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MASTER THESIS

The Cameron Martin space

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Abstract

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The Cameron Martin space

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This thesis delves into the detailed study of Cameron-Martin Spaces associated with centered Gaussian measures on separable Banach spaces. The Cameron-Martin Space, a Hilbert subspace of the Banach space, plays a crucial role in understanding the quasi-invariance of Gaussian measures under translation. The research emphasizes the contrast between finite-dimensional and infinite-dimensional cases, where in the latter, Gaussian measures in different directions tend to be mutually singular, leading to significant implications for infinite-dimensional analysis.

Key contributions of this thesis include a rigorous definition of the Cameron-Martin Space via the covariance operator and its equivalence to other characterizations. Several foundational properties, such as the isomorphism between the Cameron-Martin Space and the closure of the dual space in L^2 space, are established. The thesis also explores the practical implications of these spaces in the context of stochastic processes, particularly highlighting the role of Cameron-Martin Spaces in the study of Gaussian measures and their transformations.

The results obtained are pivotal for furthering the understanding of Gaussian measures in infinite dimensions and provide a foundation for future work in functional analysis and probability theory.

Contents

Chapter 1

Preliminaries

1 Functional Analysis

Let X be a vector space. The most abstract spaces that we deal with in this thesis are the locally convex spaces. First of all, recall that a set $A \subset X$ is said to be *convex* if for every two points in A the whole segment connecting them is contained in A. More precisely:

$$x, y \in A \Rightarrow tx + (1-t)y \in A, \quad \forall t \in [0,1].$$

Generally, the definition of a locally convex space is based on the notion of seminorm.
A nonnegative function p on a real linear space X is called a *seminorm* if $p(\lambda x) = |\lambda| p(x)$ and $p(x+y) \leq p(x) + p(y)$ for all $\lambda \in \mathbb{R}$ and all vectors $x, y \in X$. A family of seminorms $\mathcal{P} = (p_i)_{i \in I}$ on X is said to be separating the points if for every nonzero element $x \in X$ there exists an index $i \in I$ such that $p_i(x) > 0$.

Definition 1.1. *A real linear space X is called* a locally convex space *if it is equipped with a family of seminorms $\mathcal{P} = (p_i)_{i \in I}$ on X separating the points. The open sets of this space are arbitrary unions of finite intersections of sets of the form*

$$\{x : p_i(x-y) < \varepsilon\}, \quad i \in I,\, y \in X,\, \varepsilon > 0\}.$$

Observe that for any fixed $y \in X$ (so locally) finite intersections of the above convex sets are also convex, so the corresponding local base consists of convex sets. This property justifies the name locally convex space.

Now, recall that a topological vector space is a vector space equipped with a topology, where the addition and the scalar product of the vector space are continuous operations with respect to the topology. In locally convex spaces, the notion of Fréchet space is very relevant and powerful.

Definition 1.2. *A topological vector space X is called* Fréchet space *if it satisfies the following three properties:*

- *It is locally convex*
- *Its topology can be induced by a translation-invariant metric, that is, a metric $d : X \times X \to \mathbb{R}$ such that $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$.*
- *Some (or equivalently, every) translation-invariant metric on X inducing the topology of X is complete.*

Note that there is no natural notion of distance between two points of a Fréchet space: many different translation-invariant metrics may induce the same topology.

If you are not familiar with these concepts it helps to have in mind a normed space instead of locally convex space.

1.1 Normed Spaces

A normed space $(X, \|\cdot\|)$ is a linear space X, equipped with a function $\|\cdot\| : X \to \mathbb{R}$, called norm, satisfying the following properties: for every $x, y \in X$ and $t \in \mathbb{R}$,

(i) $\|x\| \geq 0$

(ii) $\|x\| = 0 \iff x = 0$

(iii) $\|tx\| = |t| \cdot \|x\|$

(iv) $\|x + y\| \leq \|x\| + \|y\|$

A Banach space is a normed space which is complete under its induced metric function $d : X \times X \to \mathbb{R},\ d(x, y) = \|x - y\|$.

The main examples of Banach spaces that we will deal with here are the following: the Euclidean spaces $\mathbb{R}^n$, $n \geq 1$, the space of Lebesgue integrable functions, the linear operators between two Banach spaces and in particular the dual space of a normed space.

Recall that if $(X, \|\cdot\|)$ has a countable dense subset, then it is called separable. Convergence in $(X, \|\cdot\|)$ means that if $x_n, x \in X$ then $x_n \to x$ in X if and only if $\|x_n - x\| \to 0$. The most important subsets in a normed space X are the balls. The open ball with center x and radius r is the set $D(x,r) = \{y \in X : \|y - x\| < r\}$ and the closed ball with center x and radius r is the set $B(x,r) = \{y \in X : \|y - x\| \leq r\}$. Also we set $B_X = B(0,1)$.

Proposition 1.1. *In a normed space X the functions $\|\cdot\|$ and $+, \cdot$ are continuous functions.*

A linear operator between two normed spaces X and Y is a function $T : X \to Y$ satisfying the linearity property

$$T(\lambda x_1 + \mu x_2) = \lambda T(x_1) + \mu T(x_2)$$

for all $x_1, x_2 \in X$ and $\lambda, \mu \in \mathbb{R}$. Sometimes we will write Tx instead of $T(x)$.

A linear operator $T : X \to Y$ is called bounded if there exists a constant $c > 0$ such that $\|Tx\|_Y \leq c\|x\|_X$ for all $x \in X$ (we will usually denote the norms without the subscript). If T is bounded, we define the norm of T, $\|T\|$, as the smallest constant c for which the previous inequality holds for all $x \in X$. So we have the useful inequality $\|Tx\| \leq \|T\| \cdot \|x\|$, $x \in X$. Another useful property for the norm of T is the following

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\| = 1} \|Tx\|.$$

Theorem 1.1. *Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and $T : X \to Y$ be a linear operator. The following are equivalent:*

(i) T is continuous.

(ii) T is continuous at $0 \in X$.

(iii) T is continuous at some point $x_0 \in X$.

(iv) There exists $M < \infty$ such that $\|Tx\|_Y \leq M\|x\|$ for all $x \in X$.

(v) The set $\{\|Tx\|_Y : x \in X$ and $\|x\| \leq 1\}$ is bounded.

(vi) T is uniformly continuous.

1.2 Dual Space

Let $(X, \|\cdot\|)$ be a normed space. The dual space of X is the linear space X^* of all continuous linear functions (bounded linear functionals) $f : X \to \mathbb{R}$. We usually endow X^* with a norm related with the norm of X defined as:

$$\|f\| = \sup_{\|x\|=1} |f(x)| = \sup_{\|x\| \leq 1} |f(x)|.$$

Theorem 1.2. *Hahn-Banach Theorem (analytic form)*

Let X be a normed space, Y a subspace of X and $f : Y \to \mathbb{R}$ a bounded linear functional $f : Y \to \mathbb{R}$. There exists a linear extension $\widetilde{f} : X \to \mathbb{R}$ of f, with $\|\widetilde{f}\| = \|f\|$.

Proposition 1.2. *Let X be a normed space and $x \in X$. Then*

$$\|x\| = \sup_{\|f\|=1} |f(x)| = \max_{\|f\|=1} |f(x)|$$

Theorem 1.3. *Hahn-Banach Separation Theorem*

Let X be a normed space, A, B nonempty disjoint closed and convex subsets of X. If B is compact then there exists $f \in X^*$ and $c_1, c_2 \in \mathbb{R}$ such that $f(a) < c_1 < c_2 < f(b)$ for all $a \in A$ and $b \in B$.

1.3 Hilbert Spaces

A (Real) Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a linear space H, equipped with a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$, called inner product, satisfying the following properties: for every $x, y, z \in H$ and $t \in \mathbb{R}$,

(i) $\langle x, x \rangle \geq 0$

(ii) $\langle x, x \rangle = 0 \iff x = 0$

(iii) $\langle x, y \rangle = \langle y, x \rangle$

(iv) $\langle tx, y \rangle = t\langle x, y \rangle$

(v) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

and H is complete under the norm $\|x\| = \sqrt{\langle x, x \rangle}$. First observe the most important equality:

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$

for all $x, y \in X$.

We say that two elements x, y in X are perpendicular to each other if their inner product is zero i.e. $\langle x, y \rangle = 0$ and we write $x \perp y$. A family $\{e_i : i \in I\} \subset X$ is called orthonormal if

$$\langle e_i, e_j \rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

So, if x, y are perpendicular one has the Pythagorean theorem:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

In a normed space X the parallelogram law is an equation relating norms:

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2 \quad \forall x, y \in X.$$

Remark 1.1. *Every inner product space satisfies the parallelogram law.*

Lemma 1.1. *(polarization identity) For any norm satisfying the parallelogram law, that norm can be induced by a unique inner product satisfying the equality:*

$$\langle x, y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

for all $x, y \in X$.

Definition 1.3. *Let X be a separable space with inner product. An orthonormal sequence $\{e_n : n \in \mathbb{N}\}$ is called orthonormal basis of X, if*

$$X = \overline{span\{e_n : n \in \mathbb{N}\}}.$$

Where $spanA$ is the set of all linear combinations of elements in A.

Proposition 1.3. *Every separable space X with inner product has orthonormal basis.*

Proposition 1.4. *Let X be an inner product space and $\{e_n : n \in \mathbb{N}\}$ orthonormal basis of X. Then, if $x \in X$ we have:*

(i) $x = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$

(ii) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$

Proposition 1.5. *(Cauchy-Schwarz inequality)*
Let $\big(X, \langle \cdot, \cdot \rangle\big)$ be an inner product space. For every $x, y \in X$,

$$|\langle x, y \rangle| \leq \sqrt{\langle x, x \rangle} \cdot \sqrt{\langle y, y \rangle}$$

and the equality holds if and only if x, y are linearly dependent.

Proposition 1.6. *Let X be an inner product space. The inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ is a continuous function. If $x_n \to x$ and $y_n \to y$ in X then $\langle x_n, y_n \rangle \to \langle x, y \rangle$.*

Theorem 1.4. *(Riesz representation theorem)*
Let H be a Hilbert space and $f \in H^*$. There exists a unique $h \in H$ such that $f(x) = \langle x, h \rangle$ for all $x \in H$.

1.4 Dual Operator

Theorem 1.5. *(Dual Operator) Given a linear bounded operator $T : H_1 \to H_2$ between two Hilbert spaces there exists a unique linear bounded operator $T^* : H_2 \to H_1$, called dual operator of T, which satisfies the relation*

$$\langle Tx, y \rangle = \langle x, T^* y \rangle$$

for all $x \in H_1$ and $y \in H_2$, with norm $\|T^*\| = \|T\|$.

From the definition we have some useful properties:

(i) $(T^*)^* = T$.

(ii) $(T_1 T_2)^* = T_2^* T_1^*$.

We say that a bounded linear operator $T : H \to H$ is:

(i) positive if $\langle Tx, x \rangle \geq 0$ for all $x \in H$.

(ii) self-adjoint if $T^* = T$.

Notice that if H is a Hilbert space and $A : H \to H$ is a linear bounded operator then every linear operator of the form A^*A or AA^* is positive.

Theorem 1.6. *(Square Root)*
Let H be a Hilbert Space and $T : H \to H$ be a linear operator. The following statements are equivalent:

(i) T is positive.

(ii) There exists $B : H \to H$ positive such that $T = B^2$.

(iii) There exists $S : H \to H$ such that $T = S^* S$.

Sometimes we put $\sqrt{T} := S$ and we introduce the notation $|T| := \sqrt{T^* T}$.

1.5 Weak Topology

Let $(X, \|\cdot\|)$ be a normed space, (x_n) be a sequence in X and $x \in X$. In this section, we would like to introduce a new type of convergence, the weak convergence. So, to avoid confusion, we will say that (x_n) strongly converges to x if $\|x_n - x\| \to 0$ instead of converges. The weak convergence is defined through linear functionals as below:

Definition 1.4. *Let (x_n) be a sequence in a normed space X and $x \in X$. We say that (x_n) converges weakly to x and we write $x_n \xrightarrow{w} x$ if for all $f \in X^*$ we have*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

In that case, we say that (x_n) is weakly convergent and x is weak limit of (x_n).

The main properties of weakly convergence are in the following lemma.

Lemma 1.2. *Let (x_n) be a weakly convergent sequence in a normed space X and a $x \in X$ such that $x_n \xrightarrow{w} x$. Then:*

(i) The weak limit of (x_n) is unique.

(ii) Every subsequence of (x_n) weakly converges to x.

(iii) The sequence (x_n) is bounded.

The relation between the convergence in X and weakly convergence is the following :

Proposition 1.7. *Let X be a normed space. If (x_n) converges strongly to $x \in X$, then it converges also weakly to the same element.*

Proposition 1.8. *(Mazur) Let X be a normed space and A be a convex subset of X. Then*

$$\overline{A}^{w} = \overline{A}^{\|\cdot\|}.$$

1.6 Compact Operators

Definition 1.5. *Let E, F be two Banach spaces. A linear function $T : E \to F$ is called* compact *if $\overline{T(B_E)}$ is compact subset of F.*

Observe that every compact linear function is bounded linear operator since the subset $\overline{T(B_E)}$ is compact so it is also bounded.

In the next theorem we have some useful characterizations of compact linear operators.

Theorem 1.7. *Let E, F be two Banach spaces, $T : E \to F$ linear function. The following are equivalent:*

(i) T is compact.

(ii) For every bounded subset $A \subset E$, the set $\overline{T(A)}$ is relatively compact (i.e. $T(A)$ is relatively compact).

(iii) For every bounded sequence (x_n) of E, the sequence (Tx_n) has $\|\cdot\|-$covergent subsequence.

A stronger characterization can be proved with the extra hypothesis that the domain is Hilbert space.

Proposition 1.9. *Let H be a Hilbert space, F be a Banach space and $T : H \to F$ a linear function. The function T is compact if and only if for every weakly null sequence (x_n) in H , i.e. $x_n \xrightarrow{w} 0$, we have $\|Tx_n\| \to 0$.*

1.7 Trace class operators

Definition 1.6. *Let H be a separable Hilbert space. A bounded linear operator A is called trace class if $|A|$ has an orthonormal basis formed by its eigenvectors corresponding to the eigenvalues a_n such that*

$$\sum_{n=1}^{\infty} a_n < \infty$$

Here are some of the main results for trace class operators

Proposition 1.10. *(i) For any trace class operator A, the sum of the series $\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_H$ does not depend on the choice of the orthonormal basis (e_n).*

(ii) A self-adjoint linear bounded operator A is trace class precisely when for every orthonormal basis (e_n) in H the series

$$\sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_H$$

converges.

(iii) A self-adjoint linear bounded operator A is trace class if and only if there exists an orthonormal basis (e_n) in H consisting of the eigenvectors of A corresponding to the eigenvalues a_n such that

$$\sum_{n=1}^{\infty} |a_n| < \infty$$

So we can define the trace of a such an operator with the formula

$$\mathrm{trace} A := \sum_{n=1}^{\infty} \langle Ae_n, e_n \rangle_H$$

(which is independent of the orthonormal basis (e_n))

2 Measure Theory

2.1 Basic Knowledge

Let X be a topological space, $\mathcal{A}$ be a $\sigma-$field (or $\sigma-$algebra) and μ be a measure on the measurable space $(X, \mathcal{A})$. We denote by $\mathcal{B}(X)$ the Borel $\sigma-$field which is generated by the open sets.

