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Morley's Categoricity Theorem

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ABSTRACT

Morley's categoricity theorem stands as a cornerstone in model theory, with many experts considering it the beginning of modern model theory. A complete theory T in a countable language is κ-categorical if it has a unique (up to isomorphism) model of cardinality κ. Morley, with his PhD thesis "Categoricity in Power", published in 1962, positively answered the conjecture of Łoś stating that if T is κ-categorical for some uncountable κ, then it is κ-categorical for any uncountable κ. This theorem is now known as the categoricity theorem. The ideas used to prove it now play a central role in model theory and still shape the direction of the field. We will follow a recent proof given by Lachlan and Baldwin, which presents many ideas and definitions that are still at the forefront of research, the way it is presented in " Model Theory: An Introduction" by David Marker.

SUBJECT AREA: Model Theory

KEYWORDS: Uncountably Categorical Theory, Algebraic Closure, Type

ΠΕΡΙΛΗΨΗ

Το θεώρημα κατηγορικότητας του Morley αποτελεί ακρογωνιαίο λίθο στη θεωρία μοντέλων, με πολλούς ειδικούς να το θεωρούν την αρχή της σύγχρονης θεωρίας μοντέλων. Μια πλήρης θεωρία T σε μια αριθμήσιμη γλώσσα είναι κ-κατηγορική εάν έχει ένα μοναδικό (προς ισομορφισμό) μοντέλο πληθικότητας κ. Ο Morley, με τη διδακτορική του διατριβή ”Categoricity in Power”, που δημοσιεύθηκε το 1962, απάντησε θετικά στην εικασία του Łoś η οποία δήλωνε ότι αν T είναι κ-κατηγορική για κάποιο μη αριθμήσιμο πληθάριθμο κ, τότε είναι κ-κατηγορική για οποιαδήποτε μη αριθμήσιμο πληθάριθμο κ. Αυτό το θεώρημα είναι πλέον γνωστό ως το θεώρημα κατηγορικότητας. Οι ιδέες που χρησιμοποιήθηκαν για να το αποδείξουν παίζουν τώρα κεντρικό ρόλο στη θεωρία μοντέλων και εξακολουθούν να καθορίζουν τον τομέα. Θα ακολουθήσουμε μια μεταγενέστερη απόδειξη που δόθηκε από τους Lachlan και Baldwin, η οποία παρουσιάζει πολλές ιδέες και ορισμούς που εξακολουθούν να είναι στην αιχμή της έρευνας, όπως παρουσιάζεται στο βιβλίο ”Model Theory: An Introduction” του David Marker.

ΘΕΜΑΤΙΚΗ ΠΕΡΙΟΧΗ: Θεωρία Μοντέων

ΛΕΞΕΙΣ ΚΛΕΙΔΙΑ: Υπεραριθμήσιμη Κατηγορική Θεωρία, Αλγεβρική Κλειστότητα, Πολυτύπος

CONTENTS

1 Introduction 15

1.1 Language, Structures, Truth . . . 15

1.3 Theories . . . 17

1.4 Embeddings . . . 18

1.5 Parameters . . . 20

1.6 Types . . . 21

1.7 Prime and Homogeneous Models, Stable Theories . . . 24

2 Vaughtian Pairs 29

2.1 Vaught's Two Cardinal Theorem . . . 29

2.2 Omitting Types on ω-stable Theories . . . 33

2.3 Sequences of Indiscernibles and Skolem Hull . . . 34

3 Strongly Minimal Sets 39

3.1 Finding a Strongly Minimal Formula . . . 39

3.2 Algebraic Closure . . . 41

3.3 Extending Partial Isomorphism of Strongly Minimal Sets . . . 44

REFERENCES 47

1. INTRODUCTION

In order to state what Morley's theorem is about, we need to establish the basic notation and tools we will use. The following is a short introduction to first-order logic, covering the basics usually taught in an undergraduate course before we can jump into more advanced terminology at the end of this chapter.

1.1 Language, Structures, Truth

Definition 1.1.1. A first-order language $\mathcal{L}$ is defined as the collection of:

- a set of function symbols $\mathcal{F}$ and positive integers n_f for each $f \in \mathcal{F}$;
- a set of relation symbols $\mathcal{R}$ and positive integers n_R for each $R \in \mathcal{R}$;
- a set of constant symbols $\mathcal{C}$. The following sets are common for all first-order languages.
- a set of the logical symbols $=, \rightarrow, \wedge, \vee, \neg, \forall, \exists$.
- a set of variables $\mathcal{V} = \{v_1, \ldots, v_n \ldots\}$

Notes:

1. The natural numbers n_f and n_R denote the arity of the functional or relational symbol.
2. Instead of using the variables names in $\mathcal{V}$, we use sometimes use x, y, z for convenience.

Definition 1.1.2. An *$\mathcal{L}$-structure* $\mathcal{M}$ defined as the collection of:

i) a nonempty set M called the universe of $\mathcal{M}$;

plus the interpretation of the non-logical symbols of $\mathcal{L}$ over $\mathcal{M}$:

ii) a function $f^{\mathcal{M}} : M^{n_f} \rightarrow M$ *for each* $f \in \mathcal{F}$;

iii) a set $R^{\mathcal{M}} \subseteq M^{n_R}$ *for each* $R \in \mathcal{R}$;

iv) an element $c^{\mathcal{M}} \in M$ *for each* $c \in \mathcal{C}$.

We refer to $f^{\mathcal{M}}, R^{\mathcal{M}},$ and $c^{\mathcal{M}}$ as the interpretations of the symbols $f, R,$ and c in $\mathcal{M}$, respectively. The interpretation of logical symbols $=, \rightarrow, \wedge, \vee, \neg, \forall, \exists$ does not vary in different structures, in fact they have the same meaning we are all used to across all structures described in a first-order language. Lastly the interpretation of variables is a function from

$\mathcal{V} \to M$ but it is not fixed for a structure and we will always declare how we interpret a set of variables.

We use combinations of the symbols of $\mathcal{L}$ to form terms. Their interpretation in any structure can be informally described as names for some elements of M, in the case when no variable is used in the combination, or when variables are used, as names of functions from $M^m \to M$ for $m \in \mathbb{N}$.

Definition 1.1.3. The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that

i) $c \in \mathcal{T}$ for each constant symbol $c \in \mathcal{C}$,

ii) each variable symbol $v_i \in \mathcal{T}$ for $i = 1, 2, \ldots$,

iii) if $t_1, \ldots, t_{n_f} \in \mathcal{T}$ and $f \in \mathcal{F}$, then $f(t_1, \ldots, t_{n_f}) \in \mathcal{T}$.

Example 1.2. Let $\mathcal{L} = \{+, \cdot, 0, 1\}$ be the language of rings. In the structure of reals, one can think of the term $1 + 1$ as a name for the element 2, while $v_1 + v_1 + v_1$ is a name for the function $x \mapsto 3x.$

Definition 1.2.1. We will now define a term given an interpretation of its variables, like giving input to a function. Let t be a term, we will denote $(v_{i_1}, \ldots, v_{i_m})$the variables used in t (note this might be empty), and $\overline{a} = (a_1, \ldots, a_m) \in M$ how we will interpret those variables. Let s be a sub-term of t, we inductively define $s^{\mathcal{M}}(\overline{a})$ as follows:

(i) If s is a constant symbol c, then $s^{\mathcal{M}}(\overline{a}) = c^{\mathcal{M}}$.

(ii) If s is the variable v_{i_j}, then $s^{\mathcal{M}}(\overline{a}) = a_j$.

(iii) If s is the term $f(t_1, \ldots, t_{n_f})$, where f is a function symbol of $\mathcal{L}$ and $t_1, \ldots, t_{n_f}$ are terms, then

$$s^{\mathcal{M}}(\overline{a}) = f^{\mathcal{M}}(t_1^{\mathcal{M}}(\overline{a}), \ldots, t_{n_f}^{\mathcal{M}}(\overline{a})).$$

Terms are a stepping stone to defining formulas and the notion of truth in a structure.

Definition 1.2.2. We say that ϕ is an atomic $\mathcal{L}$-formula if ϕ is either

(i) $t_1 = t_2$, where t_1 and t_2 are terms, or

(ii) $R(t_1, \ldots, t_{n_R})$, where $R \in \mathcal{R}$ and $t_1, \ldots, t_{n_R}$ are terms.

The set of $\mathcal{L}$-formulas is the smallest set $\mathcal{W}$ containing the atomic formulas such that

(i) if $\phi \in \mathcal{W}$, then $\neg\phi \in \mathcal{W}$,

(ii) if ϕ and $\psi \in \mathcal{W}$, then $(\phi \wedge \psi) \in \mathcal{W}$ and $(\phi \vee \psi) \in \mathcal{W}$, and

(iii) if $\phi \in \mathcal{W}$, then $(\exists v_i)(\phi)$ and $(\forall v_i)(\phi)$ are in $\mathcal{W}$.

We define the scope of a quantifier $(\forall v_i)(\phi)$ as the all the variables v_i found in ϕ. A variable v_i that appears in a formula ψ is bound if it is in the scope of a quantifier; otherwise, it is considered free. A sentence is a formula that has only bounded variables.

Definition 1.2.3. Let ϕ be a formula with free variables from $\overline{v} = (v_{i_1}, \ldots, v_{i_m})$, and let $\bar{a} = (a_{i_1}, \ldots, a_{i_m}) \in M^m$. We inductively define $\mathcal{M} \models \phi(\overline{a})$ as follows:

(i) If ϕ is $t_1 = t_2$, then $\mathcal{M} \models \phi(\bar{a})$ if $t_1^{\mathcal{M}}(\bar{a}) = t_2^{\mathcal{M}}(\bar{a})$.

(ii) If ϕ is $R(t_1, \ldots, t_{n_R})$, then $\mathcal{M} \models \phi(\bar{a})$ if $(t_1^{\mathcal{M}}(\bar{a}), \ldots, t_{n_R}^{\mathcal{M}}(\bar{a})) \in R^{\mathcal{M}}$.

(iii) If ϕ is $\neg\psi$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \not\models \psi(\overline{a})$.

(iv) If ϕ is $(\psi \wedge \theta)$, then $\mathcal{M} \models \phi(\bar{a})$ if $\mathcal{M} \models \psi(\bar{a})$ and $\mathcal{M} \models \theta(\bar{a})$.

(v) If ϕ is $(\psi \vee \theta)$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a})$ or $\mathcal{M} \models \theta(\overline{a})$.

(vi) If ϕ is $\exists v_j \psi(\bar{v}, v_j)$, then $\mathcal{M} \models \phi(\bar{a})$ if there is $b \in M$ such that $\mathcal{M} \models \psi(\overline{a}, b)$.

(vii) If ϕ is $\forall v_j \psi(\overline{v}, v_j)$, then $\mathcal{M} \models \phi(\overline{a})$ if $\mathcal{M} \models \psi(\overline{a}, b)$ for all $b \in M$.

Notice that if ϕ is a sentence $\mathcal{M} \models \phi$, no assignment of variables influences its truth in a structure, so it's either true or false. Let ϕ be a formula with free variables. We can view the fact $\mathcal{M} \models \phi(\overline{a})$ or $\mathcal{M} \not\models \phi(\overline{a})$, as a property of $\overline{a}$. Informally, we can say that if ϕ is a sentence, it describes a property of a structure, a rule; if it has free variables, then it describes a property of tuples of M^n.

We will use the notation $\phi(v_1, \ldots, v_n)$ to denote the free variables occurring in ϕ. Also, will might write $\mathcal{M} \models \phi(\overline{a})$ for $\overline{a} = (a_1, \ldots, a_m)$ where $m > n$. In this case, we will end up using the sub-tuple $\overline{a}_n = (a_1, \ldots, a_n)$. Also we will write $\phi(v_1, \ldots, v_n)$ even if ϕ is a sentence, since for any $\overline{a} \in M^n$, $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{M} \models \phi$.

1.3 Theories

Given $\mathcal{L}$, we have a vast pool of structures to study. We usually want to consider only the ones that follow specific rules, i.e., sentences.

Definition 1.3.1. Let $\mathcal{L}$ be a language. An $\mathcal{L}$-theory T a set of $\mathcal{L}$-sentences, the axioms. We say that $\mathcal{M}$ is a model of T and write $\mathcal{M} \models T$ if $\mathcal{M} \models \phi$ for all sentences $\phi \in T$.

