with Ricci curvature satisfying $|Ric_{g_\varepsilon}| \leq \varepsilon$

As $\varepsilon \to 0$ the diameter of surgery $S^2 \times \mathbb{R}^2$ attached to torus decreases, and the radius $S^1(\delta)$ as well. Then as $\varepsilon \to 0$

$$(M_1, g_\varepsilon) \to (T^3, g_{\text{eucl}})$$

Suppose the neighborhood $S^2 \times \mathbb{R}^2$ converges to the point $p_0 \in T^3$ as $\varepsilon \to 0$ Take a family of normal mimimizing geodesics $\gamma_s : [-1, 1] \to T^3$ which do not intersect p_0 and also

$$\inf_t dist(\gamma_s(t), p_0) = dist(\gamma_s(0), p_0) = \mu$$

Where, μ is an arbitary constant. For predescribed small ε the geodesics γ_s are $\varepsilon-$ close , in the GH topology to normal minimizing geodesics γ_s^ε in (M_1, g_ε). Thus we can take normal neighborhoods $N_{2\mu'}(\gamma_s)$ for $\mu' > \mu$ about the centers of γ_s do not split topologically because geodesics $\{\gamma_s\}$ are within distance 2μ to the neighborhood $S^2 \times \mathbb{R}^2$ For more details of this counterexample you can check [An]

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