It is necessary for the following theory that linear functions are measurable. So the $\sigma-$field that we will use often is the *Cylindrical* $\sigma-$field which makes all linear functions measurable.

Definition 1.7. *We call cylidrical sets (or cylinders) the sets in a locally convex space X with the dual X^*, which have the form*

$$C = \{x \in X : (\ell_1(x), ..., \ell_n(x)) \in C_0\}, \quad \ell_i \in X^*, \quad i = 1, \dots, n, \quad n \in \mathbb{N}$$

where $C_0 \in \mathcal{B}(\mathbb{R}^n)$ is called a base of C.

Some notations about cylindrical sets are the following.
Denote by $\mathcal{E}(X)$ the $\sigma-$ field generated by all cylindrical subsets of X. In other words, $\mathcal{E}(X)$ is the minimal $\sigma-$field , with respect to which all continuous linear functionals on X are measurable.
Let F be some family of functionals on a set X. Let us denote by $\mathcal{E}(X, F)$ the minimal $\sigma-$field of subsets of X, with respect to which all functionals in F are measurable. In other words, $\mathcal{E}(X, F)$ is generated by the sets of the form $\{f < c\}$, $f \in F$, $c \in \mathbb{R}$. In particular, $\mathcal{E}(X) = \mathcal{E}(X, X^*)$.

Lemma 1.3. *A set E belongs to $\mathcal{E}(X)$ precisely when it has the form*

$$E = \{x \in X : (\ell_1(x), ..., \ell_n(x), \dots) \in B\},$$

where $\ell_i \in X^*, i \in \mathbb{N}, \quad B \in \mathcal{B}(\mathbb{R}^\infty)$.

Let's discuss a little about product spaces.
Let μ_n be probability measures defined on $\sigma-$fields $\mathcal{C}_n$ in spaces X_n. Put $X = \prod_{n=1}^\infty X_n$. Let $\mathcal{C} = \otimes_{n=1}^\infty \mathcal{C}_n$ be the $\sigma-$field generated by all sets of the form $C = C_1 \times C_2 \times \cdots \times C_n \times X_{n+1} \times X_{n+2} \times \cdots$, where $C_i \in \mathcal{C}_i$. Recall that the countable product $\otimes_{n=1}^\infty \mu_n$ is a probability measure μ on $\mathcal{C}$ (called a *product-measure*) defined by $\mu(C) = \mu_1(C_1) \cdots \mu_n(C_n)$ for the sets C of the form above. It is readily seen that this set function is well-defined. A well-known theorem in measure theorem states that μ is countably additive (which is not obvious) on the algebra generated by such sets, hence it uniquely

extends to a measure on $\mathcal{C}$ also denoted by $\otimes_{n=1}^{\infty}\mu_n$ and called the product of the μ_n's. The construction of countable products enables one to define arbitrary products $\otimes_i\mu_i$ of probability measures on $\sigma-$fields $\mathcal{C}_i$ in spaces X_i. To this end, it suffices to note that the $\sigma-$field $\otimes_i\mathcal{C}_i$, generated by the sets $\prod_i C_i$, where $C_i \in \mathcal{C}_i$ and only finitely many of the C_i's differ from X_i, consists of the sets of the form $E = C \times Y$, where $C \in \otimes_{n=1}^{\infty}\mathcal{C}_{i_n}$ and $Y = \prod_{j\neq i_n} X_j$. Hence we may put $\otimes_i\mu_i(E) = \otimes_{n=1}^{\infty}\mu_{i_n}(C)$.

In this thesis, we will usually identify functions, in the Lebesgue integrals, which are almost everywhere equal, i.e. $f = g$ if $f = g$ $\mu-$almost everywhere.

Definition 1.8. *The image of the measure (or the pushforward measure of) μ on a measurable space $(X, \mathcal{M}_X)$ under a measurable mapping $f : X \to Y$ to a measurable space $(Y, \mathcal{M}_Y)$ is denoted by $f_*\mu$ or $\mu \circ f^{-1}$ and defined by the equality*

$$f_*\mu(B) = \mu \circ f^{-1}(B) = \mu(f^{-1}(B)), \quad B \in \mathcal{M}_Y.$$

A function ϕ on Y is integrable with respect to $f_*\mu$ precisely when the function $\phi \circ f$ is $\mu-$integrable on X. In this case, the following identity called *the change of variable formula* holds true:

$$\int_Y \phi(y) f_*\mu(dy) = \int_X \phi(f(x))\mu(dx).$$

If X is a locally convex space, $Y = X$ and $\mathcal{M}_X = \mathcal{E}(X)$ or $\mathcal{M}_X = \mathcal{B}(X)$, then for any $h \in X$ the mapping $x \mapsto x + h$ is measurable. The image of the measure μ under this mapping is denoted by μ_h and is called the *shift of the measure to the vector h*. By definition

$$\mu_h(A) = \mu(A - h).$$

It is clear that $\mathcal{E}(X)$ is contained in the Borel $\sigma-$field of $\mathcal{B}(X)$. However, for separable Banach spaces the equality holds true.

Proposition 1.11. *Let $\mathcal{B}$ be a separable Banach space and let μ and ν be two probability Borel measures on $\mathcal{B}$. If $f_*\mu = f_*\nu$ for every $f \in \mathcal{B}^*$, then $\mu = \nu$.*

2.2 Fourier Transform

Definition 1.9. *Let X be a locally convex space and let μ be a measure on $\mathcal{E}(X)$. The* Fourier tranform *(or characteristic functional) $\tilde{\mu}$ of the measure μ is defined by the formula*

$$\tilde{\mu} : X^* \to \mathbb{C}, \quad \tilde{\mu}(f) = \int_X \exp\left(if(x)\right)\mu(dx)$$

Remark 1.2. *When X is a Hilbert space, from Riez representation theorem we know that X^* is isomorphic to X so we usually use the corresponding element in X instead of $f \in X^*$ in the Fourier transform.*

Proposition 1.12. *Any two measures on $\mathcal{E}(X)$ with equal Fourier transforms coincide.*

Since $\mathcal{E}(X) = \mathcal{B}(X)$ when X is a separable Banach space we obdain the following result.

Corollary 1.1. *Let μ and ν be any two probability measures on a separable Banach space $\mathcal{B}$. If $\tilde{\mu}(f) = \tilde{\nu}(f)$ for every $f \in \mathcal{B}^*$, then $\mu = \nu$.*

2.3 Bochner Intergral

In this section, we present a way of defining integrals in a Banach space, which will be essential for the proofs of Cameron-Martin space.

Let $(X, \mathcal{A}, \mu)$ be a measurable space and $\mathcal{B}$ be a Banach space. The Bochner integral of a function $f : X \to \mathcal{B}$ is defined in much the same way as the Lebesgue integral. The main idea is that we "approach" f by simple functions.

A simple function is defined to be a finite sum of the form

$$s_n(x) = \sum_{k=1}^{n} \chi_{E_k}(x) b_k,$$

where the E_k are disjoint members of the $\sigma-$field $\mathcal{A}$, the b_k are distinct elements of $\mathcal{B}$, and χ_E is the characteristic function of E. If $\mu(E_k)$ is finite whenever $b_k \neq 0$, then the simple function is *integrable*, and the integral is then defined by

$$\int_X \left[\sum_{k=1}^{n} \chi_{E_k}(x) b_k \right] \mu(dx) = \sum_{k=1}^{n} \mu(E_k) b_k$$

A measurable function $f : X \to \mathcal{B}$ is *Bochner integrable* if there exists a sequence of integrable simple functions s_n such that

$$\lim_n \int_X \|f(x) - s_n(x)\|_{\mathcal{B}} \mu(dx) = 0,$$

where the integral on the left-side is an ordinary Lebesgue integral.

In this case, the *Bochner integral* is defined by

$$\int_X f(x)\mu(dx) = \lim_n \int_X s_n(x)\mu(dx).$$

It can be shown that the sequence $\left(\int_X s_n(x)\mu(dx)\right)_n$ is a Cauchy sequence in the Banach space $\mathcal{B}$, hence the limit on the right exists; furthermore, the limit is independent of the approximating sequence of simple functions (s_n).

A useful property connecting Bochner integrable functions and bounded linear operators is the following:

Proposition 1.13. *If $T : \mathcal{B} \to \mathcal{D}$ is a bounded linear operator between Banach spaces $\mathcal{B}$ and $\mathcal{D}$, and $f : X \to \mathcal{B}$ is Bochner integrable, then $Tf : X \to \mathcal{D}$ Bochner integrable and*

$$\int_A Tf(x)\mu(dx) = T\int_A f(x)\mu(dx)$$

for all measurable subsets $A \in \mathcal{A}$.

2.4 Martingales

Let $\mathcal{B}$ be a Banach space and $(\Omega, \mathcal{F}, P)$ be a probability space.

Theorem 1.8. *Let $X : \Omega \to \mathcal{B}$ be a Bochner integrable function and $\mathcal{F}$ a Borel field of measurable Ω sets. Then there exists a function $\mathcal{E}^s\{X|\mathcal{F}\}(\cdot) : \Omega \to \mathcal{B}$ which is Bochner integrable, strongly measurable relative to $\mathcal{F}$, unique a.e., and*

$$\int_A X(\omega)dP = \int_A \mathcal{E}^s\{X|\mathcal{F}\}(\omega)dP \quad \text{for all } A \in \mathcal{F}.$$

We can now define a (discrete) Martingale.

Definition 1.10. *Let $X_n : \Omega \to \mathcal{B}$ be integrable in the sense of Bochner for $n \in \mathbb{N}$ and $\mathcal{F}$ be a Borel field of measurable subsets of Ω for $n \in \mathbb{N}$. Let $\mathcal{F}_n \subset \mathcal{F}_m$ if $n < m$. Suppose X_m is strongly measurable relative to $\mathcal{F}_m$ or equal almost everywhere to such a function. If $\mathcal{E}^s\{X_m|\mathcal{F}_n = X_n\}$ with probability 1 when $n < m$ then $\{X_n, \mathcal{F}_n, n \geq 1\}$ is a (strong) $\mathcal{B}-$martingale.*

The main result we will use about maritngales is Doob's convergence theorem

Theorem 1.9. *Let $\mathcal{B}$ be a reflexive Banach space and $\{X_n, \mathcal{F}_n, n \geq 1\}$ be a $\mathcal{B}-$martingale.*
If $\sup_n E\|X_n\| < \infty$, then there is a strongly measurable $\mathcal{B}-$valued random variable X such that $\|X_n(\omega) - X(\omega)\| \to 0$ as $n \to \infty$ with probability 1.

3 Gaussian Measures

3.1 Gaussian measures on real line

Definition 1.11. *A Borel probability measure $\gamma = \gamma_1$ on $\mathbb{R}^1$ is called Gaussian if it is either Dirac measure δ_a at some point a or has density*

$$p(\cdot, a, \sigma^2) : t \mapsto \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{(t-a)^2}{2\sigma^2}\right)$$

with respect to Lebesgue measure with $a \in \mathbb{R}^1$ and $\sigma > 0$. In the latter case, the measure γ is called nondegenerate.

The parameters a and σ^2 are called the mean variance of γ, respectively. The quantity σ is called the mean-square deviation. For any Dirac measure (i.e., a probability measure at a point) we put $\sigma = 0$. The mean and variance of a Gaussian measure γ admit the following representation:

$$a = \int_{-\infty}^{\infty} t\, \gamma(dt), \quad \sigma^2 = \int_{-\infty}^{\infty} (t-a)^2\, \gamma(dt).$$

The measure with density $p(\cdot, 0, 1)$ is called standard. A mean zero Gaussian measure is called centered or symmetric.

Definition 1.12. *A Gaussian random variable is a variable with Gaussian distribution.*

A Gaussian random variable with a centered distribution is called centered or symmetric. Clearly, an arbitrary Gaussian random variable with mean a and variance σ^2 can be represented as $\sigma\xi + a$, where ξ is a random variable with the standard Gaussian distribution. Gaussian distributions are often called normal.

The Fourier transform (the characteristic functional) of the Gaussian measure γ with parameters (a, σ^2) is

$$\tilde{\gamma}(y) := \int_{\mathbb{R}^1} \exp(iyx)\, \gamma(dx) = \exp\left(iay - \frac{1}{2}\sigma^2 y^2\right).$$

3.2 Multivariate Gaussian distributions

About $\mathbb{R}^n$ we have the following theory.

Definition 1.13. *A Borel probability measure γ on $\mathbb{R}^n$ is called Gaussian if for every linear functional f on $\mathbb{R}^n$, the induced (image) measure $\gamma \circ f^{-1}$ is Gaussian.*

Recall that the Fourier transform $\tilde{\mu}$ of a finite Borel measure μ on $\mathbb{R}^n$ is defined by the formula

$$\tilde{\mu} : \mathbb{R}^n \to \mathbb{C}^1, \quad \tilde{\mu}(y) = \int_{\mathbb{R}^n} \exp[i\langle y, x\rangle]\mu(dx)$$

and that measures on $\mathbb{R}^n$ are uniquely determined by their Fourier transforms.

Proposition 1.14. *A measure γ on $\mathbb{R}^n$ is Gaussian if and only if its Fourier transforms has the form*

$$\tilde{\gamma}(y) = \exp\left(i\langle y, a\rangle - \frac{1}{2}\langle Ky, y\rangle\right) \tag{1.1}$$

where a is a vector in $\mathbb{R}^n$ and K is a nonnegative matrix. The measure γ has a density if and only if the matrix K is invertible. In this case, the density of the measure γ is given by

$$x \to \frac{1}{\sqrt{(2\pi)^n detK}} \exp\{-\frac{1}{2}\langle K^{-1}(x-a), x-a\rangle\}.$$

Proof. Let f be a linear function on $\mathbb{R}^n$. Using the change of variables, one evaluates the Fourier transform of the measure $\nu = \gamma \circ f^{-1}$ as follows:

$$\tilde{\nu}(t) = \int_{\mathbb{R}^1} \exp(its)\nu(ds) = \int_{\mathbb{R}^n} \exp(itf(x))\gamma(dx).$$

Let us denote the vector representing the functional f by the same symbol. From (1.1) we get

$$\tilde{\nu}(t) = \int_{\mathbb{R}^n} \exp(i\langle tf, x\rangle)\gamma(dx) = \tilde{\gamma}(tf) = \exp\left(it\langle a, f\rangle - \frac{1}{2}t^2\langle Kf, f\rangle\right),$$

which means that the measure ν is Gaussian.