As in definition, 1.2.3 structures satisfy either $\phi(\overline{a})$ or $\neg\phi(\overline{a})$ for any ϕ and $\overline{a}$. However, a theory T can have both $\phi, \neg\phi \in T$. Thus, T has no models that satisfy all its sentences at the same time. We call a theory satisfiable (unsatisfiable) if it has at least a model (or none).

Definition 1.3.2. Let T be an $\mathcal{L}$-theory and ϕ an $\mathcal{L}$-sentence. We say that ϕ is a logical consequence of T and write $T \models \phi$ if $\mathcal{M} \models \phi$ whenever $\mathcal{M} \models T$, i.e., it is true for all models of T.

Definition 1.3.3. Let $\mathcal{M}$ be an $\mathcal{L}$-structure. We define the the theory of $\mathcal{M}$

$$\text{Th}(\mathcal{M}) = \{\phi \mid \phi \text{ is a sentence and } \mathcal{M} \models \phi\}.$$

Definition 1.3.4. An $\mathcal{L}$-theory T is called complete if for any $\mathcal{L}$-sentence ϕ either $T \models \phi$ or $T \models \neg\phi$.

By definition, a complete theory doesn't have any contradictions. Notice, for $\mathcal{M}$ an $\mathcal{L}$-structure, the theory Th$(\mathcal{M})$ is complete and satisfiable.

It is true that when a theory T is satisfiable, then any finite subset $\Delta \subset T$ does not have a contradiction, as any model of T is a model of Δ. The converse is also true. This is known as the Compactness theorem. We say T is finitely satisfiable if every finite subset $\Delta \subset T$ is satisfiable.

Theorem 1.3.5 (Compactness Theorem)**.** T is satisfiable if and only if every finite subset of T is satisfiable. Specifically, if T is a finitely satisfiable $\mathcal{L}$-theory and κ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality at most κ.

The importance of this theorem cannot be understated as almost exclusively every time we want to know if a theory is satisfiable, we resort to it. Also, the proof of the theorem, which the reader can find in [1], constructs a model for T of size at most κ. This is used to create small models for a theory. Using the 1.3.5, we can also create arbitrarily large models of T.

Theorem 1.3.6. Let T be an $\mathcal{L}$-theory with infinite models. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ.

In the following section, we present a refinement of the 1.3.6.

1.4 Embeddings

Now that we have focused on models of an $\mathcal{L}$-theory instead of all the $\mathcal{L}$-structures. It is time to introduce another crucial way to organize and group $\mathcal{L}$-structures.

Definition 1.4.1. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, we let $\phi(\mathcal{M}) = \{\overline{x} \in M^n \mid M \models \phi(\overline{x})\}$. We say that $X \subseteq M^n$ is definable if and only if there is an $\mathcal{L}$-formula $\phi(v_1, \ldots, v_n)$ that $\phi(\mathcal{M}) = X$.

Definition 1.4.2. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures with universes M and N, respectively. An $\mathcal{L}$-embedding $\eta : \mathcal{M} \to \mathcal{N}$ is a one-to-one map $\eta : M \to N$:

(i) $\eta(f^{\mathcal{M}}(a_1, \ldots, a_{n_f})) = f^{\mathcal{N}}(\eta(a_1), \ldots, \eta(a_{n_f}))$ for all $f \in \mathcal{F}$ and $a_1, \ldots, a_{n_f} \in M$;

(ii) $(a_1, \ldots, a_{m_R}) \in R^{\mathcal{M}}$ if and only if $(\eta(a_1), \ldots, \eta(a_{m_R})) \in R^{\mathcal{N}}$ for all $R \in \mathcal{R}$ and $a_1, \ldots, a_{m_R} \in M$;

(iii) $\eta(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for $c \in \mathcal{C}$.

If $M \subseteq N$ and the inclusion map[1] is an $\mathcal{L}$-embedding, we say that $\mathcal{M}$ is a substructure of $\mathcal{N}$or $\mathcal{N}$ is an extension of $\mathcal{M}$. We sometimes say $\mathcal{M}$ is a substructure of $\mathcal{N}$ without $M \subseteq N$. We mean that under $\eta : \mathcal{M} \to \mathcal{N}$ as described above, $\eta(\mathcal{M})$ is a substructure of $\mathcal{N}$.

The following theorem shows that not only the interpretation of $\mathcal{L}$-symbols is retained in $\mathcal{N}$[2]under η, but it also retains the interpretation of some weak formulas.

Theorem 1.4.3. Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$, $\overline{a} \in M$, and $\phi(\overline{v})$ is a quantifier-free formula. Then, $\mathcal{M} \models \phi(\overline{a})$ iff $\mathcal{N} \models \phi(\overline{a})$. In other words, the properties of $\overline{a}$ that are quantifier-free formulas are preserved.

This result can be viewed from the perspective of the definable sets of the structures. Let ϕ be a quantifier-free formula, then $\phi(\mathcal{M}) = \phi(\mathcal{N}) \cap M$, that if there are new elements $\overline{b}$ satisfying $\phi(\overline{b})$, then $\overline{b} \in N \setminus M$.

An $\mathcal{L}$-embedding that is also onto is called an $\mathcal{L}$-isomorphism; in other words, the two structures are the same under the function η and consequently preserve all properties of $\overline{a} \in M$.

Theorem 1.4.4. If η is an onto $\mathcal{L}$-embedding then $\mathcal{M} \models \phi(a_1, \dots, a_n)$ if and only if $\mathcal{N} \models \phi(\eta(a_1), \dots, \eta(a_n))$ for all formulas ϕ, not only quantifier-free ones.

Definition 1.4.5. An embedding $j : M \to N$ between two $\mathcal{L}$-structures is called elementary if for every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and any $a_1, \dots, a_n \in M$,

$$M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(j(a_1), \dots, j(a_n)).$$

In other words, j preserves all the properties of elements of M to N. We write $\mathcal{M} \prec \mathcal{N}$ when the inclusion map is elementary and say that $\mathcal{N}$ is an elementary extension of $\mathcal{M}$. We sometimes say $\mathcal{M} \prec \mathcal{N}$ without $M \subseteq N$. We mean that under $j : \mathcal{M} \to \mathcal{N}$ as described above, $j(\mathcal{M})$ is an elementary substructure of $\mathcal{N}$.

Notice that $\mathcal{M}$ and $\mathcal{N}$ have the same theory as sentences are $\mathcal{L}$-formulas.

Corollary 1.4.6. Suppose that $j : \mathcal{M} \to \mathcal{N}$ is an isomorphism. Then j is elementary. Also, $\mathcal{M}$ and $\mathcal{N}$ have the same theory.

Definition 1.4.7. Suppose that $(I, <)$ is a linear order. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for $i \in I$. We say that $(\mathcal{M}_i : i \in I)$ is a chain of $\mathcal{L}$-structures if $\mathcal{M}_i \subseteq \mathcal{M}_j$ for $i < j$. If $\mathcal{M}_i \prec \mathcal{M}_j$ for $i < j$, we call $(\mathcal{M}_i : i \in I)$ an elementary chain.

Theorem 1.4.8. Suppose that $(I, <)$ is a linear order and $(\mathcal{M}_i : i \in I)$ is an elementary chain. Then, $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ is an elementary extension of each $\mathcal{M}_i$.

[1]the injection $f : M \to N$ defined by $f(m) = m$ for all $m \in M$.
[2]$R^{\mathcal{M}} = R^{\mathcal{N}} \cap M$ and $f^{\mathcal{M}} = f^{\mathcal{N}} \cap M^{n_f+1}$

1.5 Parameters

Definition 1.5.1. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $A \subseteq M$. Let $\mathcal{L}_A = \mathcal{L} \cup \{c_\alpha \mid a \in A\}$, we say that A is a set of parameters; that is, we deal with A as constants. Under the interpretation of $c_\alpha \mapsto \alpha$ $\mathcal{M}$ is an $\mathcal{L}_A$-structure. Let $\text{Th}_A(\mathcal{M})$ be the set of all $\mathcal{L}_A$-sentences true in $\mathcal{M}$. We also have new definable subsets under the parameters A. We call $X \subseteq M^n$ an A-definable set or definable over A if there is a formula ψ and $\overline{b} \in A^m$ such that $X = \{\overline{a} : \psi(\overline{a}, \overline{b})\}$ defines X.

Note, if $\mathcal{N} \models \text{Th}_A(\mathcal{M})$, the interpretation of c_α must satisfy the same formulas $\phi(v)$ as $a \in M$. This generalizes to any tuple of parameters.

Definition 1.5.2. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure. We call $\text{Th}_M(\mathcal{M})$ the elementary diagram of $\mathcal{M}$, and we write $\text{Diag}_{\text{el}}(M)$, which is the following set in the language $\mathcal{L}_M$:

$$\{\phi(m_1, \ldots, m_n) : \mathcal{M} \models \phi(m_1, \ldots, m_n), \phi \text{ an } \mathcal{L}\text{-formula}\}.$$

We can see the elementary diagram $\text{Diag}_{\text{el}}(\mathcal{M})$ as the full description of the model $\mathcal{M}$ because all the elements in M are now in the expanded language $\mathcal{L}_M$ and so they can be referenced in sentences.

Theorem 1.5.3. If $\mathcal{N} \models \text{Diag}_{\text{el}}(\mathcal{M})$, $\mathcal{M} \prec \mathcal{N}$.

Proof. Let $j : M \to N$ be $j(m) = c_m^{\mathcal{N}}$, i.e., maps the interpretation of c_m in $\mathcal{M}$, which is m to the interpretation of c_m in $\mathcal{N}$. Notice j is an embedding. If m_1, m_2 two distinct elements of M, then $c_{m_1} \neq c_{m_2} \in \text{Diag}_{\text{el}}(\mathcal{M})$ then $j(m_1) \neq j(m_2)$. Using a similar argument, we can show that j is a function. Assume that j is not elementary, then $\mathcal{M} \models \phi(\overline{m})$ and $\mathcal{N} \models \neg\phi(j(\overline{m}))$ for $\overline{m} \in M^n$ and an $\mathcal{L}$-formula ϕ. Let $c_{\overline{m}} = (c_{m_1}, \ldots, c_{m_n})$, then $\phi(c_{\overline{m}}) \in \text{Diag}_{\text{el}}(\mathcal{M})$, so $\mathcal{N} \models \phi(c_{\overline{m}})$, a contradiction to the assumption. $\square$

In the following sections, when $a \in M$ is a parameter we will use a as a constant symbol instead of c_a as we did above.

Theorem 1.5.4 (Tarski-Vaught Test)**.** Suppose that $\mathcal{M}$ is a substructure of $\mathcal{N}$. Then, $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$ if and only if, for any tuple of elements $\overline{a} \in M^n$ and every $\phi(v, \overline{w})$ such that $\mathcal{N} \models \exists v \phi(v, \overline{a})$, then $\mathcal{M} \models \exists v \phi(v, \overline{a})$.

So, an easy way to check if an $\mathcal{L}$-embedding is elementary is to check if all existential properties of any tuple of elements are preserved in the substructure. In other words, the substructure contains all the witnesses of existential formulas.

With enough symbols in the language, we can force all substructures of models of T to contain the witnesses to existential formulas, whatever the size of the model.

Definition 1.5.5. We say that an $\mathcal{L}$-theory T has built-in Skolem functions if for all $\mathcal{L}$-formulas $\phi(v, w_1, \ldots, w_n)$, there is a function symbol f such that

$$T \models \forall \overline{w}\,((\exists v\, \phi(v, \overline{w})) \to \phi(f(\overline{w}), \overline{w})).$$

In other words, the language has enough function symbols to witness all existential statements.

Theorem 1.5.6. Let T be an $\mathcal{L}$-theory. There are $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ an $\mathcal{L}^*$-theory such that T^* has built-in Skolem functions, we call T^* a Skolemization of T. The following properties hold for T^* and $\mathcal{L}^*$:

- If $\mathcal{M} \models T$, then we can expand $\mathcal{M}$ to $\mathcal{M}^* \models T^*$. The opposite is also true.
- $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.
- Let $\mathcal{M}$ be a substructure of $\mathcal{N} \models T^*$ then $M \prec N$.