Conversely, suppose that all such measures are Gaussian. Denote their means and variances by $a(f)$ and $\sigma(f)^2$, respectively. Then the following equalities hold true:

$$a(f) = \int_{\mathbb{R}^1} t\, \gamma \circ f^{-1}(dt) = \int_{\mathbb{R}^n} f(x)\gamma(dx),$$

$$\sigma(f)^2 = \int_{\mathbb{R}^1} (t - a(f))^2\, \gamma \circ f^{-1}(dt) = \int_{\mathbb{R}^n} (\langle f, x\rangle - a(f))^2\, \gamma(dx).$$

Hence the function $f \mapsto a(f)$ is linear, and the function $f \mapsto \sigma(f)^2$ is a nonnegative quadratic form. Therefore, there exist a vector a and a nonnegative self-adjoint operator K such that $a(f) = \langle f, a\rangle$ and $\sigma(f)^2 = \langle Kf, f\rangle$. This yields (1.1). The assertion about densities reduces to the one dimensional case, since we can use the coordinates to the eigenvector of the matrix K. □

Corollary 1.2. *Let γ be a Gaussian measure on $\mathbb{R}^n$ with the Fourier transform (1.1). Then*

$$a = \int_{\mathbb{R}^n} x\gamma(dx), \tag{1.2}$$

$$\langle Ku, v\rangle = \int_{\mathbb{R}^n} \langle u, x-a\rangle\langle v, x-a\rangle\gamma(dx), \quad \forall u, v \in \mathbb{R}^n. \tag{1.3}$$

The vector a given by equality (1.2) is called the *mean* of the Gaussian measure γ, and the operator K defined by means of (1.3) is called its *covariance operator*.

Clearly, Gaussian measures on $\mathbb{R}^n$ can be described as the images of the *standard Gaussian measure* on $\mathbb{R}^n$ (i.e., the product of n copies of the standard Gaussian measure on $\mathbb{R}^1$) under affine mappings $x \mapsto \sqrt{K}x + a$.

On the linear subspace $\sqrt{K}(\mathbb{R}^n)$ we define the inner product

$$\langle u, v\rangle_\gamma := \langle \sqrt{K}^{-1}u, \sqrt{K}^{-1}v\rangle.$$

The unit ball with respect to this inner product (i.e., the ellipsoid $\sqrt{K}(U)$, where U is the closed unit ball in $\mathbb{R}^n$) is called the *ellispoid of concentration* of the Gaussian measure γ. The ellipsoid of concentration can be written as

$$\sqrt{K}(U) = \{h \in \mathbb{R}^n : |h|_\gamma \leq 1\},$$

$$|h|_\gamma = \sup\{\langle \ell, h\rangle : \ell \in \mathbb{R}^n, \quad \int_{\mathbb{R}^n} \langle \ell, x-a\rangle^2\gamma(dx) \leq 1\}.$$

3.3 Gaussian measures on locally convex spaces

Let us now discuss about some more abstract spaces. First, we define the Gaussian measure on a locally convex space.

Definition 1.14. *(i) Let E be a linear space and let F be some linear space of linear functions on E, separating the points in E. A probability measure γ on $\mathcal{E}(E, F)$ is called Gaussian if, for any $f \in F$, the measure $\gamma \circ f^{-1}$ is Gaussian.*

(ii) Let X be a locally convex space. A probability measure γ defined on the $\sigma-$field $\mathcal{E}(X)$, generated by X^*, is called Gaussian if, for any $f \in X^*$, the induced measure $\gamma \circ f^{-1}$ on $\mathbb{R}^1$ is Gaussian. The measure γ is called centered (or symmetric) if all the measures $\gamma \circ f^{-1}$, $f \in X^*$, are centered.

(iii) A random vector is called Gaussian if it induces a Gaussian measure.

Lemma 1.4. *Let γ be a Gaussian measure on locally convex space X and let $T : X \to Y$ be a linear mapping to a locally convex space such that $\ell \circ T \in X^*$ for any $\ell \in Y^*$. Then the measure $\gamma \circ T^{-1}$ is Gaussian on Y. The same is true for the affine mapping $x \mapsto Tx + a$, where $a \in Y$.*

Proof. The proof is immediate from the definition. □

Recall that the Fourier transform (the characteristic functional) of a measure μ defined on the $\sigma-$field $\mathcal{E}(X)$ in a locally convex space X is given by the formula

$$\tilde{\mu} : X^* \to \mathbb{C}^1, \quad \tilde{\mu}(f) = \int_X \exp\left(if(x)\right)\mu(dx).$$

Example 1.1. *Let X be a locally convex space. For every measure μ on $\mathcal{E}(X)$ and every $h \in X$, we have*

$$\tilde{\mu_h}(f) = e^{if(h)}\tilde{\mu}(f), \quad \forall f \in X^*$$

Proof. Use the change of variables formula and then the linearity of f.

□

This example shows that μ_h and μ have a close relation since a measure is determined by its Fourier transform. So many results concerning a measure are the same as its centered measure.

Theorem 1.10. *A measure γ on a locally convex space X is Gaussian if and only if its Fourier transform has the form*

$$\tilde{\gamma}(f) = \exp\left(iL(f) - \frac{1}{2}B(f,f)\right), \tag{1.4}$$

where L is a linear function on X^* and B is a symmetric bilinear function on X^* such that the quadratic form $f \mapsto B(f,f)$ is nonnegative.

Proof. Let γ be a Gaussian measure. It follows from the definition that $f \in L^2(X,\gamma)$ for any $f \in X^*$. Hence one can put

$$L(f) = \int f(x)\gamma(dx), \quad f \in X^*$$

$$B(f,g) = \int [f(x) - L(f)][g(x) - L(g)]\gamma(dx), \quad f,g \in X^*.$$

It is clear that L is a linear function and B is a bilinear symmetric function. In addition, the quadratic form $f \mapsto B(f,f)$ is nonnegative. Now equality (1.4) follows by the elementary properties of the one dimensional Gaussian distributions.

Conversely, let γ be a measure on $\mathcal{E}(X)$ with the Fourier transform given by (1.4). By the change of variables formula, we get

$$\int \exp(iyt)\gamma \circ f^{-1}(dt) = \int \exp\left(iyf(x)\right)\gamma(dx) = \tilde{\gamma}(f(y\cdot)) = \exp\left[iyL(f) - \frac{1}{2}y^2 B(f,f)\right],$$

where it follows that γ is Gaussian. □

Corollary 1.3. *A Gaussian measure γ on a locally convex space X is centered precisely when $\gamma(A) = \gamma(-A)$ for any $A \in \mathcal{E}(X)$. This is equivalent to the relation $L = 0$ in 1.4.*

Proof. It suffices to note that the Fourier transform to the measure $A \to \gamma(-A)$ is the function which is complex conjugate to $\tilde{\gamma}$ and that measures on $\mathcal{E}(X)$ with equal Fourier transforms coincide. □

Corollary 1.4. *The product $\gamma_1 \otimes \gamma_2$ of the Gaussian measures γ_1 and γ_2 on locally convex spaces X_1 and X_2 is Gaussian measure on $X_1 \times X_2$. In the case where $X_1 = X_2 = X$, the convolution $\gamma_1 * \gamma_2$ of the measures γ_1 and γ_2, defined as the image of the measure $\gamma_1 \otimes \gamma_2$ under the mapping $X \times X \to X$, $(x,y) \to x + y$ is Gaussian as well.*

Proposition 1.15. *Let γ be a Gausssian measure on a Banach space $\mathcal{B}$ and, for every $\phi \in \mathbb{R}$ define the "rotation" $R_\phi : \mathcal{B}^2 \to \mathcal{B}^2$ by*

$$R_\phi(x,y) = (x\sin\phi + y\cos\phi, a\cos\phi - y\sin\phi).$$

Then one has $(\gamma \otimes \gamma) \circ R_\phi = \gamma \otimes \gamma$, i.e. $\gamma \otimes \gamma$ is invariant under rotations.

Proof. Since a measure is characterised by its Fourier transform, it suffices to check that $\widetilde{\gamma \otimes \gamma} \circ R_\phi = \widetilde{\gamma \otimes \gamma}$. □

The following theorem is useful to show that the function $x \mapsto \|x\|^2$ is s integrable for any Gaussian measure. Actually, we are going to show much more, namely that there always exists a constant $a > 0$ such that $\exp(a\|x\|^2)$ is integrable.
In other words, the norm of any Banach-space valued Gaussian random variable has Gaussian tails, just like in the finite-dimensional case. Amazingly, this result uses the Gaussianity of the measure only indirectly through the rotation invariance shown before and even this property is only used for rotations by the angle $\phi = \frac{\pi}{4}$.

Theorem 1.11. *(Fernique) Let γ be a Gaussian measure on a separable Banach space $\mathcal{B}$. Then, there exists $a > 0$ such that $\int_{\mathcal{B}} \exp(a\|x\|^2)\gamma(dx) < \infty$.*

Proof. Note first that, from the previous proposition, one has for any two positive numbers t and s the bound

$$\begin{aligned}\gamma(\|x\|\le s)\gamma(\|x\|\le t) &= \int_{\|x\|\le s}\int_{\|y\|>t}\gamma(dx)\gamma(dy)\\ &= \int_{\|\frac{x-y}{\sqrt{2}}\|\le s}\int_{\|\frac{x+y}{\sqrt{2}}\|>t}\gamma(dx)\gamma(dy)\\ &= \int_{\|x\|>\frac{t-s}{\sqrt{2}}}\int_{\|y\|>\frac{t+s}{\sqrt{2}}}\gamma(dx)\gamma(dy)\\ &= \gamma\Big(\|x\|>\frac{t-s}{\sqrt{2}}\Big)^2.\end{aligned}$$

In order to go from the first to the second line, we have used the fact that the triangle inequality implies

$$\min\{\|x\|,\|y\|\}\ge\frac{1}{2}(\|x+y\|-\|x-y\|)$$

so that $\|x+y\|>\sqrt{2}t$ and $\|x-y\|\le\sqrt{2}s$ do indeed imply that both $\|x\|$ and $\|y\|$ are greater than $\frac{t-s}{\sqrt{2}}$. Since $\|x\|$ is $\gamma-$almost surely finite, there exists some $s>0$ such that $\gamma(\|x\|\le s)\ge\frac{3}{4}$. Set now $t_0=s$ and define s_n for $n>0$ recursively by the relation $t_n=\frac{t_{n+1}-s}{\sqrt{2}}$. It follows from the above inequality that

$$\gamma(\|x\|>t_{n+1})\le\gamma\left(\|x\|>\frac{t_{n+1}-s}{\sqrt{2}}\right)^2/\gamma(\|x\|\le s)\le\frac{4}{3}\gamma(\|x\|>t_n)^2.$$

Setting $y_n=\frac{4}{3}\gamma(\|x\|>t_{n+1})$, this yields the recursion $y_{n+1}\le y_n^2$ with $y_0\le\frac{1}{3}$. Applying this inequality repeatedly, we obtain

$$\gamma(\|x\|>t_n)=\frac{3}{4}y_n\le\frac{3}{4}y_0^{2^n}\le\frac{1}{4}3^{-1-2^n}\le 3^{-2^n}$$

On the other hand, one can check explicitly that $t_n=\frac{\sqrt{2}^{n+1}-1}{\sqrt{2}-1}s\le 2^{n/2}\cdot(2+\sqrt{2})s$, so that in particular $t_{n+1}\le 2^{n/2}\cdot 5s$. This shows that one has the bound

$$\gamma(\|x\|>t_n)\le 3^{-\frac{t_{n+1}^2}{25s^2}}$$

implying that there exists a universal constant $a>0$ such that the bound $\gamma(\|x\|>t)\le\exp(-2at^2/s^2)$ holds for every $t>s$. Integrating by parts, we finally obtain

$$\begin{aligned}\int_{\mathcal{B}}\exp\Big(\frac{a\|x\|^2}{s^2}\Big)\gamma(dx) &\le e^a+\frac{2a}{s^2}\int_s^\infty te^{a\frac{t^2}{s^2}}\gamma(\|x\|>t)dt\\ &\le e^a+2a\int_1^\infty te^{-at^2}dt<\infty\end{aligned}$$

which is the desired result. □

For Hilbert spaces, we have a nice and useful representation for the Fourier transform.

Theorem 1.12. *Let γ be a Gaussian measure on a separable Hilbert space X and let X^* be identified with X through Riesz representation. Then there exist a vector $a \in X$ and a self-adjoint nonnegative trace class operator K such that the Fourier transform of the measure γ equals*

$$x \mapsto \exp\left(i\langle a, x\rangle - \frac{1}{2}\langle Kx, x\rangle\right). \tag{1.5}$$

Conversely, for every pair (A, k) of the type mentioned above, the function (1.5) is the Fourier transform of a Gaussian measure on the space X. In addition, a is the mean of the measure γ and K is its covariance operator.

Proof. Suppose that γ is a Gaussian measure. Then its Fourier transform is given by equality (1.4), where $x \mapsto L(x)$ is a linear function and $x \mapsto B(x, x)$ is a quadratic form. By Lebesgue's theorem, the function $\tilde{\gamma}$ is continuous. This implies the continuity of both functions L and B. Therefore, there exists a vector a and a bounded self-adjoint nonnegative operator K such that

$$L(x) = \langle a, x\rangle, \quad B(x, x) = \langle Kx, x\rangle.$$

It is clear that the operator K is compact. Indeed, if $x_n \to 0$ in the weak topology, then $\tilde{\gamma}(x_n) \to 1$ by Lebesgue's theorem. Hence $\|\sqrt{K}x_n\| \to 0$. Now we will see that K is a trace class operator.
By Fernique's theorem about the exponential integrability we have that

$$\int_X \langle x, x\rangle \gamma(dx) < \infty.$$

Assuming that $a = 0$ (which simplifies the formulas), for any orthonormal basis $\{e_n\}$ in X, we get

$$\sum_{n=1}^{\infty} \langle Ke_n, e_n\rangle = \sum_{n=1}^{\infty} \int_X \langle x, e_n\rangle^2 \gamma(dx) = \int_X \langle x, x\rangle \gamma(dx) < \infty.$$

It is readily seen that a is the mean of γ and K is its covariance operator.

Conversely, let a be a vector in X and let K be a self-adjoint nonnegative trace class operator on X. Denote by e_n the eigenvectors of K corresponding to the eigenvalues k_n. Let $\{\xi_n\}$ be a sequence of independent standard Gaussian random variables. Note that the series

$$\xi(\omega) = a + \sum_{n=1}^{\infty} \sqrt{k_n}\xi_n(\omega)e_n$$

converges in X for almost all ω. Indeed, by the monotone convergence theorem, $\sum_{n=1}^{\infty} k_n \xi_n(\omega)^2 < \infty$ a.s., since $\sum_{n=1}^{\infty} k_n < \infty$. Denote by γ the distribution of the random vector ξ in X. It is easy to see that γ is a Gaussian measure and its Fourier transform is given by (1.5).

□

Corollary 1.5. *If $dim X = \infty$, then the function $\phi(x) = \exp\left(-\frac{1}{2}\|x\|^2\right)$ cannot be the Fourier transform of any countably additive measure on X.*

Corollary 1.6. *A Gaussian measure γ on a locally convex space X is centered precisely when $\gamma(A) = \gamma(-A)$ for any $A \in \mathcal{E}(X)$. This is equivalent to the relation $L = 0$ in (1.4).*

Corollary 1.7. *The product $\gamma_1 \otimes \gamma_2$ of two Gaussian measures γ_1 and γ_2 on locally convex spaces X_1 and X_2 is a Gaussian measures on $X_1 \times X_2$. In the case where $X_1 = X_2 = X$, the convolution $\gamma_1 * \gamma_2$ of the measures γ_1 and γ_2, defined as the image of the measure $\gamma_1 \otimes \gamma_2$ under the mapping $X \times X \to X, \quad (x, y) \mapsto x + y$ is Gaussian as well.*

The next theorem of Hajek and Feldman is one of the central results in the theory of Gaussian measures.

Theorem 1.13. *Any two Gaussian measures on one and the same locally convex space are either equivalent or mutually singular.*

Theorem 1.14. *(Hellinger's Theorem). Let μ and ν be two probability measures on a measurable space $(\Omega, \mathcal{B})$, and let λ be a measure on $\mathcal{B}$ such that $\mu \ll \lambda$, and $\nu \ll \lambda$. Then the number*

$$H(\mu, \nu) := \int \sqrt{\frac{d\mu}{d\lambda}} \sqrt{\frac{d\nu}{d\lambda}} d\lambda$$

is independent of λ and the following inequalities hold true:

$$2(1 - H(\mu, \nu)) \leq \|\mu - \nu\| \leq 2\sqrt{1 - H(\mu, \nu)^2}$$

where $\|\mu - \nu\| = \|\frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda}\|_{L^1(\lambda)}$ is the variation distance.

Proof. Let $p = \frac{d\mu}{d\lambda}$ and $q = \frac{d\nu}{d\lambda}$. Then $\|\mu - \nu\| = \|p - q\|_{L^1(\lambda)}$. Note that

$$(\sqrt{p} - \sqrt{q})^2 \leq |p - q| = |\sqrt{p} - \sqrt{q}||\sqrt{p} + \sqrt{q}|.$$

Integrating this inequality with respect to λ we have

$$\int (\sqrt{p} - \sqrt{q})^2 d\lambda \leq \int |p - q| d\lambda = \int |\sqrt{p} - \sqrt{q}||\sqrt{p} + \sqrt{q}| d\lambda \leq \sqrt{\int |\sqrt{p} - \sqrt{q}|^2 d\lambda} \cdot \sqrt{\int |\sqrt{p} + \sqrt{q}|^2 d\lambda}$$

By expanding the sqares we get the desired result. □

Lemma 1.5. *$\mu \perp \nu$ if and only if $H(\mu, \nu) = 0$, if and only if $\|\mu - \nu\| = 2$, for μ, ν probability measures.*

3.4 Hermite polynomials

Definition 1.15. *For any $k = 0, 1, \dots$, the* Hermite *(or* Chebyshev-Hermite *polynomials H_k on the real line are defined by the formula*

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp(\frac{x^2}{2}) \frac{d^k}{dx^k} \exp(\frac{-x^2}{2}).$$

For any multiindex $\alpha = (k_1, \dots, k_n)$ with nonnegative integer entries, the Hermite poynomials H_α on $\mathbb{R}$ is defined by

$$H_\alpha(x_1, \dots, x_n) = H_{k_1}(x_1) \cdots H_{k_n}(x_n),$$

where H_{k_i} are the one dimensional Hermite polynomials.

Hermite polynomials on the real line are obtained by orthogonalized the sequence of the powers of x in $L^2(\mathcal{B}, \gamma)$ with respect to the Gaussian measure. These polynomials can be introduced in several different ways, e.g., by means of the expansion

$$\exp\left(\lambda x - \frac{1}{2}\lambda^2\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\sqrt{k!}} H_k(x).$$

To see this, note that $\exp\left(\lambda x - \frac{1}{2}\lambda^2\right)$ can be written as

$$\exp\left(\frac{1}{2}x^2 - \frac{1}{2}(x-\lambda)^2\right) = \exp\left(\frac{1}{2}x^2\right) \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{d^k}{d\lambda^k} \exp\left(-\frac{1}{2}(x-\lambda)^2\right)\Bigg|_{\lambda=0}$$

Lemma 1.6. *Hermite polynomials on $\mathbb{R}^1$ have the following properties:*

(i) $\{H_k\}$ *is an orthonormal basis in the space $L^2(\mathcal{B}, \gamma_1)$, where γ_1 is the standard Gaussian measure on $\mathbb{R}^1$.*

(ii) $H_k'(x) = \sqrt{k} H_{k-1}(x) = x H_k(x) - \sqrt{k+1} H_{k+1}(x), \quad k \geq 1.$

(iii) for any numbers $\lambda_1, \dots, \lambda_n$, one has

$$\prod_{i=1}^{n} \sqrt{k_i!} H_{k_i}(\lambda_i) = \frac{\partial^{k_1 + \cdots + k_n}}{\partial t_1^{k_1} \cdots \partial t_n^{k_n}} \exp\left(\sum_{i=1}^{n} t_i \lambda_i - \frac{1}{2} \sum_{i=1}^{n} t_i^2\right)\Bigg|_{t_1 = \cdots = t_n = 0}$$

Proposition 1.16. *The system of the Hermite polynomials*

$$H_{k_1,\dots,k_n}, \quad k_i = 0, 1, \dots$$

on $\mathbb{R}^n$ *is an orthonormal basis in* $L^2(\mathcal{B}, \gamma_n)$, *where* γ *is the standard Gaussian measure on* $\mathbb{R}^n$.

Proof. This follows from the fact that, given measures $\mu_i, \quad i = 1, \dots, n$, on spaces X_i are orthonormal bases $\{\phi_j^i\}$ in $L^2(X_i, \mu_i)$, the system of functions

$$\phi_{k_1,\dots,k_n}(x_1, \dots, x_n) = \phi_{k_1}(x_1) \cdots \phi_{k_n}(x_n)$$

forms an orthonormal basis in $L^2(\bigotimes_{i=1}^n X_i, \otimes_{i=1}^n \mu_i)$. $\square$

Chapter 2

Cameron Martin Space

1 Motivation

Given a centered Gaussian measure γ on a separable Banach space $\mathcal{B}$, it is possible to associate to it in a canonical way a Hilbert space $\mathcal{H}_\gamma \subset \mathcal{B}$, called the Cameron–Martin space of γ. The main importance of the Cameron–Martin space is that it characterizes precisely those directions in B in which translations leave the measure µ "quasi-invariant" in the sense that the translated measure has the same null sets as the original measure. In general, the space $\mathcal{H}_\gamma$ will turn out to be strictly smaller than $\mathcal{B}$. Actually, this is always the case as soon as $\dim \mathcal{H}_\gamma = \infty$ and, even worse, we will see that in this case one necessarily has $\gamma(\mathcal{H}_\gamma) = 0$! Contrast this to the case of finite dimensional Lebesgue measure which is invariant under translations in any direction! This is a striking illustration of the fact that measures in infinite-dimensional spaces have a strong tendency of being mutually singular.

In the next chapters it is assumed (when not mentioned) that we are in a Borel measure space $(\mathcal{B}, \gamma)$ where $\mathcal{B}$ is a separable Banach space and γ is a centered Gaussian measure.

2 Definitions

Let γ be a centered Gaussian measure on a separable Banach space $\mathcal{B}$.
We define the Covariance operator of γ by the map $\mathcal{C}_\gamma : \mathcal{B}^* \times \mathcal{B}^* \to \mathbb{R}$

$$\mathcal{C}_\gamma(\ell_1, \ell_2) = \int_{\mathcal{B}} \ell_1(x)\ell_2(x)\gamma(dx). \tag{2.1}$$

We will give some various ways of defining Cameron-Martin space. First we present our main definition and then we give the others which are equivalent as propotions.

Definition 2.1. *The Cameron-Martin space $\mathcal{H}_\gamma$ of γ is the completion of the linear subspace $\mathring{\mathcal{H}}_\gamma \subset \mathcal{B}$ defined by*

$$\mathring{\mathcal{H}}_\gamma = \{h \in \mathcal{B} : \exists h^* \in \mathcal{B}^* \text{ with } \mathcal{C}_\gamma(h^*, \ell) = \ell(h) \; \forall \ell \in \mathcal{B}^*\}$$

under the inner product $\langle h, k \rangle_\gamma = \mathcal{C}_\gamma(h^*, k^*)$.

Let's see some remarks concerning the definition.

(i) $\mathring{\mathcal{H}}_\gamma$ is indeed a linear subspace of $\mathcal{B}$ since

$$\mathcal{C}_\gamma(h_1^* + h_2^*, \ell) = \mathcal{C}_\gamma(h_1^*, \ell) + \mathcal{C}_\gamma(h_2^*, \ell) = \ell(h_1) + \ell(h_2) = \ell(h_1 + h_2)$$
$$\mathcal{C}_\gamma(ah^*, \ell) = a\mathcal{C}_\gamma(h^*, \ell) = a\ell(h) = \ell(ah)$$

using the linearity of $\mathcal{C}_\gamma$ and ℓ.

(ii) Even though the map $\iota : h \mapsto h^*$ from the definition of $\mathring{\mathcal{H}}_\gamma$ may not be one to one, the norm $\|h\|_\gamma = \sqrt{\mathcal{C}_\gamma(h^*, h^*)} = \|h^*\|_{L^2}$ is well-defined.
To see this, assume that for a given $h \in \mathring{\mathcal{H}}_\gamma$, there are two corresponding elements h_1^* and h_2^* in $\mathcal{B}^*$. Then, defining $k = h_1^* + h_2^*$, one has

$$\mathcal{C}_\gamma(h_1^*, h_1^*) - \mathcal{C}_\gamma(h_2^*, h_2^*) = \mathcal{C}_\gamma(h_1^*, k) - \mathcal{C}_\gamma(h_2^*, k) = k(h) - k(h) = 0$$

showing that $\|h\|_\gamma$ does indeed not depend on the choice of h^*.
Now using the identity

$$\mathcal{C}_\gamma(h^*, k^*) = \frac{1}{4}\{\|h + k\|_\gamma^2 - \|h - k\|_\gamma^2\}$$

we can conlude that also the inner product $\langle h, k \rangle_\gamma = \mathcal{C}_\gamma(h^*, k^*)$ is also well defined.

(iii) If we view $\mathcal{B}^*$ as a subset of $L^2(\mathcal{B}, \gamma)$ (by identifying linear functionals that agree γ-almost surely), then the correspondence $\iota : h \mapsto h^*$ is injective and linear using the equalities of the first remark.
To see this, if h_1^*, h_2^* are two elements in $\mathcal{B}^*$ associated to the same element $h \in \mathcal{B}$, then

$$\int_{\mathcal{B}} h_1^*(x)\ell(x)\, \gamma(dx) = \ell(h) = \int_{\mathcal{B}} h_2^*(x)\ell(x)\, \gamma(dx) \quad \forall \ell \in \mathcal{B}^* \Rightarrow$$
$$h_1^* = h_2^* \quad \gamma - \text{almost everywhere} \Rightarrow h_1^* = h_2^* \quad \text{in } L^2(\mathcal{B}, \gamma)$$

Also if $h^* = \iota(h)$ we have $\|h^*\|_{L^2} = \|h\|_\gamma$ so ι is isometry.

(iv) The initial space $\mathring{\mathcal{H}}_\gamma$ can be embedded linearly and isometrically as a dense subspace to $\mathcal{H}_\gamma$. Also is "unique" in a sense that if $(K, \langle\cdot,\cdot\rangle_\gamma)$ is a Hilbert space and $U : \mathring{\mathcal{H}}_\gamma \to K$ a linear isometry with dense image, then U can be extended to a surjective linear isometry from $\mathcal{H}_\gamma$ to K.
We can consider $\mathcal{H}_\gamma$ as a dense subset of the Banach space $\mathcal{B}$ and define addition, scalar multiplication and inner product as follows:
If $h, k \in \mathcal{H}_\gamma$ and $a \in \mathbb{R}$, there exists sequences $(h_n),\ (k_n)$ in $\mathring{\mathcal{H}}_\gamma$ converging to h, k with the $\|\cdot\|_\gamma$ norm we have:

$$h + k = \lim_n (h_n + k_n),$$
$$ah = \lim_n (ah_n),$$
$$\langle h, k\rangle_\gamma = \lim_n \langle h_n, k_n\rangle_\gamma.$$

(v) Using the above remark one can show that for every $h \in \mathcal{H}_\gamma$ there exists $h^* \in \mathcal{B}^*$ with
$\mathcal{C}_\gamma(h^*, \ell) = \ell(h) \quad \ell \in \mathcal{B}^*$.

First of all, we show that it is a subspace of $\mathcal{B}$ despite the completion procedure and that all non-zero elements of $\mathcal{H}_\gamma$ have strictly positive norm:

Proposition 2.1. *There exists a constant $\|\mathcal{C}_\gamma\| < \infty$ such that for all $h, k \in \mathcal{B}$ we have:*

$$\langle h, h\rangle_\gamma \geq \|\mathcal{C}_\gamma\|^{-1} \cdot \|h\|^2.$$

Proof. One has the chain of inequalities

$$\|h\|^2 = \sup_{\ell \in \mathcal{B}^* - \{0\}} \frac{\ell(h)^2}{\|\ell\|^2} = \sup_{\ell \in \mathcal{B}^* - \{0\}} \frac{\mathcal{C}_\gamma(h^*, \ell)^2}{\|\ell\|^2} \leq \sup_{\ell \in \mathcal{B}^* - \{0\}} \frac{\mathcal{C}_\gamma(h^*, h^*) \cdot \mathcal{C}_\gamma(\ell, \ell)}{\|\ell\|^2} \leq \|\mathcal{C}_\gamma\| \langle h, h\rangle_\gamma,$$

which yields the bound on the norms

□

This inequality shows that a Cauchy sequence in $(\mathcal{B}, \|\cdot\|_\gamma)$ is also Cauchy in $(\mathcal{B}, \|\cdot\|)$. Since $(\mathcal{B}, \|\cdot\|)$ is complete, we can conclude that also $(\mathcal{B}, \|\cdot\|_\gamma)$ is complete. So $\mathcal{H}_\gamma$ lives in $\mathcal{B}$.

Proposition 2.2. *There is a canonical isomorphism $\iota : h \mapsto h^*$ between $\mathcal{H}_\gamma$ and the closure $\mathcal{B}^*_\gamma$ of $\mathcal{B}^*$ in $L^2(\mathcal{B}, \gamma)$. In particular, $\mathcal{B}^*_\gamma$ is separable.*

Proof. We have already shown that $\iota : \mathcal{H}_\gamma \to L^2(\mathcal{B}, \gamma)$ is an isomorphism onto its image.
We need to show that all of $\mathcal{B}^*$ belongs to the image of ι.