Using 1.5.3, 1.3.6, and 1.5.4 1.5.6 we get the following theorems about elementary extensions and elementary substructures.

Theorem 1.5.7 (Upward Löwenheim-Skolem). Let $\mathcal{M}$ be an infinite $\mathcal{L}$-structure and κ be an infinite cardinal such that $\kappa \geq |\mathcal{M}| + |\mathcal{L}|$. Then, there is an $\mathcal{L}$-structure $\mathcal{N}$ of cardinality κ and an elementary embedding $j : \mathcal{M} \to \mathcal{N}$.

Theorem 1.5.8 (Downward Löwenheim-Skolem). Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and $X \subseteq M$. There is an elementary submodel $\mathcal{N}$ of $\mathcal{M}$ such that $X \subseteq N$ and $|\mathcal{N}| \leq |X| + |\mathcal{L}| + \aleph_0$.

1.6 Types

Definition 1.6.1. From now on, assume that $\mathcal{L}$ is a countable language and T is a complete theory. Let $\mathcal{M} \models T$ and a set of parameters $A \subseteq M$. Let p be a set of $\mathcal{L}_A$-formulas all in the same variables $v_1, \ldots, v_n$. We call p an n-type if $p \cup \text{Th}_A(\mathcal{M})$ is satisfiable. What we mean for the set of $\mathcal{L}_A$ formulas to be satisfiable is there is $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ and $\overline{a} \in N^n$ that satisfies all $\phi \in p$. We can also view $p \cup \text{Th}_A(\mathcal{M})$ as a theory in an expanded language with new constant symbols c_i , $0 < i \leq n$, and the new theory being $\{\phi(\overline{c}) \mid \text{for all } \phi \in p\} \cup \text{Th}_A(\mathcal{M})$.

Definition 1.6.2. We say that p is a complete n-type if $\phi \in p$ or $\neg\phi \in p$ for all $\mathcal{L}_A$-formulas ϕ with free variables from $v_1, \ldots, v_n$, otherwise we call p incomplete.

We let $S_n^{\mathcal{M}}(A)$ be the set of all complete n-types of the model $\mathcal{M}$ over the parameters A. Each tuple of elements $\overline{a} \in M^n$ has a complete type over parameters A denoted

$$\text{tp}^{\mathcal{M}}(\overline{a}/A) = \{\phi(v_1, \ldots, v_n) \in \mathcal{L}_A : \mathcal{M} \models \phi(a_1, \ldots, a_n)\}.$$

If p is an n-type over A, we say that $\overline{a} \in M^n$ realizes p if $\mathcal{M} \models \phi(\overline{a})$ for all $\phi \in p$. If p is not realized in $\mathcal{M}$, we say that $\mathcal{M}$ omits p.

By definition, a set p of $\mathcal{L}_A$ formulas is a type if there is a model of $\text{Th}_A(\mathcal{M})$ that is realized. We can make this more specific so the model is an elementary extension of $\mathcal{M}$.

Definition 1.6.3. We define $[\phi] = \{p \in S_n^{\mathcal{M}}(A) \mid \phi \in p\}$, which is all the types that contain the formula ϕ.

Theorem 1.6.4. Let M be an $\mathcal{L}$-structure, $A \subseteq M$, and p an n-type over A. There is $\mathcal{N}$ an elementary extension of $\mathcal{M}$ such that p is realized in $\mathcal{N}$.

It is often hard to provide isomorphisms and elementary embeddings between structures; partial elementary embeddings are used to express partial isomorphism between subsets of the models. These are important as they can be expanded to full embeddings later.

Definition 1.6.5. If $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}$-structures and $B \subseteq M$, we say that the one-to-one $f : B \to N$ is a partial elementary map if

$$\mathcal{M} \models \phi(\bar{b}) \Leftrightarrow \mathcal{N} \models \phi(f(\bar{b}))$$

for all $\mathcal{L}$-formulas ϕ and all finite sequences $\bar{b}$ from B. Note that ϕ can be a sentence, so $\mathcal{M}$ and $\mathcal{N}$ share the same theory.

There is a connection between theories with parameters and partial elementary embeddings.

Theorem 1.6.6. Let $\mathcal{M}$ and $\mathcal{N}$ be models of T.

1. If $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ for $A \subseteq M$, there is a partial elementary map $f : A \to N$.
2. If there is partial elementary embedding between $\mathcal{M}$ and $\mathcal{N}$, $f : A \to B$ where $A \subseteq M$ and $B \subseteq N$, then $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ and $\mathcal{M} \models \text{Th}_B(\mathcal{N})$.

Proof.

1. We can take the interpretation of the set parameters A in $\mathcal{N}$, in the language $\mathcal{L}_A$, as the partial elementary map.
2. To prove $\mathcal{N} \models \text{Th}_A(\mathcal{M})$ we can interpret the the parameters A as the elements B as they are matched in f. This way an $\mathcal{L}_A$-sentence $\mathcal{M} \models \psi(\bar{a})$ iff $\mathcal{N} \models \psi(\bar{b})$ because f is elementary.

□

This way, we can view A and B as isomorphic copies and we can identify them as just A. The following corollary says that if two models have the same theory with parameters (if any), they have the same set of types.

Corollary 1.6.7. Assume $\text{Th}_A(\mathcal{M}) = \text{Th}_A(\mathcal{N})$. Then $S_n^{\mathcal{M}}(A) = S_n^{\mathcal{N}}(A)$.

All models of the same theory (with parameters or not) share the same fixed number of types. The maximum number of types is $2^{|\mathcal{L}_A|}$, in the case without parameters $2^{\aleph_0}$.

Types influence how many different models a theory can have for a given cardinality. We can think of different models realizing different types or in different quantities ($\aleph_0, \aleph_1, \ldots$ realizations for p).

The main focus of this exposition is to count the non-isomorphic models of cardinality κ for specific theories.

Definition 1.6.8. A complete theory T in a countable language $\mathcal{L}$ is κ-categorical if it has a unique (up to isomorphism) model of cardinality κ.

Theorem 1.6.9. Let $|\mathcal{L}| \leq \kappa$, then the maximum number of models of cardinality κ a theory can have is 2^{κ}.

Under normal conditions, we cannot use language to refer to an element with a specific type as it would an infinite sentence with all $\phi \in p$. However, if the type is isolated, we can reference it with a single formula.

Definition 1.6.10. We say $p \in S_n^{\mathcal{M}}(A)$ is isolated if there is an $\mathcal{L}_A$-formula $\phi(\overline{v}) \in p$ such that for all $\mathcal{L}_A$-formulas $\psi(\overline{v})$

$$\psi(\overline{v}) \in p \Leftrightarrow \mathrm{Th}_A(\mathcal{M}) \models \phi(\overline{v}) \rightarrow \psi(\overline{v}).$$

In other words, the property ϕ determines all the other properties of p; no other type has the property ϕ.

Lemma 1.6.11. Suppose that $A \subseteq B \subseteq \mathcal{M} \models T$ and every $\bar{b} \in B^m$ realizes an isolated type in $S_m^{\mathcal{M}}(A)$. Suppose that $\bar{a} \in \mathcal{M}^n$ realizes an isolated type in $S_n^{\mathcal{M}}(B)$. Then, $\bar{a}$ realizes an isolated type in $S_n^{\mathcal{M}}(A)$.

We are ready to state the main theorem we are aiming for, which was proven by Morley.

Theorem 1.6.12. Let T be a complete theory in a countable language with infinite models and $\kappa \geq \aleph_1$. T is κ-categorical iff T is λ-categorical for any $\lambda \geq \aleph_1$.

We will prove 1.6.12 through the characterization of κ-categorical theories given by Baldwin and Lachlan[2]. We encourage the reader to see the original proof by Morley in [2].

We now state the Baldwin-Lachlan characterization, although we have not defined the two properties, ω-stability and Vaughtian pairs.

Theorem 1.6.13. Let $\kappa \geq \aleph_1$ and T a complete theory in a countable $\mathcal{L}$. T is κ-categorical iff T has no Vaughtian pairs and is ω-stable.

Proof. This immediately implies Morley's theorem, as the second part of the characterization does not depend on κ.

□

In the following sections, we will focus our attention on defining the prerequisites for 1.6.13 and proving both directions of the theorem. We start at the next chapter with ω-stability.

1.7 Prime and Homogeneous Models, Stable Theories

Definition 1.7.1. We say that $\mathcal{M} \models T$ is a prime model of T if for all such that $\mathcal{N} \models T$, $\mathcal{M} \prec \mathcal{N}$ holds.

The same definition can be adjusted with parameters.

Definition 1.7.2. Let $\mathcal{M} \models T$ and $A \subseteq M$; we say $\mathcal{M}_0 \models \text{Th}_A(\mathcal{M})$ is prime over A if for every $\mathcal{N} \models \text{Th}_A(\mathcal{M})$, $\mathcal{M}_0 \prec \mathcal{N}$ holds. Equivalently using 1.6.6, we have that $\mathcal{M}_0 \models T$ is prime over $A \subseteq M_0$ if whenever $\mathcal{N} \models T$ and $f : A \to N$ is a partial elementary function, there is an elementary $f^* : M_0 \to N$ extending f.

To summarize the above definitions, a prime model of T over a set of parameters A can be embedded in every model with a copy of A and the same theory, T.

Definition 1.7.3. Let $\mathcal{M} \models T$ be a first-order theory and $A \subseteq M$. We say that isolated types are dense in $S_n^{\mathcal{M}}(A)$ if for all $\mathcal{L}_A$-formulas ϕ exist an isolated type $p \in S_n^{\mathcal{M}}(A)$ such that $p \in [\phi]$.

Not all theories have prime models or prime models over parameters. The following T theories, known as κ-stable, have a special connection with prime models.

Definition 1.7.4. Let T be a complete theory in a countable language, and let κ be an infinite cardinal. We say that T is κ-stable if whenever $\mathcal{M} \models T$, $A \subseteq M$, and $|A| = \kappa$, then $|S_n^{\mathcal{M}}(A)| = \kappa$.

Intuitively, the number of parameters $|A|$ dictates the number of types a theory would have over them. Each parameter $a \in A$ adds one unique type, isolated by the formula $v = a$, so having $|A|$ as parameters always yields at least $|A| \leq |S_n^{\mathcal{M}}(A)|$ complete types. So, stable theories yield the least possible types for a $|A| = \kappa$. We traditionally use ω-stable instead of $\aleph_0$-stable theories. We give a case of a theory that is not ω-stable.

Example 1.8. Take $(\mathbb{Q}, <)$ to be the rationals with the ordering relation in the language $\mathcal{L} = \{<\}$, Th($\mathbb{Q}$) is not ω-stable. Take $\mathbb{Q}$ as the set of parameters. However $|S_1^{\mathbb{Q}}(\mathbb{Q})| = 2^{\aleph_0}$ because for each Dedekind cut (L, U), there is a type $\{q < v \mid q \in L\} \cup \{v < q \mid q \in U\}$ that expresses a real number.

Theorem 1.8.1. Let T be a complete theory in a countable language. If T is ω-stable, then T is κ-stable for all infinite cardinals κ.

Proof. For the sake of contradiction, assume that T is ω-stable but not κ-stable for some specific cardinal κ. Since T is not κ-stable there exists a model M and A a set of parameters with $|A| = \kappa$ such that $|S_n^{\mathcal{M}}(A)| > \kappa$ for some n.