For $h \in \mathcal{B}^*$, define $h_* \in \mathcal{B}$ by

$$h_* = \int_{\mathcal{B}} h(x)x\gamma(dx).$$

This integral converges since $\|x\|^2$ is integrable by Fernique's theorem. Since one has the identity $\ell(h_*) = \int_{\mathcal{B}} \ell(h(x)x)\gamma(dx) = \int_{\mathcal{B}} h(x)\ell(x)\gamma(dx) = \mathcal{C}_\gamma(h,\ell)$ for all $\ell \in \mathcal{B}^*$, it follows that $h_* \in \mathring{\mathcal{H}}_\gamma$ and $h = \iota(h_*)$. To show that is onto $\mathcal{B}^*_\gamma$, take $z \in \mathcal{B}^*_\gamma$ and by definiton there exists $f_n \in \mathcal{B}^*$ with $f_n \longrightarrow z$ in $L^2(\mathcal{B},\gamma)$ and by the previous incusion there exists $h_n \in \mathcal{H}_\gamma$ such that $h^*_n := \iota(h_n) = f_n \longrightarrow z$.
Since (h^*_n) is Cauchy and $\|h_n - h_m\|_\gamma = \|h^*_n - h^*_m\|_{L^2}$ (because ι is isometry),we have that (h_n) is also Cauchy and converges to some $h \in \mathcal{H}_\gamma$ by the completness of $\mathcal{H}_\gamma$.
By continuity of ι we have $h^*_n = \iota(h_n) \longrightarrow \iota(h)$, so $\iota(h) = z$.
The separability of $\mathcal{H}_\gamma$ then follows immediately from the fact that $L^2(\mathcal{B},\gamma)$ is separable whenever $\mathcal{B}$ is separable since its Borel $\sigma-$field is countably generated.

□

Remark 2.1. *The space $\mathcal{B}^*_\gamma$ is called the reproducing kernel Hilbert space for γ (or just reproducing kernel for short).*

We now give a second way one can represent Cameron-Martin space.

Proposition 2.3. *If we put $|h|_\gamma = \sup\{\ell(h) : \ell \in \mathcal{B}^*,\ \mathcal{C}_\gamma(\ell,\ell) \le 1\}$ then we have*

$$\|h\|_\gamma = |h|_\gamma \quad \forall h \in \mathcal{H}_\gamma \quad \textit{and}$$

$$\mathcal{H}_\gamma = \{h \in \mathcal{B} : |h|_\gamma < \infty\}.$$

Proof. We first prove the fist equality.
Using Cauchy-Swartz inequality for every $\ell \in \mathcal{B}^*$ and $\mathcal{C}_\gamma(\ell,\ell) \le 1$ we have

$$\ell(h) = \mathcal{C}_\gamma(h^*,\ell) \le \sqrt{\mathcal{C}_\gamma(h^*,h^*)}\sqrt{\mathcal{C}_\gamma(\ell,\ell)} \le \|h\|_\gamma.$$

So $\|h\|_\gamma \ge \sup\{\ell(h) : \ell \in \mathcal{B}^*,\ \mathcal{C}_\gamma(\ell,\ell) \le 1\}$
For the inverse inequality first observe that for $h \in \mathcal{H}_\gamma$ one has $h^*(h) = \mathcal{C}_\gamma(h^*,h^*) = \|h\|^2_\gamma$ where h^* is the corresponding element of h in $\mathcal{B}^*$.
For $h = 0$ it is trivial. Otherwise take $\ell = \frac{1}{\|h\|_\gamma} h^*$ and see that $\ell(h) = \|h\|_\gamma$ and $\mathcal{C}_\gamma(\ell,\ell) = 1$.
So $\|h\|_\gamma \le sup\{\ell(h) : \ell \in \mathcal{B}^*,\ \mathcal{C}_\gamma(\ell,\ell) \le 1\}$. Notice that we proved that the sup is actually max.
We now prove the second equality. The inclusion $\mathcal{H}_\gamma \subset \{h \in \mathcal{B} : |h|_\gamma < \infty\}$ is obvius by the previous equality of the norms. For the other inclusion suppose that there exist $h \in \{h \in \mathcal{B} : |h|_\gamma < \infty\}$ but $h \notin \mathcal{H}_\gamma$. Apply separation theorem for $\mathcal{H}_\gamma$ which is closed and convex as a Hilbert space and the

compact set $\{h\}$ and take $f \in \mathcal{B}^*$ and $c \in \mathbb{R}$ such that $f(x) < c < f(h) \quad \forall x \in \mathcal{H}_\gamma$.
Now define the linear map $T : (\mathcal{B}^*_\gamma, \mathcal{C}_\gamma(\cdot,\cdot)) \to \mathbb{R} \quad \ell \mapsto \ell(h)$.
We have that $|T(\ell)| = |\ell(h)|$ is bounded for all ℓ in the unit ball of $(\mathcal{B}^*_\gamma, \mathcal{C}_\gamma(\cdot,\cdot))$. Indeed, $h \in \{h \in \mathcal{B} : |h|_\gamma < \infty\}$,therefore $\sup\{|\ell(h)| : \ell \in \mathcal{B}^*,\ \mathcal{C}_\gamma(\ell,\ell) \le 1\} = \sup\{\ell(h) : \ell \in \mathcal{B}^*,\ \mathcal{C}_\gamma(\ell,\ell) \le 1\} = |h|_\gamma < \infty$.
From the Riez represention theorem there exists $g \in \mathcal{B}^*_\gamma$ such that $\ell(h) = \mathcal{C}_\gamma(g,\ell) \quad \forall \ell \in \mathcal{B}^*$.
$\mathcal{B}^*$ is dense in $\mathcal{B}^*_\gamma$ and from proposition 2.2 there exists $g_n \in \mathcal{H}_\gamma$ such that $g_n^* \in \mathcal{B}^*$ and $g_n^* \longrightarrow g$.
So $f(h) = \mathcal{C}_\gamma(g,f) = \lim \mathcal{C}_\gamma(g_n^*, f) = \lim f(g_n) \le c < f(h)$ which leads to contradiction.

□

A useful remark to have in mind is the following.
The above way of defining Cameron Martin space is the most popular. More precisely, they start by defining the covariance operator $\mathcal{R}_\gamma$ exactly as $\mathcal{C}_\gamma$ was defined. More precisely, the operator $\mathcal{R}_\gamma : \mathcal{B}^* \to \mathcal{B}^{**}$, defined by the formula

$$\mathcal{R}_\gamma f(g) := \int_{\mathcal{B}} f(x) g(x) \gamma(dx).$$

Then, make a remark that $\mathcal{R}_\gamma$ can also be considered as function $\mathcal{R}_\gamma : \mathcal{B}^*_\gamma \to \mathcal{B}$ as a Bochner integral through the formula

$$\mathcal{R}_\gamma f = \int_{\mathcal{B}} f(x) x \gamma(dx)$$

as exactly in the proof of proposition 2.2. The reason why $\int_{\mathcal{B}} f(x) x \gamma(dx)$ can represent $\mathcal{R}_\gamma f$ is because of the identity

$$\mathcal{R}_\gamma f(g) = g\left(\int_{\mathcal{B}} f(x) x \gamma(dx)\right), \quad \forall g \in \mathcal{B}^*$$

The key point here is that γ is on a parable Banach space and more generally is a Radon Gaussian measure. Generally, this cannot happen in locally convex spaces, so that is the main reason for taking the assumption that the initial space is separable Banach. So, for now on $\mathcal{R}_\gamma$ is just the inverse of ι in proposition 2.2 and $\mathcal{R}_\gamma$ is a function from $\mathcal{B}^*_\gamma$ to $\mathcal{H}_\gamma$.

Theorem 2.1. *Let $\mathcal{B}$ a separable Banach space and γ be a Gaussian measure with the covariance operator having the representation*

$$\mathcal{R}_\gamma = SS^*$$

where $S : H \to \mathcal{B}$ is a continuous linear operator from some Hilbert space H and $S^* : \mathcal{B}^* \to H$ is its adjoint. The following equality holds:

$$\mathcal{H}_\gamma = S(H)$$

Proof. From the equality $\langle S^* f, S^* f\rangle_H = \mathcal{C}_\gamma(f,f)$, the operator S^* extends to a continuous operator from $\mathcal{B}^*_\gamma$ to H, hence $\mathcal{R}_\gamma(\mathcal{B}^*_\gamma) \subset S(H)$ from the representation of $\mathcal{R}_\gamma$. The inverse inclusion follows from the Hahn-Banach theorem, since the set $SS^*(U) = \mathcal{R}_\gamma$, where U is the closed unit ball in $\mathcal{B}^*_\gamma$, is compact in $\mathcal{B}$. If we had a unit vector $v \in H$ with $S(v) \notin \mathcal{R}_\gamma(U)$, then we could find a functional $f \in \mathcal{B}^*$ such that $f(S(H)) > 1$, $f(SS^*(g)) \leq 1$ for all $g \in U$. Then $\langle S^* f, v\rangle_H = f(S(v)) > 1$, whence $\|S^* f\|_H > 1$. On the other hand, $\langle S^* f, S^* g\rangle_H \leq 1$ for all $g \in U$, whence $\|S^* f\|_H \leq 1$, which is a contradiction. □

Observe that we can always find such a factorization of $\mathcal{R}_\gamma$. Indeed, one can put $H = \mathcal{B}^*_\gamma$ and $Sf = \mathcal{R}_\gamma f$. Then S^* is the identical embedding g of $\mathcal{B}^*$ into $\mathcal{B}^*_\gamma$.

Theorem 2.2. *Let H be a Hilbert Space and let γ be a Gaussian measure with covariance operator K_γ. Let $J : H \to H$ be an injective linear mapping such that factorization*

$$K_\gamma = JJ^*$$

holds.
Then the Cameron-Martin space can be expressed as $\mathcal{H}_\gamma = J(H)$, while the scalar product and norm in $\mathcal{H}_\gamma$ admit representations

$$\langle h_1, h_2\rangle_\gamma = \langle J^{-1}h_1, J^{-1}h_2\rangle_H$$
$$\|h\|_\gamma = \|J^{-1}h\|_H.$$

3 Main Results

Proposition 2.4. *If μ and ν are two Gaussian measures on $\mathcal{B}$ such that $\mathcal{H}_\mu = \mathcal{H}_\nu$ and such that $\|h\|_\mu = \|h\|_\nu$ for every $h \in \mathcal{H}_\mu$, then the measures are identical.*

Proof. It is easily derived by Theorem 1.4 and the equality of the norms in the Cameron-Martin spaces implies equality of the quadratic forms in their Fourier transform. □

For this reason, a Gaussian measure on $\mathcal{B}$ is sometimes given by specifying the Hilbert space structure $(\mathcal{H}_\mu, \|\cdot\|_\mu)$.

Lemma 2.1. *If $\tilde{\mathcal{B}} \subset \mathcal{B}$ is a continuously embedded Banach space with full measure, i.e. $\gamma(\tilde{\mathcal{B}}) = 1$ then the embedding $\mathcal{B}^* \hookrightarrow \mathcal{B}^*_\gamma$ extends to an embedding $\tilde{\mathcal{B}}^* \hookrightarrow \mathcal{B}^*_\gamma$. And deduce from this that the restriction of γ to $\tilde{\mathcal{B}}$ is again a Gaussian measure.*

Proposition 2.5. *The law of any element $h^* = \iota(h) \in \mathcal{B}^*_\gamma$ is a centred Gaussian with variance $\|h\|^2_\gamma$. Furthermore, any two elements h^*, k^* have covariance $\langle h, k\rangle_\gamma$.*

Proof. We already know from the definition of a Gaussian measure that the law of any element of $\mathcal{B}^*$ is a centred Gaussian. Let now h^* be any element of $\mathcal{B}^*_\gamma$ and let h_n be a sequence in $\mathcal{B}^*_\gamma \cap \mathcal{B}^*$ such that $h_n \to h^*$ in $\mathcal{B}^*_\gamma$. We can furthermore choose this approximating sequence such that
$\|h_n\|_{L^2} = \|h^*\|_{L^2} = \|h\|_\gamma$, so that the law of each of the h_n is equal to $\mathcal{N}(0, \|h\|^2_\gamma)$.
Since L^2-convergence implies convergence in law, we conclude that the law of h^* is also given by $\mathcal{N}(0, \|h\|^2_\gamma)$. The statement about the covariance then follows by polarization, since

$$\mathbf{E}(h^*k^*) = \frac{1}{2}(\mathbf{E}(h^* + k^*)^2 - \mathbf{E}(h^*)^2 - \mathbf{E}(k^*)^2) = \frac{1}{2}(\|h+k\|^2_\gamma - \|h\|^2_\gamma - \|k\|^2_\gamma) = \langle h, k\rangle_\gamma$$

by the previous statement. □

Proposition 2.6. *Let γ be a centered Gaussian measure on a Banach space $\mathcal{B}$ and let $g \in \mathcal{B}^*_\gamma$. Then the measure ν on $\mathcal{B}$ given by the density*

$$\rho(x) = \exp(g(x) - \frac{1}{2}\sigma(g)^2)$$

with respect to the measure γ is a Gaussian measure with the Fourier transform

$$\tilde{\nu}(f) = \exp\Big(i\langle g, f\rangle_\gamma - \frac{1}{2}\sigma(f)^2\Big). \tag{2.2}$$

Proof. Since g is Gaussian, the function $\exp|g|$ is integrable with respect to the measure γ. Hence the function ρ defines a finite measure ν. Let $f \in \mathcal{B}^*$. Put

$$k = \exp[-\frac{1}{2}\sigma(g)^2].$$

Let us consider the following function of real argument z:

$$\phi(z) = k\int_X \exp[i(f(x) - zg(x))]\gamma(dx).$$

By virtue of the inclusion $f - zg \in \mathcal{B}^*_\gamma$ one has

$$\begin{aligned}\phi(z) &= k\exp\Big(-\frac{1}{2}\int(f-zg)^2 d\gamma\Big)\\ &= k\exp[-\frac{1}{2}\sigma(f)^2 - \frac{1}{2}z^2\sigma(g)^2 + z\langle g, f\rangle_\gamma].\end{aligned}$$

It remains to note that ϕ admits a holomorphic extension to the complex plane. By Lebesgue's theorem, $\phi(z) \to \tilde{v}(f)$ as $z \to i$. On the other hand, the limit on the right-hand side of the foregoing equality coincides with the right-hand side of the equality (2.2). $\square$

Now let X be a separable Hilbert Space. Note that if the measure γ is centered, then the operator K in representation (1.5) is determined by the identity

$$\langle Ku, v\rangle_X = \int_X \langle u, x\rangle_X \cdot \langle v, x\rangle_X \gamma(dx). \tag{2.3}$$

Therefore, the closure of $X = X^*$ in $L^2(X,\gamma)$ coincides with the completion of X with respect to the norm $\|\sqrt{K}x\|_X$. Hence, if $\{e_n\}$ is the orthonormal basis in X formed by the eigenvectors of K, corresponding to the eigenvalues k_n, then this completion can be identified with the weighted Hilbert space of sequences

$$\Big\{(x_n) : \sum_{n=1}^{\infty} k_n x_n^2 < \infty\Big\}.$$

The operator K has the natural extension to the completion X_γ^* and one has $\mathcal{H}_\gamma = K(X_\gamma^*)$. However, if K is considered only on the initial space X, then

$$\mathcal{H}_\gamma = \sqrt{K}(X).$$

For arbitrary a, in the natural coordinates associated with $\{e_n\}$, formula (1.5) reads as follows:

$$\tilde{\gamma}(y) = \exp\Big(i\sum_{n=1}^{\infty} a_n y_n - \frac{1}{2}\sum_{n=1}^{\infty} k_n y_n^2\Big).$$

Let $a = 0$. Then for a cylindrical set $C = P_n^{-1}(B)$, where P_n is the orthogonal projection onto the linear span H_n of the first n vectors e_i (identified with $\mathbb{R}^n$) and $B \in \mathcal{B}(H_n)$, one has

$$\gamma(C) = \int_B \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi k_j}} \exp(-\frac{1}{2k_j}x_j^2)dx_1 \cdots dx_n.$$

Let γ be a centered Gaussian measure on a separable Hilbert space X and K be defined by (2.3). When we identify X^* with X, the operator K coincides with $\mathcal{R}_\gamma$, but in order to avoid confusion we do not denote K by $\mathcal{R}_\gamma$ (in addition, $\mathcal{R}_\gamma$ is defined on the larger space X_γ^*). If X is infinite-dimensional, then the domain of the operator $\sqrt{K}$ has measure zero. However, for every $v \in X$, one can define $\sqrt{K}v$ as the element $v^* \in X_\gamma^*$ corresponding to the vector $\sqrt{K}v \in \mathcal{H}_\gamma$. This notation is consistent, i.e., if $v = \sqrt{K}u$, where $u \in X$, then v^* coincides an element of X_γ^* with the continuous linear functional on X generated by u. Indeed, for any $y \in X$, one has

$$\int_X v^*(x) \cdot \langle y, x \rangle_X \gamma(dx) = \langle \sqrt{K} v, y \rangle_X = \langle Ku, y \rangle_X = \int_X \langle u, x \rangle_X \cdot \langle y, x \rangle_X \gamma(dx).$$

Note also that $\langle v^*, w^* \rangle_{L^2(\gamma)} = \langle \sqrt{K} v, \sqrt{K} w \rangle_{\mathcal{H}_\gamma} = \langle v, w \rangle_X$.

The following is known as *Cameron Martin theorem* and is the central theorem about the Cameron Martin space.

Theorem 2.3. *(Cameron Martin theorem)*
For $h \in \mathcal{B}$, define the shift measure $\gamma_h(A) = \gamma(A - h)$.
Then, the measure γ_h is absolutely continuous with respect to γ if and only if $h \in \mathcal{H}_\gamma$.
In this case the Radon-Nikodym density is given by the expression

$$\frac{d\gamma_h}{d\gamma}(x) = \mathcal{D}_h(x) = \exp(h^*(x) - \frac{1}{2}\|h\|_\gamma^2)$$

where h^* is the corresponding element of h in the reproducing kernel.

Proof. Fix $h \in \mathcal{H}_\gamma$ and let $h^* \in L^2(\mathcal{B}, \gamma)$ be the corresponding element of the reproducing kernel. Since the law of h^* is Gaussian the map $x \mapsto \exp(h^*(x))$ is integrable. Since furthermore the variance of h^* is given by $\|h\|_\gamma^2$, the function

$$\mathcal{D}_h(x) = \exp(h^*(x) - \frac{1}{2}\|h\|_\gamma^2)$$

is strictly positive, belongs to $L^1(\mathcal{B}, \gamma)$, and integrates to 1. It is therefore the Radon-Nikodym derivative of a measure γ_h' that is absolutely continuous with respect to γ. To check that one has indeed $\gamma_h' = \gamma_h$, it suffices to show that their Fourier transforms coincide.
From proposition 2.2 we get

$$\tilde{\gamma_h'}(\ell) = \exp(i\langle \ell, h^* \rangle_\gamma - \frac{1}{2}\sigma^2(\ell)) = \exp(i\mathcal{C}_\gamma(\ell, h^*) - \frac{1}{2}\|h\|_\gamma^2) = \exp(i\ell(h) - \frac{1}{2}\mathcal{C}_\gamma(\ell, \ell))$$

On the other hand, we have

$$\begin{aligned}\tilde{\gamma}_h(\ell) &= \int_{\mathcal{B}} \exp(i\ell(x))\gamma_h(dx) \\ &= \int_{\mathcal{B}} \exp(i\ell(x+h)\gamma(dx) \\ &= e^{i\ell(h)} \int_{\mathcal{B}} \exp(i\ell(x))\gamma(dx) \\ &= \exp(i\ell(h) - \frac{1}{2}\mathcal{C}_\gamma(\ell,\ell)).\end{aligned}$$

Showing that $\gamma'_h = \gamma_h$. To show the converse, note first that one can check by an explicit calculation that $\|\mathcal{N}(0,1) - \mathcal{N}(h,1)\|_{TV} \geq 2 - 2\exp(-\frac{h^2}{8})$. Indeed, we will use the bound from below of the theorem 1.14. Calculating Hellinger's quantity we have:

$$\begin{aligned}H(\mathcal{N}(0,1),\mathcal{N}(h,1)) &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{4}) \exp(-\frac{(x-h)^2}{4})\lambda(dx) \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{4}) \exp(\frac{h}{2}x) \exp(-\frac{h^2}{4})\lambda(dx) \\ &= \exp(-\frac{h^2}{4}) \int_{\mathbb{R}} \exp(\frac{h}{2}x)\gamma_1(dx) \\ &= \exp(-\frac{h^2}{4}) M_Z(\frac{h}{2}) = \exp(-\frac{h^2}{8}).\end{aligned}$$

Fix now some arbitrary $n > 0$. If $h \notin \mathcal{H}_\gamma$, then from 2.3, there exists $\ell \in \mathcal{B}^*$ with $\mathcal{C}_\gamma(\ell,\ell) = 1$ such that $\ell(h) \geq n$. Since the image $\ell_*\gamma$ of γ under ℓ is $\mathcal{N}(0,1)$ and the image $\ell_*\gamma_h$ of γ_h under ℓ is $\mathcal{N}(-\ell(h),1)$, this shows that

$$\|\gamma - \gamma_h\|_{TV} \geq \|\ell_*\gamma - \ell_*\gamma_h\|_{TV} = \|\mathcal{N}(0,1) - \mathcal{N}(-\ell(h),1)\|_{TV} \geq 2 - 2\exp(-\frac{n^2}{8}).$$

Since this is true for every n, we conclude that $\|\gamma - \gamma_h\|_{TV} = 2$, thus showing that they are mutually singular.

□

In summary, we have the following results:
For $h \in \mathcal{B}$ we have

(i) If $h \notin \mathcal{H}_\gamma$ then the measures γ_h and γ are mutually singular.

(ii) If $h \in \mathcal{H}_\gamma$ then the measures γ_h and γ are equivalent.

In particular

$$\mathcal{H}_\gamma = \{h \in \mathcal{B} : \gamma_h \sim \gamma\}$$

As a consequence, we have the following characterization of the Cameron–Martin space

Proposition 2.7. *The Cameron Martin space $\mathcal{H}_\gamma \subset \mathcal{B}$ is the intersection of all (measurable) linear subspaces of full measure. However, if $\mathcal{H}_\gamma$ is infinite-dimensional, then one has $\gamma(\mathcal{H}_\gamma) = 0$.*

Proof. Let V be a subspace of $\mathcal{B}$ with full measure i.e. $\gamma(V) = 1$ and fix $h \in \mathcal{H}_\gamma$. It follows from the theorem 2.3 that the affine space $V - h$ also has full measure. Since $(V - h) \cap V = \emptyset$ unless $h \in V$, one must have $h \in V$, so that $\mathcal{H}_\gamma \subset \bigcap\{V \subset \mathcal{B} : \gamma(V) = 1\}$.

Conversly, take an arbitrary $x \notin \mathcal{H}_\gamma$ and let us construct a linear space $V \subset \mathcal{B}$ of full measure, but not containing x. Since $x \notin \mathcal{H}_\gamma$, one has $\|h\|_\gamma = \infty$ with $\|\cdot\|_\gamma$ extended to $\mathcal{B}$ as in 2.3. Therefore, we can find a sequence $\ell_n \in \mathcal{B}^*$ such that $\mathcal{C}_\gamma(\ell_n, \ell_n) \leq 1$ and $\ell_n(x) \geq n$. Defining the norm $|y|^2 = \sum_n n^{-2}(\ell_n(y))^2$, we see that

$$\int_{\mathcal{B}} |y|^2 \gamma(dy) = \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{\mathcal{B}} (\ell_n(y))^2 \gamma(dy) \leq \frac{\pi^2}{6}$$

so that the linear space $V = \{y : |y| < \infty\}$ has full measure. However, $|x| = \infty$ by construction, so that $x \notin V$.

To show that $\gamma(\mathcal{H}_\gamma) = 0$ if $\dim \mathcal{H}_\gamma = \infty$, consider an orthonormal sequence $e_n \in \mathcal{B}_\gamma^*$ so that the random variables $\{e_n(x)\}$ are i.i.d. $\mathcal{N}(0, 1)$ distributed. By 2.3 and the second Borel-Cantelli lemma, it follows that $\|x\|_\gamma \geq \sup_n |e_n(x)| = \infty$ for $\gamma-$almost every x, so that the claim follows. □

Proposition 2.8. *Let μ and ν be two centered Gaussian measures*

(i) If $\mu \sim \nu$, then $\mathcal{H}_\mu = \mathcal{H}_\nu$ as sets.

(ii) If $\mu \perp \nu$ then $\mu_a \perp \nu_b$ for all $a, b \in \mathcal{B}$.

(iii) If $\mu \sim \nu$ then $\mathcal{B}_\mu^* = \mathcal{B}_\nu^*$

Proof. (i) We will use this alternative definition: $\mathcal{H}_\gamma = \{h \in \mathcal{B} : \gamma_h \sim \gamma\}$. Let $h \in \mathcal{H}_\mu$. We have $\mu_h \sim \mu$ and $\mu \sim \nu$. So $\mu_h \sim \nu$. It can be easily seen that $\mu \sim \nu_{-h}$ and we conclude that $\nu \sim \nu_{-h}$. Finally $-h \in \mathcal{H}_\nu$ which means $h \in \mathcal{H}_\nu$. By symmetry, we have the equality.

(ii) It is sufficient to show the case $b = 0$. If $a \in \mathcal{H}_\mu$ then $\mu_a \sim \mu \perp \nu$ thus $\mu_a \perp \nu$.
If $a \notin \mathcal{H}_\mu$, then, by proposition 2.7 there exists a linear space $L \in \mathcal{E}(\mathcal{B})$ such that $\mu(L) = 1$ and $a \notin L$. Then $\mu_a(L + a) = 1$. On the other hand $\nu_a(L + a) = 0$ according to Corollary 2.4 below.

(iii) Clearly, equivalent measures have equal collections of measurable functions and sequences of measurable functions convergent in measure. If a sequence $\{f_n\}$ from $\mathcal{B}^*$ converges in $L^2(\mathcal{B}, \mu)$, then it converges in measure μ, hence also in measure ν, whence the convergence in $L^2(\mathcal{B},\nu)$ follows by the Gaussian property. Interchanging μ and ν, we get the claim.

□

Proposition 2.9. *For any $p \in (1,\infty)$, $\quad r > p$ and any function $f \in L^r(\mathcal{B},\gamma)$, the mapping*

$$\left(\mathcal{H}_\gamma, \|\cdot\|_\gamma\right) \to L^p(\mathcal{B},\gamma), \quad h \mapsto f(\cdot + h)$$

is continuous. If $p = 1$, then the same is true for all functions $f \in L^\infty(\mathcal{B},\gamma)$.

Proof. We will use a part of the theorem 2.3 as follows. For every cylindrical function f of the form $f(x) = \phi(l_1(x),\dots,l_n(x))$, where $\phi \in C_b(\mathbb{R}^n)$ and $l_i \in X^*$, the mapping above is continuous by Lebesgue's theorem. Let $f \in L^r(\mathcal{B},\gamma)$, where $r > p$. There exists a sequence $\{f_j\}$ of cylindrical functions of the indicated type that converges to f in $L^r(\mathcal{B},\gamma)$. There exist numbers t and s such that $t > 1$, $tp = r$, $t^{-1} + s^{-1} = 1$. Recall that the element $h^* = \iota(h) \in X^*_\gamma$ corresponding to the vector h is a centered Gaussian random variable on $(\mathcal{B},\gamma)$. Hence,

$$\int \exp(sh^*)d\gamma = \exp\left(\frac{s^2}{2}\|h\|_\gamma^2\right).$$

According to Cameron Martin's theorem and Hölder's inequality, we get

$$\begin{aligned}\int |f(x+h) - f_j(x+h)|^p \gamma(dx) &= \int |f(z) - f_j(z)|^p \exp(h^*(z) - \frac{1}{2}\|h\|_\gamma^2)\gamma(dx)\\ &\le \left\{\int |f(z) - f_j(z)|^p \gamma(dx)\right\}^{1/t} \cdot \left\{\int \exp(sh^*(z) - \frac{s}{2}\|h\|_\gamma^2)\gamma(dx)\right\}^{1/s}\\ &\le \left\{\int |f(z) - f_j(z)|^p \gamma(dx)\right\}^{1/t} \cdot \exp(\frac{s-1}{2}\|h\|_\gamma^2)\end{aligned}$$

which tends to zero, as $j \to \infty$, uniformly in $h \in \mathcal{H}_\gamma$, with $\|h\|_\gamma \le R$, for every fixed $R > 0$. Therefore, the mapping $h \mapsto f(\cdot + h)$ is continuous. In the case when $p = 1$ and $f \in L^\infty(\mathcal{B},\gamma)$, the proof is analogous with the only difference that the sequence $\{f_j\}$ should be taken to be convergent to f in measure with $\sup|f_j| \le \|f\|_\infty$, and instead of Hölder's inequality one has to use the Lebesgue-Vitali theorem on uniformly integrable sequences. □

Corollary 2.1. *Let A be a set of positive $\gamma-$measure. Then there exists $c > 0$ such that $cB_\gamma[0,1] \subset A - A$, where $B_\gamma[0,1]$ is the unit closed ball of $\mathcal{H}_\gamma$.*

Proof. The function $h \mapsto \gamma\Big((A+h)\cap A\Big)$ is positive at zero and continuous by the previous proposition applied to the indicator function of A, since

$$\gamma\Big((A+h)\cap A\Big) = \int \chi_A(x-h)\chi_A(x)\gamma(dx)$$

Therefore, there exists $c > 0$ such that $\gamma\Big((A+h)\cap A\Big) > 0$ for all $h \in cB_\gamma[0,1]$. Clearly, $h \in A - A$ for such h. □

Proposition 2.10. *The unit closed ball of $\mathcal{H}_\gamma$, $B_\gamma[0,1]$, is relatively compact in $\mathcal{B}$ and hence the embedding $\mathcal{H}_\gamma \hookrightarrow \mathcal{B}$ is compact.*

Proof. By inequality in proposition 2.1, the ball $B_\gamma[0,1]$ is bounded in $\mathcal{B}$. Let us prove that it is weakly closed. To this aim, consider a net (h_a) in $B_\gamma[0,1]$ weakly converging to $h \in \mathcal{B}$. For every $f \in \mathcal{B}^*$ with $\|f\|_{L^2}$ the inequality $|f(h)| \leq 1$ holds because $f(h) = \lim_a f(h_a)$ and $\|f\|_{\mathcal{B}^*} \leq \|f\|_{L^2}$. Hence $B_\gamma[0,1]$ is weakly closed, then it is closed in $\mathcal{B}$ since it is convex from the Mazur theorem 1.8. To prove that the embedding of $\mathcal{H}_\gamma$ in $\mathcal{B}$ is compact, it is sufficient to prove that $B_\gamma[0,r]$ is compact in $\mathcal{B}$ for some $r > 0$. Fix any compact set $K \subset \mathcal{B}$ with $\gamma(K) > 0$, by corollary 2.1 there exists $r > 0$ such that $B_\gamma[0,r]$ is contained in the compact $K - K$. So $B_\gamma[0,r]$ is compact.