1. Notice there are κ $\mathcal{L}_A$-formulas. If every formula ϕ is in at most κ many types, then there are $|S_n^{\mathcal{M}}(A)| = \kappa$. Thus, there is a $\mathcal{L}_A$-formula $\phi_\emptyset$, which is in more than κ many types. Recall $[\phi] = \{p \in S_n^{\mathcal{M}}(A) \mid \phi \in p\}$. So $|[\phi_\emptyset]| > \kappa$

2. Assume we have any $\mathcal{L}_A$ formula $|[\phi]| > \kappa$. Assume for the sake of contradiction there is no ψ formula that can divide $[\phi]$ into $|[\phi \wedge \psi]| > \kappa$ and $|[\phi \wedge \neg\psi]| > \kappa$. So always, one of the two sets is bigger than κ and the other less or equal to κ. Let $B_0 = [\phi]$ and $B_{\alpha+1} = B_\alpha \cap [\psi_{\alpha+1}]$ with ψ_i belonging to a well-ordering of all $\mathcal{L}_A$ formulas such $|[\phi \wedge \psi_i]| > \kappa$ and let $p = \{\psi_i\}$ denote that set. Notice that p is a complete set of $\mathcal{L}_A$ formulas because of our assumption. If $|B_\alpha| > \kappa$ then $|B_{\alpha+1}| > \kappa$. If $\beta \leq \kappa$ is a limit ordinal, then $\bigcap_{\alpha<\beta} B_\alpha \neq \emptyset$, because every finite subset of $\{\phi, \psi_1, \ldots \psi_\beta\}$ is satisfiable. Also, for $\beta < \kappa$, $|B_\beta| > \kappa$ otherwise $|[\phi]| \leq \kappa$,

$$[\phi] = \bigcup_{i<\beta} [\phi \wedge \neg\psi_i] \cup B_\beta$$

3. So because B_κ is non-empty set of types, but all types in B_κ are complete, as noted above so p is the only one. However this is a contradiction because

$$[\phi] = \bigcup_{\psi_i \in p} [\phi \wedge \neg\psi_i] \cup \{p\}$$

so the cardinality of $|[\phi]| = \kappa \cdot \kappa + 1$, a contradiction. So we have that any ϕ with $|[\phi]| > \kappa$ can be divided by some ψ to $|[\phi \cap \psi]| > \kappa$ and $|[\phi \cap \neg\psi]| > \kappa$.

4. We will build a binary tree of formulas $(\varphi_\sigma : \sigma \in 2^{<\omega})$ such that:

   i) if $\sigma \subset \tau$ then $\varphi_\tau \models \varphi_\sigma$;

   ii) $\varphi_{\sigma,i} \models \neg\varphi_{\sigma,1-i}$;

   iii) $\|\varphi_\sigma\| > \aleph_0$.

   We start by letting $\varphi_\emptyset$ be the formula we found on the first part, such that $|[\phi_\emptyset]| > \kappa$. Given φ_σ where $[\varphi_\sigma]| > \kappa$, by the third part we can find ψ such that $|[\varphi_\sigma \wedge \psi]| > \kappa$ and $|[\varphi_\sigma \wedge \neg\psi]| > \kappa$.
   Let $\varphi_{\sigma,0} = \varphi_\sigma \wedge \psi$ and $\varphi_{\sigma,1} = \varphi_\sigma \wedge \neg\psi$. This is a complete binary tree. We now argue that for each infinite branch $f \in 2^\omega$ there is a countable type associated

$$p_f \in \bigcap_{m=0}^{\infty} \left[\varphi_{f|m}\right],$$

   where $f|m$ is f restricted to the first m bits. We need to show p_f is a type. We know that for any m, $[\phi_{f|m}] \neq \emptyset$, so by the compactness theorem we have that $\bigcap_{m=0}^{\infty} \left[\varphi_{f|m}\right] \neq \emptyset$. Notice that if $f \neq g$ then $p_f \neq p_g$. Assume f and g split at the n-step, i.e. for some ψ, $\psi \in p_f$ and $\neg\psi \in p_g$. However the number of branches are $2^{\aleph_0}$ so there are at least that many types using the parameters A, because in the tree we used only countably many formulas each using finitely many parameters. So let $A_0 \subset A$ be the parameters used by the formulas of the tree, we have $|A_0| = \aleph_0$ and $|S^n(A_0)| = 2^{\aleph_0}$, this is a contradiction to ω-stability.

□

Lemma 1.8.2. Let T be a complete theory in a countable language. If T is ω-stable, then for all $\mathcal{M} \models T$ and $A \subseteq M$, the isolated types in $S_n^{\mathcal{M}}(A)$ are dense.

Proof. If the isolated types in $S_n^{\mathcal{M}}(A)$ are not dense, this means that there exists an $\mathcal{L}_A$-formula ϕ such that $[\phi]$ contains no isolated types. Notice that if $|[\phi]| \in \mathbb{N} \setminus \{0\}$, then all $p \in [\phi]$ are isolated, so it has to be infinite. Because of that we can find a $\mathcal{L}_A$-formula ψ such that $|[\phi \wedge \psi]| \geq \aleph_0$ and $|[\phi \wedge \neg\psi]| \geq \aleph_0$; each one has no isolated types, so we can apply again the same idea. This allows us to build a complete binary tree. Having that tree we can complete the proof as the one in 1.8.1. □

Theorem 1.8.3. Suppose that T is ω-stable. Let $\mathcal{M} \models T$ and $A \subseteq M$. There is $\mathcal{M}_0 \prec \mathcal{M}$, a prime model over A. Moreover, we can choose $\mathcal{M}_0$ so that every element of $\mathcal{M}_0$ realizes an isolated type over A.

Proof. To build the elementary submodel of $\mathcal{M}$, which is prime over A, we need to start investigating the substructures of M that contain A. We will find an ordinal δ and build a sequence of sets $(A_\alpha : \alpha < \delta)$ where $A_\alpha \subseteq M$ and

i) $A_0 = A$;

ii) if α is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$;

iii) if no element of $M \setminus A_\alpha$ realizes an isolated type over A_α, then let $\delta = \alpha$ this is our universe, which are elements realizing isolated types. Otherwise, include any a_α realizing an isolated type over A_α, and let $A_{\alpha+1} = A_\alpha \cup \{a_\alpha\}$. Let $\mathcal{M}_0$ be the substructure of $\mathcal{M}$ with universe A_δ.

We will now prove that the substructure $\mathcal{M}_0 \prec \mathcal{M}$. To prove this, we use the 1.5.4. Suppose that $\mathcal{M} \models \varphi(v, \overline{a})$, where $\overline{a} \in A_\delta$. By 1.8.2, the isolated types in $S^{\mathcal{M}}(A_\delta)$ are dense. Thus, if there is $b \in M$ such that $\mathcal{M} \models \varphi(b, \overline{a})$ there is a $c \in M$ with $\mathcal{M} \models \phi(c, \overline{a})$ and tp$^{\mathcal{M}}(c/A_\delta)$ is isolated. By choice of δ, $c \in A_\delta$. Thus, $\mathcal{M}_0 \prec \mathcal{M}$. Now, we need to show that $\mathcal{M}_0$ is a prime model over A. Suppose that $\mathcal{N} \models T$ and $f : A \to \mathcal{N}$ is partial elementary. We will construct a sequence of functions $f = f_0 \subseteq \cdots \subseteq f_\alpha \subseteq \cdots \subseteq f_\delta$, where $f_\alpha : A_\alpha \to \mathcal{N}$ is elementary, and ultimately extending the domain of f to A_δ.

- If α is a limit ordinal, we let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$.
- Assume that $f_\alpha : A_\alpha \to \mathcal{N}$ partial elementary, we know because of the way we constructed $\mathcal{M}_0$ we know that there exists a formula $\varphi(v, \overline{a})$, that isolates tp$^{\mathcal{M}_0}(a_\alpha/A_\alpha)$. Because f_α is partial elementary, we have that $\varphi(v, f_\alpha(\overline{a}))$ isolates tp$^{\mathcal{N}}(f_\alpha(a_\alpha)/f_\alpha(A_\alpha))$. Also, because f_α is partial elementary, there is $b \in \mathcal{N}$ with $\mathcal{N} \models \varphi(b, f_\alpha(\overline{a}))$. Thus, $f_{\alpha+1} = f_\alpha \cup \{(a_\alpha, b)\}$ is elementary.

So the last union will be $f_\delta : \mathcal{M}_0 \to \mathcal{N}$ and elementary, proving that $\mathcal{M}_0$ is prime over A. We have that every $\overline{a} \in \mathcal{M}_0$ realizes an isolated type over $M = A_\delta$. However, we want every element $\overline{a} \in \mathcal{M}$ to be isolated in A. Here, an argument of just removing too many parameters does not work because we can actually remove parameters that define the isolating formula. We can prove that using ordinal induction and 1.6.11. $\square$

The last property we will explore in this section is homogeneity, which is a model's property to extend local similarities.

Definition 1.8.4. Let κ be an infinite cardinal. We say that $\mathcal{M} \models T$ is κ-homogeneous if whenever $A \subset M$ with $|A| < \kappa, f : A \to M$ is a partial elementary map, and $a \in M$, there is $f^* \supseteq f$ such that $f^* : A \cup \{a\} \to M$ is partial elementary. We say $\mathcal{M}$ is homogeneous if it is $|M|$-homogeneous.

Note that $|A \cup \{a\}| < \kappa$ still, so we can repeat the process using ordinals until we get $f^* : B \to M$, where $|B| = \kappa$ and cannot be extended. If $\mathcal{M}$ is homogeneous, then with the ordering of M, we have $B = M$.

The next theorem will play an important role by having an easy, sufficient condition for countable models to be isomorphic. Here, homogeneity is a key property in constructing the isomorphism by using a method called back-and-forth.

Theorem 1.8.5. Let T be a complete theory in a countable language. Suppose that $\mathcal{M}$ and $\mathcal{N}$ are countable homogeneous models of T and $\mathcal{M}$, and $\mathcal{N}$ realize the same types in $S_n(T)$ for $n \geq 1$. Then $\mathcal{M} \cong \mathcal{N}$.

Proof. To construct an isomorphism $f : \mathcal{M} \to \mathcal{N}$, we define a sequence of partial elementary maps $f_0 \subset f_1 \subset \cdots$, each with a finite domain each strictly larger than the previous. A chain of finite subsets, the domains, and ranges of those functions, does not guarantee that the limit of this sequence $f = \bigcup_{i=0}^{\infty} f_i$ will be a bijection, so we must carefully construct this chain to include all elements of $\mathcal{M}$ and $\mathcal{N}$. Let $a_0, a_1, \ldots$ be an enumeration of elements in $\mathcal{M}$, and $b_0, b_1, \ldots$ be an enumeration of elements in $\mathcal{N}$. We will ensure that a_i belongs to dom(f_{2i+1}) and b_i belongs to img(f_{2i+2}). In this way, we establish that dom$(f) = \mathcal{M}$ and that f is a surjective elementary map from $\mathcal{M}$ onto $\mathcal{N}$.

- $s = 0$: Let $f_0 = \emptyset$. Because T is complete f_0 is partial elementary.

We inductively assume that f_s is partial elementary. Let $\overline{a}$ be the domain of f_s and $\overline{b} = f_s(\overline{a})$.

- If $s + 1 = 2i + 1$: We want to extend the domain of f_{s+1} by adding a_i to it. If we have already add that element we proceed with no change to our function. Let $p =$ tp$^{\mathcal{M}}(\overline{a}, a_i)$. Because $\mathcal{M}$ and $\mathcal{N}$ realize the same types, we can find $\overline{c}, d \in \mathcal{N}$ such that tp$^{\mathcal{N}}(\overline{c}, d) = p$. Also tp$^{\mathcal{N}}(\overline{c}) =$ tp$^{\mathcal{M}}(\overline{a})$, as any $\phi(\overline{v}) \in$ tp$^{\mathcal{M}}(\overline{a}, a_i)$ is in tp$^{\mathcal{M}}(\overline{c}, d)$. Because f_s is partial elementary, we have that tp$^{\mathcal{M}}(\overline{a}) =$ tp$^{\mathcal{N}}(\overline{b})$. Thus, tp$^{\mathcal{N}}(\overline{c}) =$ tp$^{\mathcal{N}}(\overline{b})$. Because $\mathcal{N}$ is homogeneous, there is $e \in \mathcal{N}$ such that tp$^{\mathcal{N}}(\overline{b}, e) =$ tp$^{\mathcal{N}}(\overline{c}, d) = p$. Thus, $f_{s+1} = f_s \cup \{(a_i, e)\}$ is partial elementary with a_i in the domain.

- If $s + 1 = 2i + 2$: We want to extend the range of the function by adding b_i to it. Because f_s is elementary, we can apply everything we did in the previous step for the partial elementary function f_s^{-1}.