□

4 Zero-one laws

Lemma 2.2. *The collection of the functions of the form $H_{k_1}(l_1) \cdot \dots \cdot H_{k_n}(l_n)$, where the H_{k_i}'s are Hermite polynomials and $l_i \in \mathcal{B}^*$, is dense in $L^2(\mathcal{B},\gamma)$.*

Proof. The functions of the form $\phi(l_1,\dots,l_n)$, where ϕ is a bounded Borel function on $\mathbb{R}^n$, are dense in $L^2(\gamma)$. Hence the claim follows from the corresponding finite dimensional result. □

Theorem 2.4. *Suppose that a set $A \in \mathcal{E}(\mathcal{B})_\gamma$ satisfies the condition*

$$\gamma(A+h) = \gamma(A), \quad \forall h \in \mathcal{R}_\gamma(\mathcal{B}^*).$$

Then either $\gamma(A) = 1$ or $\gamma(A) = 0$. In addition, if f is a $\gamma-$measurable function such that for every $h \in \mathcal{R}_\gamma(\mathcal{B}^*)$ one has

$$f(x+h) = f(x) \quad \gamma - a.e.$$

then f coincides a.e. with a constant.

Proof. Let $h_1 = \mathcal{R}_\gamma(l_1), \ldots, h_n = \mathcal{R}_\gamma(l_n)$, where $l_i \in \mathcal{B}^*$ and the vectors h_i are orthonormal on $\mathcal{H}_\gamma$. By condition and the Cameron-Martin theorem, the function

$$F(t_1, \ldots, t_n) = \gamma(A - t_1 h_1 - \cdots - t_n h_n) = \int_A \exp\Big(\sum_{i=1}^n t_i l_i(x) - \frac{1}{2}\Big\|\sum_{i=1}^n t_i h_i\Big\|_\gamma^2\Big)\gamma(dx)$$

is constant. Therefore, for any collection $m_1, \ldots, m_n$ of nonnegative integers not vanishing simultaneously, we have

$$\frac{\partial^{m_1+\cdots+m_n} F}{\partial t_1^{m_1} \ldots \partial t_n^{m_n}}(0, \ldots, 0) = 0.$$

Since the vectors h_i are orthonormal (Lemma Hermite pol) yields

$$\int_X H_{m_1}(l_1(x)) \cdot \cdots \cdot H_{m_n}(l_n(x))\chi_A(x)\gamma(dx) = 0$$

where H_k is the k-th Hermite polynomial. Thus, the function χ_A is orthogonal to all the polynomials $H_{m_1}(l_1) \ldots H_{m_n}(l_n)$, where m_i are not zero simultaneously and $l_i \in \mathcal{B}^*$ are mutually orthogonal in $\mathcal{B}^*_\gamma$. This means that χ_A is a constant. This constant can be only 0 or 1. Considering the sets $\{f < c\}$, we get the claim for functions.

□

Corollary 2.2. *Let A be a $\gamma-$measurable ste such that*

$$\gamma(A \setminus (A + h)) = 0, \quad \forall h \in \mathcal{R}_\gamma(\mathcal{B}^*).$$

Then either $\gamma(A) = 1$ or $\gamma(A) = 0$.

Proof. Let $h \in \mathcal{R}_\gamma(\mathcal{B}^*)$. Then $-h \in \mathcal{R}_\gamma(\mathcal{B}^*)$. By the hypothesis, we have the equality $\gamma(A \setminus (A - h)) = 0$. Since $\gamma_{-h} \sim \gamma$, we get $\gamma(A + h \setminus A) = 0$, which together with the hypothesis assumption yileds $\gamma(A + h) = \gamma(A)$. □

Note that in both claims the measurability of $A + h$ follows immediately from A, since the measures γ_{-h} and γ are equivalent. The following more general fact can be seen from the proof of the previous theorem.

Corollary 2.3. *Let $\{e_n\}$ an orthonormal basis and a set A measurable with respect to the measure γ has the property that*

$$A + re_n = A \textit{ up to a set of } \gamma\textit{−measure zero}$$

for all rational r and all $n \in \mathbb{N}$. Then either $\gamma(A) = 1$ or $\gamma(A) = 0$.

Proof. The proof is easily seen from the proof of the previous theorem. □

Theorem 2.5. *Let $V \subset \mathcal{B}$ be a measurable linear subspace. Then, one has either $\gamma(V) = 0$ or $\gamma(V) = 1$.*

Proof. Let us first consider the case where $\mathcal{H}_\gamma \not\subset V$. In this case, just as in the proof of 2.7, we conclude that $\gamma(V) = 0$, for otherwise we could construct an uncountable collection of disjoint sets with positive measure.

If $\mathcal{H}_\gamma \subset V$, then we have $V + B_\gamma[0, \epsilon] = V$ for every $\epsilon > 0$, so that if $\gamma(V) > 0$, one must have $\gamma(V) = 1$ by Borell–Sudakov–Cirel'son theorem 2.10.

□

Corollary 2.4. *Let L be a linear subspace in $\mathcal{B}$. Then for any $a \notin L$ one has*

$$\gamma_*(L + a) = 0$$

where γ_* is the inner measure.

Proof. Let $C \subset L + a, \quad C \in \mathcal{E}(\mathcal{B})$. The linear span L_0 of the set $C - a \subset L$ is measurable with respect to γ. So its measure is either 0 or 1. The latter, however, is impossible, since $\gamma(L_0 - a) = \gamma(L_0 + a)$ (due to the symmetry of γ), but $(L_0) \cap (L_0 + a) = \emptyset$. It remains to note that $C \subset L_0 + a$. □

5 Measurable linear functionals

Recall that a function f on a linear space is said to be affine if $f = l + c$, where l is linear and $c \in \mathbb{R}^1$. By analogy one defines affine mappings with values in linear spaces.

Definition 2.2. *Let γ be a Gaussian measure on a Banach space $\mathcal{B}$. A function f on $\mathcal{B}$ is called a measurable linear functional (or, more precisely, $\gamma-$measureable linear functional) on $(\mathcal{B}, \gamma)$ if there exists a linear subspace L of full $\gamma-$measure and a $\gamma-$measurable and linear, in the usual sense, function f_0 on L such that $f = f_0$ $\gamma-$a.e. . The notion of a $\gamma-$measurable affine function is defined by analogy.*

Note that one can redefine a $\gamma-$measurable linear functional $f : \mathcal{B} \to \mathbb{R}^1$ in such a way that f_0 will be linear on all of $\mathcal{B}$ (such a version will be called *proper linear*). Indeed, using any Hamel basis in $\mathcal{B}$, we can extend f_0 to a linear function on $\mathcal{B}$. Clearly, all such extensions are $\gamma-$equivalent functions.

Proposition 2.11. *Let f be a $\gamma-$measurable properly linear functional on $\mathcal{B}$. Then the restriction of f to the space $\mathcal{H}_\gamma$ with the norm $\|\cdot\|_\gamma$ is continuous.*

Proof. The set $V_n = \{x : f(x) \leq n\}$ has positive measure for some $n \in \mathbb{N}$. According to 2.1, there exists $r > 0$ such that $rB_\gamma[0,1] \subset V_n - V_n$. Hence f is bounded on the unit ball $B_\gamma[0,1]$ which is equivalent to the continuity of f. □

Theorem 2.6. *Let f and g two $\gamma-$measurable linear functionals. Then they are either distinct almost everywhere or equal almost everywhere. If they are proper linear, then the second of the two possibilities above takes place precisely when $f = g$ on $\mathcal{H}_\gamma$.*

Proof. We assume that f and g are proper linear. Then $L := \{f = g\}$, is a measurable linear space. According to the zero-one law 2.5, either $\gamma(L) = 0$ or $\gamma(L) = 1$. In the latter case $\mathcal{H}_\gamma \subset L$ because $\mathcal{H}_\gamma$ is the intersection of all measurable linear subspaces of full measure. □

Corollary 2.5. *Suppose that f is a $\gamma-$measurable linear functional (or a $\gamma-$measurable proper linear functional) such that it vanishes on some dense subset of the space $\mathcal{H}_\gamma$ (equipped with the norm $\|\cdot\|_\gamma$). Then $f = 0$ $\gamma-$almost everywhere.*

6 Images of Gaussian measures

It follows immediately from the definition of a Gaussian measure and the expression for its Fourier transform that if γ is a Gaussian measure on some Banach space $\mathcal{B}$ and $A : \mathcal{B} \to \hat{\mathcal{B}}$ is a bounded linear map where $\hat{\mathcal{B}}$ is an other Banach space, then $\nu = (A^t)_*\gamma$ is a Gaussian measure on $\hat{\mathcal{B}}$ with covariance

$$\mathcal{C}_\nu(\ell, \ell') = \mathcal{C}_\gamma(A^t\ell, A^t\ell')$$

where $A^t : \hat{\mathcal{B}}^* \to \mathcal{B}^*$ is the transpose operator of A, that is the operator such that $(A^t\ell)(x) = \ell(Ax)$ for every $x \in \mathcal{B}$ and every $\ell \in \hat{\mathcal{B}}^*$.

Lemma 2.3. *For every $\ell \in \mathcal{B}^*_\gamma$ there exists a measurable linear subspace V_ℓ of $\mathcal{B}$ such that $\gamma(V_\ell) = 1$ and a linear map $\hat{\ell} : V_\ell \to \mathbb{R}$ such that $\ell = \hat{\ell}$ in V_ℓ.*

Proof. Fix $\ell \in \mathcal{B}^*_\gamma$. By definition of $\mathcal{B}^*_\gamma$ and Borel-Cantelli, we can find a sequence $\ell_n \in \mathcal{B}^*$ such that $\lim_n \ell_n(x) = \ell(x)$ for $\gamma-$almost every $x \in \mathcal{B}$. (Take for example ℓ_n such that $\|\ell_n - \ell\|_\gamma^2 \leq \frac{1}{n^2}$.) It then suffices to define

$$V_\ell = \{x \in \mathcal{B} : \lim_n \ell_n(x) \text{ exists } \},$$

and to set $\hat{\ell}(x) = \lim_n \ell_n(x)$ on V_ℓ. □

Recall now that $\mathcal{H}_\gamma$ is the intersection over all linear subspaces of $\mathcal{B}$ that have full measure under γ. This suggests that in order to determine the image of γ under a linear map, it is sufficient to know how that map acts on elements of $\mathcal{H}_\gamma$. This intuition is made precise by the following theorem:

Theorem 2.7. *Let γ be a centred Gaussian probability measure on a separable Banach space $\mathcal{B}$. Let furthermore $\mathcal{H}$ be a separable Hilbert space and let $A : \mathcal{H}_\gamma \to \mathcal{H}$ be a Hilber Schmidt operator. (That is $AA^* : \mathcal{H} \to \mathcal{H}$ is trace class.) Then, there exists a measurable map $\hat{A} : \mathcal{B} \to \mathcal{H}$ such that $\nu = \hat{A}_* \gamma$ is Gaussian with covariance $\mathcal{C}_\nu(h,k) = \langle A^* h, A^* k \rangle_\gamma$. Furthermore, there exists a measurable linear subspace $V \subset \mathcal{B}$ of full μ-measure such that $\hat{A}$ restricted to V is linear and $\hat{A}$ restricted to $\mathcal{H}_\gamma \subset V$ agrees with A.*

Proof. Let $\{e_n\}$ be an orthonormal basis for $\mathcal{H}_\gamma$ and denote by e_n^* the corresponding elements in $\mathcal{B}_\gamma^* \subset L^2(\mathcal{B}, \gamma)$ and define $S_N(x) = \sum_{n=0}^N e_n^*(x) A e_n$. Recall from lemma 2.3 that we can find spaces V_{e_n} of full measure such that e_n^* is linear on V_{e_n}. Define now a linear subspace $V \subset \mathcal{B}$ by

$$V = \left\{ x \in \bigcap V_{e_n} : \text{the sequence } \{S_N(x)\} \text{ converges in } \mathcal{H} \right\}$$

(the fact that V is linear follows from the linearity of each of the e_n^*) and set

$$\hat{A}(x) = \begin{cases} \lim_N S_N(x) & \text{for } x \in V \\ 0 & \text{otherwise} \end{cases}$$

Since the random variables $\{e_n^*\}$ are i.i.d. $\mathcal{N}(0,1)-$distributed under γ, the sequence $\{S_N\}$ forms an $\mathcal{H}-$valued martingale and one has

$$\sup_N \mathbf{E}_\gamma \|S_N(x)\|^2 = \sum_{n=0}^{\infty} \|Ae_n\|^2 \leq tr(A^* A) < \infty$$

where the last inequality is a consequence of A being Hilbert-Schmidt. It follows that $\gamma(V) = 1$ by Doob's martingale convergence theorem 1.9.

To see that $\nu = \hat{A}_* \gamma$ has the stated property, fix an arbitrary $h \in \mathcal{H}$ and note that the series $\sum e_n^* \langle Ae_n, h \rangle$ converges in $\mathcal{B}_\gamma^*$ to an element with covariance $\|A^* h\|^2$. The statement then follows from 2.3 and the fact that $\mathcal{C}_\nu(h,h)$ determines $\mathcal{C}_\nu$ by polarisation. To check that ν is Gaussian, we can compute its Fourier transform in a similar way.

□

The proof of the previous theorem can be extended to the case where the image space is a Banach space rather than a Hilbert space. However, in this case, we cannot give a straightforward characterization of those maps A that are 'admissible', since we have no good complete characterization of covariance operators for Gaussian measures on Banach spaces. However, we can take the pragmatic approach and simply assume that the new covariance determines a Gaussian measure on the target Banach space. With this approach, we can formulate the following version for Banach spaces:

Theorem 2.8. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two separable Banach spaces and let γ be a centred Gaussian probability measure on $\mathcal{B}_1$. Let $A : \mathcal{H}_\gamma \to \mathcal{B}_2$ be a bounded linear operator such that there exists a centered Gaussian measure ν on $\mathcal{B}_2$ with covariance $\mathcal{C}_\nu(h,k) = \langle A^*h, A^*k\rangle_\gamma$. Then, there exists a measurable map $\hat{A} : \mathcal{B}_1 \to \mathcal{B}_2$ such that $\nu = \hat{A}_*\gamma$ and such that there exists a measurable linear subspace $V \subset \mathcal{B}$ of full $\gamma-$measure such that $\hat{A}$ restricted to V is linear and $\hat{A}$ restricted to $\mathcal{H}_\gamma \subset V$ agrees with A.*

Proof. As a first step, we construct a Hilbert space $\mathcal{H}_2$ such that $\mathcal{B}_2 \subset \mathcal{H}_2$ as a Borel subset. Denote by $\mathcal{H}_\nu \subset \mathcal{B}_2$ the Cameron–Martin space of ν and let $\{e_n\} \subset \mathcal{H}_\nu$ be an orthonormal basis of elements such that $e_n^* \in \mathcal{B}_2^*$ for every n. (Such an orthonormal basis can always be found by using the Grahm-Schmidt procedure.) We then define a norm on $\mathcal{B}_2$ by

$$\|x\|_2^2 = \sum_{n\geq 1} \frac{e_n^*(x)^2}{n^2\|e_n^*\|^2}$$

where $\|e_n^*\|$ is the norm of e_n^* in $\mathcal{B}_2^*$. It is immediate that $\|x\|_2 < \infty$ for every $x \in \mathcal{B}_2$, so that this turns $\mathcal{B}_2$ into a pre-Hilbert space. We finally define $\mathcal{H}_2$ as the completion of $\mathcal{B}_2$ under $\|\cdot\|_2$.

Denote by ν' the image of the measure ν under the inclusion map $\iota : \mathcal{B}_2 \to \mathcal{H}_2$. It follows that the map $A' = \iota \circ A$ satisfies the assumptions of the previous theorem so that there exists a map $\hat{A} : \mathcal{B}_1 \to \mathcal{H}_2$ which is linear on a subset of full $\gamma-$measure and such that $\nu' = \hat{A}_*\gamma$. On the other hand, we know by construction that $\nu'(\mathcal{B}_2) = 1$, so that the set $\{x : \hat{A}x \in \mathcal{B}_2)$ is of full measure. Modifying $\hat{A}$ outside of this set by for example setting it to 0 and by 2.1 yields the required statement. □

In fact we will show in the next theorem that the extension $\hat{A}$ above is unique up to sets of $\gamma-$measure 0.

Theorem 2.9. *Let γ be a Gaussian measure on a separable Banach space $\mathcal{B}_1$ with Cameron–Martin space $\mathcal{H}_\gamma$ and let $A : \mathcal{H}_\gamma \to \mathcal{B}_2$ be a linear map satisfying the assumptions of the previous theorem. Then the linear measurable extension $\hat{A}$ of A is unique, up to sets of measure 0.*

As a consequence of the inverse, the precise Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$ are completely irrelevant when one considers the image of a Gaussian measure under a linear transformation. The only thing that matters is the Cameron–Martin space for the starting measure and the way in which the linear transformation acts on this space.

This is probably one of the most remarkable results in Gaussian measure theory. At first sight, it appears completely counterintuitive: the Cameron–Martin space $\mathcal{H}_\gamma$ has measure 0, so how can the specification of a measurable map on a set of measure 0 be sufficient to determine it on a set of measure 1? Part of the answer lies of course in the requirement that the extension $\hat{A}$ should be linear on a set of full measure. However, even this requirement would not be sufficient by itself to determine $\hat{A}$ since the Hahn–Banach theorem provides a huge number of different extensions of A that do not coincide anywhere except on $\mathcal{H}_\gamma$. The missing ingredient that solves this mystery is the requirement that $\hat{A}$ be not just any linear map, but a measurable linear map. This additional constraint rules out all of the non-constructive extensions of A provided by the Hahn–Banach theorem and leaves only one (constructive) extension of A. The main ingredient in the proof of the uniqueness is the Borell–Sudakov–Cirel'son inequality, a general form of isoperimetric inequality for Gaussian measures which is very interesting and useful in its own right. In order to state this result, we first introduce the notation $B_\gamma[0,\epsilon]$ for the $\mathcal{H}_\gamma$-ball of radius ϵ centered at the origin. We also denote by $A+B$ the sum of two sets defined by

$$A + B = \{a + b : a \in A, \quad b \in B\}$$

and we denote by Φ the cumulative distribution function of the normal Gaussian. With these notations at hand, we have the following:

Theorem 2.10. *(Borell–Sudakov–Cirel'son). Let γ be a Gaussian measure on a separable Banach space $\mathcal{B}$ with Cameron–Martin space $\mathcal{H}_\gamma$ and let $A \subset B$ be a measurable subset with measure $\gamma(A) = \Phi(a)$ for some $a \in \mathbb{R}$. Then, for every $\epsilon > 0$, one has the bound $\gamma(A + B_\gamma[0,\epsilon]) \geq \Phi(a+\epsilon)$.*

This theorem is remarkable since it implies that even though $\mathcal{H}_\gamma$ itself has measure 0, whenever A is a set of positive measure, no matter how small, the set $A + \mathcal{H}_\gamma$ has full measure!

Corollary 2.6. *Let $V \subset \mathcal{B}$ be a measurable linear subspace. Then, one has either $\gamma(V) = 0$ or $\gamma(V) = 1$.*

Proof. Let us first consider the case where $\mathcal{H}_\gamma \not\subset V$. In this case, just as in the proof of 2.7, we conclude that $\gamma(V) = 0$, for otherwise we could construct an uncountable collection of disjoint sets with positive measure.

If $\mathcal{H}_\gamma \subset V$, then we have $V + B_\gamma[0,\epsilon] = V$ for every $\epsilon > 0$, so that if $\gamma(V) > 0$, one must have $\gamma(V) = 1$ by Borell–Sudakov–Cirel'son theorem.

□

We have now all the necessary ingredients in place to be able to give a proof of theorme 2.9

Proof. (of theorem 2.9). Assume by contradiction that there exist two measurable extensions $\hat{A}_1$ and $\hat{A}_2$ of A. In other words, we have $\hat{A}_i x = Ax$ for $x \in \mathcal{H}_\gamma$ and there exist measurable subspaces V_i with full measure, $\gamma(V_i) = 1$, such that the restriction of $\hat{A}_i$ to V_i is linear. Denote $V = V_1 \cap V_2$ and $\Delta = \hat{A}_2 - \hat{A}_1$, so that Δ is linear on V and $\Delta|\mathcal{H}_\gamma$ (the restriction of Δ in $\mathcal{H}_\gamma$ is 0).

Let $\ell \in \mathcal{B}_2^*$ be arbitrary and consider the events $V_\ell^c = \{x : \ell(\Delta x) \leq c\}$. By the linearity of Δ, each of these events is invariant under translations in $\mathcal{H}_\gamma$, so that by Borell–Sudakov–Cirel'son theorem we have $\gamma(V_\ell^c) \in \{0,1\}$ for every choice of ℓ and c. Furthermore, for fixed ℓ, the map $c \mapsto \gamma(V_\ell^c)$ is increasing and it follows from the $\sigma-$additivity of γ that we have $\lim_{c\to-\infty} V_\ell^c = 0$ and $\lim_{c\to\infty} V_\ell^c = 1$. Therefore, there exists a unique $c_\ell \in \mathbb{R}$ such that $\gamma(V_\ell^c)$ jumps from 0 to 1 at $c = c_\ell$. In particular, this implies that $\ell(\Delta x) = c_\ell \quad \gamma$-almost surely. However, the measure γ is invariant under the map $x \mapsto -x$, so that we must have $c_\ell = -c_\ell$, implying that $c_\ell = 0$. Since this is true for every $\ell \in \mathcal{B}_2^*$, we conclude that the law of Δx is given by the Dirac measure at 0, so that $\Delta x = 0$ $\gamma-$almost surely, which is precisely what we wanted.

□

7 Examples

Example 2.1. *Let $\mathcal{B} = \mathbb{R}^n$ and $\gamma = \mathcal{N}(0,\Sigma)$. We assume that Σ is invertible. Then from theorem 2.2 we have that $J = \sqrt{\Sigma}$ is injective and the factorisation $\Sigma = JJ^*$ holds. So the Cameron-Martin space is*

$$\mathbb{R}_\gamma = \sqrt{\Sigma}(\mathbb{R}^n) = \mathbb{R}^n,$$

$$\langle h_1, h_2\rangle_\gamma = \langle \sqrt{\Sigma}^{-1} h_1, \sqrt{\Sigma}^{-1} h_2\rangle_{\mathbb{R}^n}.$$

Example 2.2. *Let γ be a centered Gaussian measure on a Hilbert space H with covariance K. We have*

$$\begin{aligned}
\mathring{\mathcal{H}}_\gamma &= \{h \in H : \exists h^* \in H^* \textit{ with } \int h^*(x)\ell(x)\gamma(dx) = \ell(h) \quad \forall \ell \in H^*\} \\
&= \{h \in H : \exists h^* \in H \textit{ with } \int \langle x, h^*\rangle\langle x, \ell\rangle \gamma(dx) = \langle h, \ell\rangle \quad \forall \ell \in H\} \\
&= \{h \in H : \exists h^* \in H \textit{ with } \langle Kh^*, \ell\rangle = \langle h, \ell\rangle \quad \forall \ell \in H\} \\
&= \{h \in H : \exists h^* \in H \textit{ with } Kh^* = h\} = K(H).
\end{aligned}$$

So the correspondence $h \to h^*$ is given by $h^* = K^{-1}h$. Now consider the spectral decomposition of K: $Ke_n = \lambda_n e_n$ with $\sum_n \lambda_n < \infty$ and $\{e_n\}$ an orthonormal basis of eigenvectors. Such a decomposition exists since we already know that K is trace class operator. Assume that $\lambda_n > 0$ (i.e. K is invertible).
We have

$$\mathcal{H}_\gamma = \{h \in H : \sum_n \lambda_n^{-1} \langle h, e_n\rangle^2 < \infty\}$$

and that $\langle h, k\rangle_\gamma = \langle K^{-1/2}h, K^{-1/2}k\rangle_H$.

Example 2.3. *Let γ_n be the standard Gaussian measure on $\mathbb{R}^1$. then the measure*

$$\gamma = \bigotimes_{n=1}^{\infty} \gamma_n$$

is centered Gaussian on $\mathbb{R}^\infty$. In addition, $X_\gamma^* \cong l^2$, $\mathcal{H}_\gamma = l^2$.

Proof. Note that the measure γ is defined on the $\sigma-$field generated by the coordinate functions $x = (x_n) \mapsto x_n$, which coincides with the Borel $\sigma-$field of the space $\mathbb{R}^\infty$. It is known that every continuous linear functional f on $\mathbb{R}^\infty$ is a finite linear combination of the coordinate functions. Therefore, for some n, the induced measure $\gamma \circ f^{-1}$ coincides with the measure $(\bigotimes_{i=1}^n \gamma_i) \circ F_n^{-1}$, where $F_n(x_1, \dots, x_n) = f(x_1, \dots, x_n, 0, 0, \dots)$. For any elements $f = (f_n)$ and $g = (g_n)$ form $(\mathbb{R}^\infty)^*$ one has

$$\mathcal{C}_\gamma(f,g) = \int \Big(\sum_{n=1}^{\infty} f_n x_n\Big)\Big(\sum_{n=1}^{\infty} g_n x_n\Big)\gamma(dx) = \sum_{n=1}^{\infty} f_n g_n.$$

This means that the closure of $(\mathbb{R}^\infty)^*$ in $L^2(\gamma)$ can be identified with l^2, since it is the set of all functions of the form $\sum_{n=1}^{\infty} c_n x_n$, where $(c_n) \in l^2$ and the series converges in $L^2(\gamma)$. It is clear that $\mathcal{H}_\gamma = l^2$ □

Example 2.4. *Let γ be the countable product of centered Gaussian measures γ_n on $\mathbb{R}^1$ with variables $\sigma_n^2 > 0$. Then γ is a centered Gaussian measure on $\mathbb{R}^\infty$. In addition, γ is a centered Gaussian measure on every weighted Hilbert space*

$$E_a = \Big\{x \in \mathbb{R}^\infty : \|x\|_a^2 = \sum_{n=1}^{\infty} a_n^2 x_n^2 < \infty\Big\},$$

where $\sum_{n=1}^{\infty} a_n^2 \sigma_n^2 < \infty$

Proof. It suffices to note that $\gamma(E_a) = 1$, which follows from convergence of the series

$$\sum_{n=1}^{\infty} \int a_n^2 x_n^2 \gamma(dx)$$

and the monotone convergence theorem. In addition, the coordinate functions generate $\mathcal{E}(E_a) = \mathcal{B}(E_a)$. □

Example 2.5. *Let γ_a, $a \in A$, be a family of Gaussian measures on locally convex spaces X_a. Then the product-measure $\gamma = \bigotimes_{a \in A} \gamma_a$ is a Gaussian measure on $X = \prod_a X_a$. The Cameron-Martin space of γ is the Hilbert direct sum of the spaces $\mathcal{H}_{\gamma_a}$, i.e.,*

$$\mathcal{H}_\gamma = \Big(h = (h_a) \in X : h_a \in \mathcal{H}_{\gamma_a}, \|h\|_{\mathcal{H}_\gamma}^2 = \sum_a \|h_a\|_{\mathcal{H}_{\gamma_a}}^2 < \infty\Big)$$

, the space X_γ^* is the collection of all functions of the form

$$x \mapsto \sum_{n=1}^{\infty} f_{a_n}(x_{a_n}), \quad a_n \in A, \, f_{a_n} \in X^2_{\gamma_{a_n}}, \sum_{n=1}^{\infty} \sigma(f_{a_n})^2 < \infty.$$

Proof. The first claim follows from the fact that the product of finitely many Gaussian measures is also Gaussian and the fact that the dual of X is the direct sum of the spaces $(X_a)^*$, i.e. the general form of a continuous linear functional on X is $f(x) = \sum_a f_a(x_a)$, where $x = (x_a)$ and finitely many $f_a \in (X_a)^*$ are nonzero. By the definition of a product measure-measure, this yields the announced description of X_γ^*. Further, for any f of the foregoing form, one has $\mathcal{R}_\gamma(f) = (\mathcal{R}_{\gamma_a}(f_a))$, whence the claim about $\mathcal{H}_\gamma$. □

Example 2.6. *Let γ be a Gaussian measure on a locally convex space X and let $T = X^*$. Define an embedding $X \to \mathbb{R}^T$ by the formula $x \mapsto x(\cdot)$, $x(t) := t(x), \quad t \in T$. Then the image of the measure γ under the embedding is a Gaussian measure γ' on $\mathbb{R}^T$ and*

$$\gamma'\Big(y \in \mathbb{R}^T : (y(t_1), \dots, y(t_n), \dots) \in B\Big) = \gamma\Big(x \in X : (t_1(x), \dots, t_n(x), \dots) \in B\Big), \quad t_i \in X^*, \quad B \in \mathcal{B}(\mathbb{R}^\infty).$$

Proof. Every continuous linear functional on $\mathbb{R}^T$ has the following form: $x \mapsto c_1 x(t_1) + \cdots + c_n x(t_n), \quad c_i \in \mathbb{R}^1, \quad t_i \in T$. The composition of such a functional with the embedding $X \to \mathbb{R}^T$ is continuous on X. Hence this composition has Gaussian distribution with respect to the measure γ'. The last claim is true by construction. □

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