□

2. VAUGHTIAN PAIRS

In this chapter, we will prove the forward direction of Baldwin's and Lahlan's characterization of unaccountably categorical theories. Specifically, the following theorem:

Theorem 2.0.1. Let T be a complete theory in a countable language with infinite models. If $\kappa \geq \aleph_1$ and T is κ-categorical, then T has no Vaughtian pairs and is ω-stable.

2.1 Vaught's Two Cardinal Theorem

We begin by finding an obstruction for categoricity for a specific uncountable cardinality κ. If such obstruction is found, we show how it is encountered in every uncountable cardinality under the condition of ω-stability.

We remind $\varphi(\mathcal{M}) = \{\bar{x} \in M^n \mid M \models \varphi(\bar{x})\}$.

Definition 2.1.1. Let $\kappa > \lambda \geq \aleph_0$. We say that an $\mathcal{L}$-theory and T has a (κ, λ)-model if there is $\mathcal{M} \models T$ and $\phi(\bar{v})$ an $\mathcal{L}$-formula such that $|M| = \kappa$ and $|\phi(\mathcal{M})| = \lambda$.

The following theorem states that a (κ, λ)-model obstructs κ-categoricity.

Lemma 2.1.2. Let $\kappa > \aleph_0$. If T is κ-categorical, then it has no (κ, λ)-model.

Proof. For the sake of contradiction, we assume there is a (κ, λ)-model $\mathcal{N}$. Using $\mathcal{N}$ and the compactness theorem, we get a model $\mathcal{M}$ such that $|\mathcal{M}| = \kappa$ and every definable set also has cardinality κ. Thus, $\mathcal{M}$ and $\mathcal{N}$ are not isomorphic, a contradiction.

To demonstrate how we build $\mathcal{M}$: We expand $\mathcal{L}$ by adding constants $\{c_i^\phi \mid i < \kappa\}$ for each formula ϕ that has infinite realizations in $\mathcal{N}$. We construct a new theory $T^* = \text{Diag}_{\text{el}}(\mathcal{N}) \cup \bigcup_\phi \{\phi(c_i)^\phi \mid i < \kappa\} \cup \{c_i^\phi \neq c_j^\psi \mid i \neq j \text{ or } \phi \neq \psi\}$, which is satisfiable using the compactness theorem, as each finite $\Delta \subset T^*$ has $\mathcal{N}$ as a model by interpreting the constants as different elements of $\mathcal{N}$ with the property $\phi(v)$. Let $\mathcal{M}$ be any model of this theory of cardinality κ. Any $\emptyset$-definable set of $\mathcal{M}$ has κ realizations.

$\square$

Definition 2.1.3. We say that $(\mathcal{N},\ \mathcal{M})$ is a Vaughtian pair of models of T if $\mathcal{M} \prec \mathcal{N}$, $M \neq N$, and there is an $\mathcal{L}_M$-formula φ such that $\varphi(\mathcal{M})$ is infinite and $\varphi(\mathcal{N}) = \varphi(\mathcal{M})$.

Vaughtian pairs are an obstruction to $\aleph_1$-categoricity. The following chain of theorems demonstrates that if T has a (κ, λ)-model for any $\kappa > \aleph_0$, then it has a Vaughtian pair of models, a countable Vaughtian pair, a countable Vaughtian pair of isomorphic models, and finally a $(\aleph_1, \aleph_0)$-model obstructing $\aleph_1$-categoricity.

Lemma 2.1.4. If T has a (κ, λ)-model with $\kappa > \lambda \geq \aleph_0$, then there exists a Vaughtian pair $(\mathcal{N},\ \mathcal{M})$ of models of T.

Proof. Let $\mathcal{N}$ be a (κ, λ)-model of T and $X = \Phi(\mathcal{N})$ be the definable set of cardinality λ. By the downward Löwenheim-Skolem theorem, we can take an elementary submodel $\mathcal{M}$ of $\mathcal{N}$ that contains X and $|\mathcal{M}| \leq |X| + |\mathcal{L}| + \aleph_0 < |\mathcal{N}|$, so it is a proper submodel. □

Now that we know that T has a Vaughtian pair, we want to make a theory T^* to capture exactly all the Vaughtian pairs of T. To do that, we need to alter our language so every model of T^* is actually a pair of models of T with the second being a proper subset of the first.

Definition 2.1.5. Let $\mathcal{L}^* = \mathcal{L} \cup \{U\}$, where U is a unary predicate symbol. If $\mathcal{M} \subseteq \mathcal{N}$ are $\mathcal{L}$-structures, we write an $\mathcal{L}^*$-structure $(\mathcal{N}, \mathcal{M})$ to designate that U is interpreted as M.

We need some work to ensure $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$. We already know that if an element in $\mathcal{N}$ has a quantifier-free property, it is preserved under substructure. From the Tarksi-Vaught theorem, we know that all existential properties are preserved under a substructure iff it is an elementary substructure. We want a way to express in our language that a property holds for $\mathcal{M}$.

If $\phi(v_1, \ldots, v_n)$ is an $\mathcal{L}$-formula, we define $\phi^U(\overline{v})$ in the new $\mathcal{L}^*$, the restriction of ϕ to U, inductively as follows:

1. If ϕ is atomic, then ϕ^U is $U(v_1) \wedge \ldots \wedge U(v_n) \wedge \phi$;
2. If ϕ is $\neg\psi$, then ϕ^U is $\neg\psi^U$;
3. If ϕ is $\psi \wedge \theta$, then ϕ^U is $\psi^U \wedge \theta^U$;
4. If ϕ is $\exists v\ \psi$, then ϕ^U is $\exists v\ U(v) \wedge \psi^U$.

Notice that by 4, if ϕ^U is true in $\mathcal{N}$, then a witness for ϕ must exist inside M, the interpretation of U.

Claim 2.1.6. The properties of elements of the submodel $\mathcal{M}$ can be expressed inside $\mathcal{N}$, i.e., $\mathcal{N} \models$"$\mathcal{M} \models \phi(\overline{a})$". If $(\mathcal{N}, \mathcal{M})$ is an $\mathcal{L}^*$-structure and $\overline{a} \in M^k$, then $\mathcal{M} \models \phi(\overline{a})$ if and only if $(\mathcal{N}, \mathcal{M}) \models \phi^U(\overline{a})$.

With our extended language, we want to capture exactly those pairs of models that U is interpreted as a substructure of $\mathcal{N}$, specifically an elementary one. Thus, we need to add some axioms.

Lemma 2.1.7. If $(\mathcal{N},\ \mathcal{M})$ is a Vaughtian pair for T, then there is a countable Vaughtian pair, i.e. a pair $(\mathcal{N}_0,\ \mathcal{M}_0)$ where $\mathcal{N}_0$ is countable.

Proof. The following axioms together with T ensure that its models are Vaughtian pairs of T. Let ϕ be a fixed $\mathcal{L}_M$ formula such that $\phi(\mathcal{M}) = \phi(\mathcal{N})$ with parameters $\overline{m_0} \in \mathcal{M}$.

1. $\overline{m_0}$ are new constants.

2. $U^{\mathcal{N}}$ is a substructure of $\mathcal{N}$. For each function symbol $f \in \mathcal{L}$, $\forall \overline{v}\, U(\overline{v}) \to U(f(\overline{v}))$.

3. $U^{\mathcal{N}}$ is an elementary substructure of $\mathcal{N}$. $\forall \overline{v} \left(\left(\bigwedge_{i=1}^{k} U(v_i) \wedge \psi(\overline{v}) \right) \to \psi^U(\overline{v}) \right)$, for any $\mathcal{L}$-formula ψ. The important detail here to distinguish the antecedent and consequent is that both state that $U(\overline{v})$, but the second one states that the witnesses of its existential properties are in U as well.

4. $\phi(\mathcal{N})$ is an infinite set. So we add each k the sentences,
$$\exists \overline{v}_1 \ldots \exists \overline{v}_k \left(\bigwedge_{i<j} \overline{v}_i \neq \overline{v}_j \wedge \bigwedge_{i=1}^{k} \phi(v_i) \right)$$
.

5. $\phi(\mathcal{N})$ is a proper subset of N. $\forall \overline{v}(\phi(\overline{v}) \to \bigwedge U(v_i))$, $\exists x \neg U(x)$.

This new theory is satisfiable by $(\mathcal{N}, \mathcal{M})$ and the language $\mathcal{L}^*$ is countable, so by the downward Löwenheim-Skolem, there is a countable elementary submodel $(\mathcal{N}_0, \mathcal{M}_0)$ that contains $\overline{m_0}$. The two models $\mathcal{N}_0$, $\mathcal{M}_0$ form a Vaughtian pair because of the axioms regarding ϕ. □

Lemma 2.1.8. Suppose that $\mathcal{M}_0 \prec \mathcal{N}_0$ are countable models of T. Now consider the model $(\mathcal{N}_0, \mathcal{M}_0)$ in $\mathcal{L} \cup \{U\}$ defined in 2.1.5, we can find an elementary extension of it $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}, \mathcal{M})$ such that $\mathcal{N}$ and $\mathcal{M}$ are countable, homogeneous, and realize the same types in $S_n(T)$. By 1.8.5, $\mathcal{M} \cong \mathcal{N}$. Specifically, if $(\mathcal{N}_0, \mathcal{M}_0)$ is a Vaughtian pair, so is $(\mathcal{N}, \mathcal{M})$.

Proof. We start with the countable pair $(\mathcal{N}_0, \mathcal{M}_0)$ and build an elementary chain of countable models working toward obtaining the homogeneous property and realizing the same types. Our chain consists of 3 sub-steps:

Claim 2.1.9. If $\bar{a} \in M_0$ and $p \in S_n(\bar{a})$ is realized in $\mathcal{N}_0$, then there is $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}', \mathcal{M}')$ such that p is realized in $\mathcal{M}'$.

Proof. Let $\Gamma(\overline{v}) = \{\phi^U(\overline{v}, \overline{a}) : \phi(\overline{v}, \overline{a}) \in p\} \cup \text{Diag}_{\text{el}}(\mathcal{N}_0, \mathcal{M}_0)$ be the type "I am like p, but myself and my properties are contained in U" in the extended language $\mathcal{L}^*$. Any finite subset of $\Gamma(\overline{v})$ is satisfiable by $(\mathcal{N}_0, \mathcal{M}_0)$. Let $\phi_1, \ldots, \phi_m \in p$, $\mathcal{N}_0 \models \exists \overline{v} \bigwedge \phi_i(\overline{v}, \overline{a})$ because it realizes the type. As a result $\mathcal{M}_0 \models \exists \overline{v} \bigwedge \phi_i(\overline{v}, \overline{a})$ since it is an elementary submodel. We can express that sentence in the extended language $\mathcal{L}^*$ satisfied by $(\mathcal{N}_0, \mathcal{M}_0) \models \exists \overline{v} \bigwedge \phi_i^U(\overline{v}, \overline{a})$. Let $(\mathcal{N}', \mathcal{M}')$ be a countable elementary extension realizing Γ by the Löwenheim-Skolem Theorem. □

Claim 2.1.10. If $\bar{b} \in N_0$ and $p \in S_n(\bar{b})$, then there is $(\mathcal{N}_0, \mathcal{M}_0) \prec (\mathcal{N}', \mathcal{M}')$ such that p is realized in $\mathcal{N}'$.

1. By iterating 2.1.9 for all $p \in S_n(T)$ that are realized in $\mathcal{N}_{3i}$, we get a new pair such that $\mathcal{M}_{3i+1}$ realizes the same types as $\mathcal{N}_{3i}$; (ω-steps)

2. We enumerate all $(\overline{a}, \overline{b}, c) \in \mathcal{M}_{3i+1}$ with the property tp$^{\mathcal{M}_{3i+1}}(\overline{a})$ = tp$^{\mathcal{M}_{3i+1}}(\overline{b})$. For any such tuple, we can find an elementary extension with $d \in \mathcal{M}_{3i+2}$ such that tp$^{\mathcal{M}_{3i+2}}(\overline{a}, c)$ = tp$^{\mathcal{M}_{3i+2}}(\overline{b}, d)$. Because we want to realize the type $\Gamma(v) = \{\phi(\overline{b}, v) : \phi(\overline{z}, v) \in$ tp$^{\mathcal{M}_{3i+1}}(\overline{a}, c)\}$ can do this using 2.1.9 as $\Gamma(v) \in S_1(\bar{a})$; ($\omega$-steps)

3. We enumerate all $(\overline{a}, \overline{b}, c) \in \mathcal{N}_{3i+2}$ with the property tp$^{\mathcal{N}_{3i+2}}(\overline{a})$ = tp$^{\mathcal{N}_{3i+2}}(\overline{b})$, and following the same method, we can find an elementary extension with a $d \in N_{3i+3}$ by 2.1.10, such that tp$^{\mathcal{N}_{3i+3}}(\overline{a}, c)$ = tp$^{\mathcal{N}_{3i+3}}(\overline{b}, d)$. (ω-steps)

Let $(\mathcal{N}, \mathcal{M}) = \bigcup_{i<\omega}(\mathcal{N}_i, \mathcal{M}_i)$. Then, $(\mathcal{N}, \mathcal{M})$ is a countable Vaughtian pair. By i), $\mathcal{M}$ and $\mathcal{N}$ realize the same types. By ii) and iii), $\mathcal{M}$ and $\mathcal{N}$ are homogeneous and hence isomorphic by 1.8.5. A note here is that we cannot skip step two or three because homogeneity for $\mathcal{M}$ does not imply homogeneity for $\mathcal{N}$ and vice versa.

□

We know that a theory like T^* has models of any cardinality, so for any cardinal κ there is $(\mathcal{N}, \mathcal{M})$ with $|\mathcal{N}| = \kappa$. However, $|\mathcal{M}|$ could be any cardinality. We are interested in creating a Vaughtian pair of models with $|N| = \aleph_1$ and $|M| = \aleph_0$ so that we can have a $(\aleph_1, \aleph_0)$ model for T.

Theorem 2.1.11. If T has a (κ, λ)-model where $\kappa > \lambda \geq \aleph_0$, then T has an $(\aleph_1, \aleph_0)$-model.

Proof. Using the previous lemmas, we have a countable Vaughtian pair of isomorphic models $(\mathcal{N}, \mathcal{M})$ as in 2.1.8 and ϕ be the formula as in the definition of Vaughtian pair. We want to build an elementary chain $(\mathcal{N}_\alpha : \alpha < \omega_1)$ such that $N_{\alpha+1} \setminus N_\alpha$ contains no elements satisfying ϕ. This is true as $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha)$ has the same theory as $(\mathcal{N}, \mathcal{M})$, so because $\forall \overline{v}(\phi(\overline{v}) \rightarrow \bigwedge U(v_i))$ is an axiom and $\mathcal{N}_\alpha$'s are elementary extensions of $\mathcal{M}$. Hence, the only elements that have the ϕ property are inside $\mathcal{M}$. Lastly having $\mathcal{N}_\alpha \cong \mathcal{N} \cong \mathcal{M}$ for every α helps us extend to $\mathcal{N}_{\alpha+1}$.

Let $\mathcal{N}_0 = \mathcal{N}$; we want two properties for our chain of models.

1. $\mathcal{N}_\alpha \cong \mathcal{N}$

2. $(\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha) \cong (\mathcal{N}, \mathcal{M})$

- For α a limit ordinal, let $\mathcal{N}_\alpha = \bigcup_{\beta<\alpha} \mathcal{N}_\beta$. Because $\mathcal{N}_\alpha$ is a union of models isomorphic to $\mathcal{N}$, every $\mathcal{N}_\alpha$ is homogeneous and realizes the same types as $\mathcal{N}$. Notice that $\mathcal{N}_\alpha$ is homogeneous because if you take any partial elementary function $f : A \rightarrow B$ with $A, B \subset \mathcal{N}_\alpha$ and finite A, B, then for any $a \in \mathcal{N}_\alpha$, there is $\beta < \alpha$ such that $A, B \subset \mathcal{N}_\beta$ and $a \in \mathcal{N}_\beta$. So using the homogeneity of $\mathcal{N}_\beta$ we can extend the function. The same argument can be used for the types in $S_n(T)$, so $\mathcal{N}_\alpha$ is countable and homogeneous realizing the same types as $\mathcal{N}$, so $\mathcal{N}_\alpha \cong \mathcal{N}$ by 1.8.5.

- For a successor ordinal, given $\mathcal{N}_\alpha \cong \mathcal{N} \cong \mathcal{M}$ we have an isomorphism $f : \mathcal{M} \to \mathcal{N}_\alpha$ we can extend $\mathcal{N}_\alpha$ to $\mathcal{N}_{\alpha+1}$ as we would extend $\mathcal{M}$ to $\mathcal{N}$. The extension of f is an isomorphism $(\mathcal{N}, \mathcal{M}) \cong (\mathcal{N}_{\alpha+1}, \mathcal{N}_\alpha)$ in $\mathcal{L}^*$, so we have $\mathcal{N}_{\alpha+1} \cong \mathcal{N}$ in the $\mathcal{L}$.

Because for every β, $\mathcal{N}_{\beta+1} \setminus \mathcal{N}_\beta$ has no elements with the property ϕ, $\mathcal{N}_\beta$ doesn't have any other such elements than the elements of $\mathcal{M}$.

Finally, the limit of the chain $\mathcal{N}^* = \bigcup_{\alpha<\omega_1} \mathcal{N}_\alpha$ has cardinality $|N^*| = \aleph_1$ the only realizations of ϕ are in M, which is countable, so $\mathcal{N}^*$ is an $(\aleph_1, \aleph_0)$-model. □

Corollary 2.1.12. If T is $\aleph_1$-categorical, then T has no Vaughtian pairs and hence no (κ, λ) models for $\kappa > \lambda \geq \aleph_0$.

The first chain of theorems showed us a "descent" from a (κ, λ) to a $(\aleph_1, \aleph_0)$-model. However, if we add the conditions of ω-stability, then we can increase $\aleph_1$ to any $\kappa > \aleph_1$, obstructing all of the uncountable categoricity with $(\kappa, \aleph_0)$-models.

2.2 Omitting Types on ω-stable Theories

The following Lemma describes a model extension that has the same omitted types.

Lemma 2.2.1. Suppose T is ω-stable, $\mathcal{M} \models T$, and $|M| \geq \aleph_1$. There is a proper elementary extension $\mathcal{N}$ of $\mathcal{M}$ such that if $\Gamma(\bar{w})$ is a countable type over $\mathcal{M}$ realized in $\mathcal{N}$, then $\Gamma(\bar{w})$ is realized in $\mathcal{M}$. By the contrapositive, if $\Gamma(\bar{w})$ is omitted in $\mathcal{M}$, then its is omitted in $\mathcal{N}$.

Proof. 1. There is an $\mathcal{L}_M$-formula $\phi(v)$ such that $|[\phi(v)]| \geq \aleph_1$ and for all $\psi(v) \in \mathcal{L}_M$ either $|[\phi(v) \wedge \psi(v)]| \leq \aleph_0$ or $|[\phi(v) \wedge \neg\psi(v)]| \leq \aleph_0$. Suppose not. Then, for each $\mathcal{L}_M$-formula $\phi(v)$ with $[\phi(v)]$ being uncountable, we can find a formula $\psi(v)$ such that $[\phi(v) \wedge \psi(v)]$ and $[\phi(v) \wedge \neg\psi(v)]$ are both uncountable. Using this fact repeatedly, we can build an infinite countable tree of formulas $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that for all $\sigma \in 2^{<\omega}$:

   - $|[\phi_\sigma]| \geq \aleph_1$;
   - $[\phi_{\sigma,0}] \cap [\phi_{\sigma,1}] = \emptyset$.

   were each branch is a unique countable type, and since we used only countable formulas, each having only finitely many elements of M as parameters, we have a countable $A \subset M$. However $|S_1^M(A)| = 2^{\aleph_0}$ since we built a complete infinite binary tree, contradicting ω-stability.

2. With ϕ as above, we consider the type $p = \{\psi(v) : \psi$ an $\mathcal{L}_M$-formula and $|[\phi(v) \wedge \psi(v)]| \geq \aleph_1\}$ that is the type that its properties have uncountable many realizations in conjunction with ϕ.

- This is a type because it is finitely satisfiable. Take $\psi_1, \dots, \psi_m \in p$ then $\mathcal{M} \models \bigwedge_{i=1}^m \psi_i(v)$, because only countable elements inside $[\phi]$ lack a property ψ_i, so $|[\phi(v) \wedge \bigvee \neg\psi_i(v)]| \leq \aleph_0$. Thus $\gamma(v) = \phi(v) \wedge \bigwedge_{i=1}^m \psi_i(v)$ must have uncountable many elements, being the complement of $\bigvee \neg\psi_i(v)$ inside $[\phi]$.
- It is also a complete type. Take any $\psi \notin p$ then $|[\phi(v) \wedge \psi(v)]| \leq \aleph_0$ so $[\phi(v) \wedge \neg\psi(v)]| \geq \aleph_1$ and $\neg\psi \in p$, assuming otherwise we get that $[\phi]$ has countable realizations.

3. Let $\mathcal{M}'$ be an elementary extension of $\mathcal{M}$ containing c, a realization of p. By 1.8.3, there is $\mathcal{N} \prec \mathcal{M}'$ prime over $M \cup \{c\}$ such that every $\overline{a} \in N$ realizes an isolated type over $M \cup \{c\}$.

4. Let $\Gamma(v)$ be any countable type over M realized by $\bar{b} \in \mathcal{N}$; this is important as these formulas can be defined in $\mathcal{M}$. The type $\Gamma(v)$ is not complete as there are uncountably many formulas with parameters from M, so $\Gamma(v) \subset \text{tp}^{\mathcal{N}}(\bar{b}/M \cup \{c\})$. Let $\mathcal{L}_M$ formula $\theta(\overline{w}, c)$ be the one that isolates $\text{tp}^{\mathcal{N}}(\bar{b}/M \cup \{c\})$. Notice the following $\mathcal{L}_{M \cup \{c\}}$sentences hold:
   - $\mathcal{N} \models \exists \overline{w} \theta(\overline{w}, c)$
   - $\mathcal{N} \models \forall \overline{w}(\theta(\overline{w}, c) \to \gamma(\overline{w}))$, for all $\gamma(\overline{w}) \in \Gamma$.

   these can be viewed as properties of c. Let $\Delta = \{\exists \overline{w} \theta(\overline{w}, v)\} \cup \{\forall \overline{w}(\theta(\overline{w}, v) \to \gamma(\overline{w})) : \gamma \in \Gamma\}$ which is a countable subset of p. We hope to find a realization of Δ in M as this will force $\Gamma(v)$ to be realized.

5. Let $\delta_0(v), \delta_1(v) \dots$ enumerate $\Delta \subset p$, each has only countable non-realizations in ϕ i.e. $|[\phi \wedge \neg\delta_i]| \leq \aleph_0$. The set $\bigcup_{i<\omega}[\phi \wedge \neg\delta_i]$ is a countable union of countable sets and so its complement under $[\phi]$ must be uncountable. This implies that there exist many realizations for Δ in $\mathcal{M}$.

□

Theorem 2.2.2. Suppose that T is ω-stable and there is an $(\aleph_1, \aleph_0)$-model of T. If $\kappa > \aleph_1$, then there is a $(\kappa, \aleph_0)$-model of T.

Proof. Let $\mathcal{M} \models T$ with $|M| \geq \aleph_1$ such that $|\phi(\mathcal{M})| = \aleph_0$ and let $\mathcal{M} \prec \mathcal{N}$ be as in 2.2.1. The type $\Gamma(v) = \{\phi(v)\} \cup \{v \neq m : m \in M \text{ and } \mathcal{M} \models \phi(m)\}$ is a countable type omitted in $\mathcal{M}$ and hence in $\mathcal{N}$. So no elements are added in the extension, $\phi(\mathcal{N}) = \phi(\mathcal{M})$. We can also find such an extension for $\mathcal{N}$ since the previous lemma applies to all uncountable cardinalities. We build an elementary chain $(\mathcal{M}_\alpha : \alpha < \kappa)$ such that $\mathcal{M}_0 = \mathcal{M}$, $\mathcal{M}_{\alpha+1} \neq \mathcal{M}_\alpha$ and for all α, $\phi(\mathcal{M}_\alpha) = \phi(\mathcal{M}_0)$. If $\mathcal{N} = \bigcup_{\alpha<\kappa} \mathcal{M}_\alpha$, then $\mathcal{N}$ is a $(\kappa, \aleph_0)$-model of T. □

2.3 Sequences of Indiscernibles and Skolem Hull

We have proved that if T is κ-categorical for some $\kappa \geq \aleph_1$ and ω-stable, then it has no Vaughtian pair. In the following part, we will prove that every such T is ω-stable. To prove this, we will focus on tuples of elements indistinguishable from each other, i.e., any such tuple satisfies the same formulas.

Definition 2.3.1. Let $(I, <)$ be an ordered set, and let $(x_i : i \in I)$ be a sequence of distinct elements of M. We say that $X = (x_i : i \in I)$ is a sequence of order indiscernibles if whenever $i_1 < i_2 < \ldots < i_m$ and $j_1 < \ldots < j_m$ are two increasing sequences from I, then $\mathcal{M} \models \phi(x_{i_1}, \ldots, x_{i_m}) \leftrightarrow \phi(x_{j_1}, \ldots, x_{j_m})$. We frequently identify X and I.

An important note is that the order $(I, <)$ is not necessarily defined inside a model by a formula ϕ.

Theorem 2.3.2. Let T be a theory with infinite models. For every infinite linear order $(I, <)$, there exists a model $\mathcal{M} \models T$ such that it contains an infinite set of order indiscernibles $(x_i : i \in I)$.

Proof.

1. We expand our vocabulary by adding constants corresponding to the elements of the order $\mathcal{L}' = L \cup \{c_i : i \in I\}$. We also increase our theory to $\Gamma = T \cup \{c_i \neq c_j : i \neq j \in I\} \cup \{\phi(c_{i_1}, \ldots, c_{i_m}) \to \phi(c_{j_1}, \ldots, c_{j_m})\}$, for all $\mathcal{L}$-formulas $\phi(\overline{v})$, where $i_1 < \cdots < i_m$ and $j_1 < \cdots < j_m$ are increasing sequences from I. Notice that in the last set of axioms, the inverse implication is also included as an axiom; this ensures the indiscernibility between every ordered tuple of size m.

2. If we find a model of Γ, the interpretations of $\{c_i : i \in I\}$ are the order indiscernibles we want. Let $\Delta \subset \Gamma$ be a finite set. Let I_0 be the finite subset of I such that if c_i occurs in Δ, then $i \in I_0$ and $\{\phi_i | \ i = 1, 2, \ldots, m\}$ be all the formulas appearing in a $\mathcal{L}'$-sentence, $\phi_i(c_{i_1}, \ldots, c_{i_m}) \to \phi_i(c_{j_1}, \ldots, c_{j_m}) \in \Delta$. We take the $\Delta' \supset \Delta$ to include all sentences that ensure indiscernibility for all tuples of constants $c_i, i \in I_0$ with respect to $\{\phi_i | \ i = 1, 2, \ldots, m\}$ Finally, take $v_1, \ldots, v_n$ as the free variables in all ϕ_i formulas, any model of Δ' is a model of Δ.

3. To find a model of Δ' we take a model $\mathcal{M} \models T$ and a $<$ linear order of its elements. In Δ', we guarantee that $\{c_i : i \in I_0\}$ are satisfying the same formulas from $\{\phi_i | \ i = 1, 2, \ldots, m\}$, but we haven't specified which. We will define a partition $F : [M]^n \to \mathcal{P}(\{1, \ldots, m\})$ that represents all the possible satisfying formulas an ordered tuple can have. If $A = \{a_1, \ldots, a_n\}$ where $a_1 < \ldots < a_n$, then $F(A) = \{i : \mathcal{M} \models \phi_i(a_1, \ldots, a_n)\}$. Because F partitions $[M]^n$ into at most 2^m sets, we can find an infinite $X \subseteq M$ homogeneous for F, using Ramsey's theorem. Let $\eta \subseteq \{1, \ldots, m\}$ such that $F(A) = \eta$ for $A \in [X]^n$. So X is an infinite set indiscernible to $\{\phi_i | \ i = 1, 2, \ldots, m\}$ for the order $<$. From this set, we can find interpretations for $c_i, i \in I_0$ that satisfy Δ' and hence Δ. The fact that X is infinite makes this proof work for every finite I_0.

□

Definition 2.3.3. In a sequence of order indiscernibles $X = (x_i : i \in I)$ in $\mathcal{M}$, every ordered n-tuple has the same complete n-type. The set of all those n-types we call type of the indiscernibles X and write as tp(X).

Theorem 2.3.4. Let T be an $\mathcal{L}$-theory. Suppose that $X = (x_i : i \in I)$ is an infinite sequence of order indiscernibles in $\mathcal{M} \models T$. If $(J, <)$ is any infinite ordered set, we can find $\mathcal{N} \models T$ containing a sequence of order indiscernibles $Y = (y_j : j \in J)$ and tp$(X) =$ tp(Y).

Proof. We expand the vocabulary to $\mathcal{L}^*$ adding constant symbols, c_j for $j \in J$; their interpretation will establish the new sequence of order indiscernibles. We also expand our theory as we did in the previous theorem, but this time we specify the formulas ordered tuples agree on.

$\Gamma = T \cup \{c_i \neq c_j : i, j \in J, i \neq j\} \cup \{\phi(c_{i_1}, \ldots, c_{i_m}) \to \phi(c_{j_1}, \ldots, c_{j_m})\} \cup \{\phi(c_{i_1}, \ldots, c_{i_m}) : i_1 < \ldots < i_m \in J$ and $\phi \in$ tp$(X)\}$.

We will use X as a witness for our Δ finite sub-theories of Γ in a straightforward way. Thus, Γ is satisfiable if $\mathcal{N} \models \Gamma$ then $(y_j : j \in J)$ is the desired sequence. □

Definition 2.3.5. Let T^* be a theory with Skolem functions 1.5.6 and a subset $A \subseteq M$ with $\mathcal{M} \models T^*$. We define $\mathcal{H}(A)$ or the Skolem hull of A to be the substructure generated from A.

Theorem 2.3.6. Let $\mathcal{L}$ be countable, and T be an $\mathcal{L}$-theory with infinite models. For all $\kappa \geq \aleph_0$, there is $\mathcal{M} \models T$ with $|M| = \kappa$ such that if $A \subseteq M$, then $\mathcal{M}$ realizes at most $|A| + \aleph_0$ types in $S_n^{\mathcal{M}}(A)$.

Proof. We will explore only the case of $n = 1$. We consider $\mathcal{L}^*$ and T^* the Skolemization of T. Take a model $\mathcal{N} \models T^*$ with a sequence of order indiscernibles I of order type $(\kappa, <)$ and take $\mathcal{M}$ to be the Skolem hull of I. Since $\mathcal{L}^*$ has countable many functions and constants, the substructure that arises has cardinality at most the cardinality of the finite subsets of $\mathcal{N}$, i.e. equal to κ. To prove this is the desired model, we take $A \subseteq M$ to be the set of parameters. Notice, however, that all the elements of $\mathcal{M}$ are terms generated from I, so each a in A, there is a term t_a and $\overline{x}_a$, a sequence from I such that $a = t_a(\overline{x}_a)$. Let $X = \{x \in I \ : \ x$ occurs in some $\overline{x}_a\}$ be the subset of indiscernibles that generate the parameters. $|A| \leq |X| \leq |A| + \aleph_0$ since every $a \in A$ yields $\overline{x_a}$ which consists of finite many elements. The main idea that follows is that we can reduce any property $\phi(v, \overline{a})$ of $m \in M$ with parameters from $\overline{a} \in A$ to a property $\phi'(\overline{v'})$ about an ordered tuple of indiscernibles.

For $y_1 < \ldots < y_n$ and $z_1 < \ldots < z_n$ tuples of order indiscernibles, we define $\overline{y} \sim_X \overline{z}$ as follows: for each $x \in X$ and each $i \in \{1, \ldots, n\}$, $y_i < x$ iff $z_i < x$ and $y_i = x$ iff $z_i = x$, which translates as $\overline{y}, \overline{z}$ are in the same positions relative to X.

Another element of our analysis is the symmetric formulas of $\phi(v_1, \ldots, v_n)$. Let σ be a permutation of the set $\{1, \ldots, n\}$. We consider ϕ_σ to be the formula ϕ where every appearance of the free variable v_i is substituted by $v_{\sigma(i)}$, i.e.,$\phi_\sigma(v_1, \ldots, v_n) = \phi(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$. The following holds for any σ:

$M \models \phi(\overline{a})$ iff $M \models \phi_\sigma(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$

Claim 2.3.7. Any elements generated by the same Skolem term $m_1 = t(\overline{y}),\ m_2 = t(\overline{z}) \in M$ realize the same type in $S_1^{\mathcal{M}}(A)$ if $\overline{y} \sim_X \overline{z}$.

Let $\psi(\overline{v}, \overline{v}_1, \ldots, \overline{v}_m) = \phi(t(\overline{v}), t_{a_1}(\overline{v}_1), \ldots, t_{a_m}(\overline{v}_m))$ and let $[\overline{y}, \overline{x}_{a_1}, \ldots, \overline{x}_{a_m}]_\sigma$ be the ordered tuple of these elements with respect to $<$ and σ the appropriate permutation.

$$\begin{aligned}
\mathcal{M} \models \phi(t(\overline{y}), a_1, \ldots, a_m) \Leftrightarrow \quad & \mathcal{M} \models \phi(t(\overline{y}), t_{a_1}(\overline{x}_{a_1}), \ldots, t_{a_m}(\overline{x}_{a_m})) && (1)\\
\Leftrightarrow \quad & \mathcal{M} \models \psi(\overline{y}, \overline{x}_{a_1}, \ldots, \overline{x}_{a_m}) && (2)\\
\Leftrightarrow \quad & \mathcal{M} \models \psi_\sigma([\overline{y}, \overline{x}_{a_1}, \ldots, \overline{x}_{a_m}]_\sigma) && (3)\\
\Leftrightarrow \quad & \mathcal{M} \models \psi_\sigma([\overline{z}, \overline{x}_{a_1}, \ldots, \overline{x}_{a_m}]_\sigma) && (4)\\
\Leftrightarrow \quad & \mathcal{M} \models \psi(\overline{z}, \overline{x}_{a_1}, \ldots, \overline{x}_{a_m}) && (5)\\
\Leftrightarrow \quad & \mathcal{M} \models \phi(t(\overline{z}), t_{a_1}(\overline{x}_{a_1}), \ldots, t_{a_m}(\overline{x}_{a_m})) && (6)\\
\Leftrightarrow \quad & \mathcal{M} \models \phi(t(\overline{z}), a_1, \ldots, a_m). && (7)
\end{aligned}$$

From $(2) \Leftrightarrow (3)$, we use the observation above to order both tuples with the same permutation since they have the same order with respect to X, and for $(3) \Leftrightarrow (4)$, we used indiscernibility.

Now $\sim_X$ is an equivalence relationship on the elements of M^n as they are terms made from elements of I, so the maximum number of equivalence classes is, at most, the number of the different placements of n elements relative to X.

We define an upper cut and a lower for $y \in I$ with respect to X to be $U_y = \{x \in X \mid x > y\}$ and $\mathcal{L}_y = \{x \in X \mid x < y\}$ and we say $y \sim_X z$ iff $U_y = U_z$ and $\mathcal{L}_y = L_z$. There are a total of $2|X| + 1$ possible cuts including, $y = x,\ x \in X$, so for any $\overline{y}$ there are $(2|X|+1)^n \leq |X| \leq |A| + \aleph_0$ different placements. □

Theorem 2.3.8. Let T be a complete theory in a countable language with infinite models, and let $\kappa \geq \aleph_1$. If T is κ-categorical, then T is ω-stable.

Proof. For the sake of contradiction, assume that T is not ω-stable, so there is $\mathcal{M} \models T$ and subset $A \subseteq M$ with $|A| = \aleph_0$ and $S_n^{\mathcal{M}}(A) > \aleph_0$. We use the Löwenheim-Skolem with $X = A$ to get a countable elementary submodel $\mathcal{M}_1$ with A included. We know that only countably many types are realized in $\mathcal{M}_1$. Using compactness, we can realize uncountably many types of $S_n^{\mathcal{M}_1}(A)$ in an elementary extension $\mathcal{N}_1$ and $|N_1| = \kappa$. To construct the second model $\mathcal{N}_2$ with $|N_2| = \kappa$, we use the 2.3.6. For each countable $B \subset N_2$, $\mathcal{N}_2$ realizes only countably many types. Let $f : \mathcal{N}_1 \to \mathcal{N}_2$ an $\mathcal{L}$-isomorphism due to κ-categoricity and $B = f(A)$ the image of A. Thus, we can compare the two models using the same set of parameters. Let $\overline{c} \in \mathcal{N}_1$ to realize a type $S_n^{\mathcal{N}_1}(A)$ not realized in $\mathcal{N}_2$. So $\overline{c}$ has a different type than $f(\overline{c})$ over A and $f(A)$, i.e. $\mathcal{N}_1 \models \phi(\overline{c}, \overline{a})$ and i.e. $\mathcal{N}_2 \not\models \phi(f(\overline{c}), f(\overline{a}))$ for some ϕ. This means that f is not an isomorphism, and hence T is not κ-categorical. □

3. STRONGLY MINIMAL SETS

Moving on to the last part of our proof, it is important to take a look back. For the first part, we relied heavily on indiscernibles to prove the first part to the Baldwin-Lachlan characterization 1.6.13. To prove the converse, when our theory has no Vaughtian pairs and is ω-stable, our methods will uncover hidden algebraic structure within every model. This structure will be the point of reference for any model. For any two models, finding a partial isomorphism between these algebraic structures yields an isomorphism between them. Conveniently, these algebraic structures can only be distinguished by cardinal size.

Theorem 3.0.1. If T is ω-stable and has no Vaughtian pairs, then it is κ-categorical any $\kappa \geq \aleph_1$.

3.1 Finding a Strongly Minimal Formula

Definition 3.1.1. If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\phi(\overline{v})$ is an $\mathcal{L}_M$-formula, we will let $\phi(\mathcal{M})$ denote the elements of M that satisfy ϕ. From now on, "definable" means "definable with parameters" unless specified. Let $D \subseteq M^n$ be a definable set with parameters. We are concerned with two notions of minimality:

1. We say that D **is minimal in** $\mathcal{M}$ if, for any definable $Y \subseteq D$, either Y is finite or $D \setminus Y$ is finite. If $\phi(\overline{v}, \overline{a})$ is the formula that defines D, then we also say that $\phi(\overline{v}, \overline{a})$ is minimal. Also, any minimal formula in M is minimal in any elementary substructure that shares the parameters of ϕ.
2. We say that D and ϕ are **strongly minimal** if ϕ is minimal in any elementary extension $\mathcal{N}$ of $\mathcal{M}$.

Strongly minimal formulas are important because they appear in every model of such T.

Lemma 3.1.2. Let T be ω-stable and $\mathcal{M} \models T$, then there is a minimal formula in $\mathcal{M}$.

Proof. Suppose not, without loss of generality, we build a tree of formulas with one variable $(\phi_\sigma : \sigma \in 2^{<\omega})$ such that:

- if $\sigma \subset \tau$, then $\phi_\tau \models \phi_\sigma$;
- $\phi_{\sigma,i} \models \neg\phi_{\sigma,1-i}$;
- $\phi_\sigma(\mathcal{M})$ is infinite.

Let $\phi_\emptyset$ be the starting formula $v = v$. Suppose we have a formula ϕ_σ such that $\phi_\sigma(\mathcal{M})$ is infinite. Because ϕ_σ is not minimal, we can find a formula ψ such that divides $\phi_\sigma(\mathcal{M})$ into two infinite sets. We do this iteratively until we get a complete binary tree where each branch defines a unique countable yet partial type in $S_1^{\mathcal{M}}(A)$ where A is the set of all parameters in the formulas in the tree, which is countable. Here, we have a contradiction as the cardinality of the branches is $2^{\aleph_0} < |S_1^{\mathcal{M}}(A)|$ is while $|A| = \aleph_0$. $\square$

Following, we prove that if T has no Vaughtian pairs, any minimal formula is strongly minimal. This is because when we have Vaughan pairs, we can express "there are infinitely many realizations of $\phi(\overline{v}, \overline{a})$ for any set of parameters."

Lemma 3.1.3. Suppose that T is an $\mathcal{L}$-theory with no Vaughtian pairs. Let $\mathcal{M} \models T$, and let $\phi(v_1, \ldots, v_k, w_1, \ldots, w_m)$ be a formula with parameters from M. There is a number n such that for each $\overline{a} \in M$ and $|\phi(\mathcal{M}, \overline{a})| > n$, then $\phi(\mathcal{M}, \overline{a})$ is infinite.

Proof. For the sake of contradiction we have $\mathcal{M} \models T$ that does not have this property for ϕ, so assume there is no such $n \in \mathbb{N}$. This means that for each n, there is $\overline{a}_n$ in M that $|\phi(\mathcal{M}, \overline{a}_n)| > n$ and not infinite, i.e., there is no maximum size for definable finite sets $\phi(\mathcal{M}, \overline{a})$ for all $a \in M$.

We will use the $\mathcal{L}^* = \mathcal{L} \cup \{U\}$ and T^* we used in 2.1.7. We can briefly describe T^* as T in addition to axioms that say for any model $\mathcal{A}$ of T^* the interpretation $U^{\mathcal{A}}$ is a proper elementary substructure of $\mathcal{A}$. We use the notation $(\mathcal{A}, \mathcal{B})$ for these models to with $\mathcal{A}$ being the model of T^* and $\mathcal{B}$ the interpretation of $U^{\mathcal{A}}$. Let $\mathcal{N}$ be an elementary extension of $\mathcal{M}$, $(\mathcal{N}, \mathcal{M})$ is a model of T^*. Let $\Gamma(\overline{w})$ be the following set; we will prove it is a type in $\mathcal{L}^*$.

1. $U(\overline{w})$;
2. $\exists \overline{v}_1 \ldots \overline{v}_n (\bigwedge_{i \neq j} \overline{v}_i \neq \overline{v}_j \wedge \bigwedge_{i \leq n} \phi(\overline{v}_i, \overline{w}))$ for each $n \in \mathbb{N}$;
3. $\forall \overline{v} \phi(\overline{v}, \overline{w}) \to U(\overline{v})$;

For each finite subset $\Delta \subset T^* \cup \Gamma(\overline{w})$, we have that all sentences of T^* are satisfied by $(\mathcal{N}, \mathcal{M})$, and the finite subset of formulas from $\Gamma(\overline{w})$, describes an element $\overline{w}$ with $U(\overline{w})$ and that all the solutions of $\phi(\overline{v}, \overline{w})$ are in U and are at least k, for some $k \in \mathbb{N}$. For every such $k \in \mathbb{N}$ there is a $a_k \in \mathcal{M}$ such that $|\phi(\mathcal{M}, a_k)| > k$, so $|\phi((\mathcal{N}, \mathcal{M}), a_k)| > k$, which satisfy the first two properties of $\Delta \subset \Gamma(w)$. For the last property, let $b \in (\mathcal{N}, \mathcal{M})$ such that $\phi(b, a_k)$, then $b \in \mathcal{M}$. If assumed otherwise then $|\phi((\mathcal{N}, \mathcal{M}), a_k)| > |\phi(\mathcal{M}, a_k)|$ which is a contradiction because M is an elementary substructure, thus $(\mathcal{N}, \mathcal{M}) \models \psi(a_k) \Leftrightarrow \mathcal{M} \models \psi(a_k)$. So $a_k \in (\mathcal{N}, \mathcal{M})$ witnesses Δ. By compactness $\Gamma(\overline{w})$ is a type, so it is realized in an elementary extension of $(\mathcal{N}, \mathcal{M}) \prec (\mathcal{N}', \mathcal{M}')$. Let $a \in (\mathcal{N}', \mathcal{M}')$ be the element realizing $\Gamma(\overline{w})$, then $\phi((\mathcal{N}', \mathcal{M}'), a) \subset \mathcal{M}' \subset \mathcal{N}'$. Because $\phi((\mathcal{N}', \mathcal{M}'), a)$ is infinite, we have that $(\mathcal{N}', \mathcal{M}')$ is a Vaughtian pair for T.

$\square$

Notice that the n in the previous lemma works for definable sets $\phi(\overline{v}, \overline{b})$ for $\overline{b}$ later introduced in some elementary extension because of the sentences $\forall \overline{w} |\phi(\overline{v}, \overline{w})| \neq k,\ k > n$; We now can express in one sentence whether a definable set is finite or not, rather than having an infinite collection of sentences, more importantly strongly minimal can be expressed as a sentence.

Theorem 3.1.4. If T has no Vaughtian pairs, then any minimal formula is strongly minimal.

Proof. Let $M \models T$ and the minimal a formula $\phi(\overline{v})$ over M, and denote D as the corresponding minimal set. Assume that it is not strongly minimal, which means in an elementary extension $\mathcal{N}$, there is a new definable set A such that $A \cap D$ and $\bar{A} \cap D$ are infinite. To this be the case, the A has to be with parameters form $\overline{b} \in \mathcal{N}$ defined by $\psi(\overline{v}, \overline{b})$. Due to 3.1.3 and because in $\mathcal{M}$, ϕ is minimal, we get the following sentence in T:
$\mathcal{M} \models \forall \overline{w}\, (\, |\psi(\mathcal{M}, \overline{w}) \cap \phi(\mathcal{M})| \leq n_1 \vee \; |\neg\psi(\mathcal{M}, \overline{w}) \cap \phi(\mathcal{M})| \leq n_2) \Rightarrow$
$\mathcal{N} \models \forall \overline{w}\, (\, |\psi(\mathcal{N}, \overline{w}) \cap \phi(\mathcal{N})| \leq n_1 \vee \; |\neg\psi(\mathcal{N}, \overline{w}) \cap \phi(\mathcal{N})| \leq n_2)$. This is a contradiction because $\overline{b} \in \mathcal{N}$ and $\big(\, |\psi(\mathcal{N}, \overline{b}) \cap \phi(\mathcal{N})| > n_1 \wedge \; |\neg\psi(\mathcal{N}, \overline{b}) \cap \phi(\mathcal{N})| > n_2\big)$. □

Corollary 3.1.5. If T is ω-stable and has no Vaughtian pairs, then for any $\mathcal{M} \models T$, there is a strongly minimal formula over $\mathcal{M}$. Since T is ω-stable we have a prime model $\mathcal{M}_0 \prec \mathcal{M}$, take ϕ to be the strongly minimal formula in $\mathcal{M}_0$, defined with parameters $\overline{m}_0 \subset M_0$. Consequently, we can always find a strongly minimal formula with parameters from the prime model of T.

3.2 Algebraic Closure

Now that we have established that every model of T, a theory that is ω-stable and without Vaughtian pairs, has a strongly minimal formula with parameters from M_0, the prime model of T, we will focus on the properties of strongly minimal sets.

Definition 3.2.1. We say an element b is algebraic over A (a set of parameters) if there is a formula $\psi(x, \overline{a})$ with $\overline{a} \in A$ such that $\psi(\mathcal{M}, \overline{a})$ is finite and $\psi(b, \overline{a})$. We also call ψ algebraic formula and $\text{tp}_1^{\mathcal{M}}(b/A)$ algebraic type. Let M be an L-structure, $D \subseteq M$ be a strongly minimal set, and $\phi(v)$ the corresponding formula (possibly defined with parameters). From now on, we will consider algebraic elements only inside D:

$$\begin{aligned} \text{acl}_D(A) &= \{b \in D : b \text{ is algebraic over } A\} \\ &= \bigcup \{\phi(\mathcal{M}) \wedge \psi(\mathcal{M}, \overline{a}) : |\phi(\mathcal{M}) \wedge \psi(\mathcal{M}, \overline{a})| < \aleph_0\} \\ &= \bigcup \{A \subset D : A \text{ is definable and finite}\}. \end{aligned}$$

To give some intuition, will also write b is generated by a set of parameters, meaning that b is algebraic over that set.

Lemma 3.2.2. The following properties hold for any strongly minimal D and $A, B \subseteq D$. We write acl(A, b) for acl$(A \cup \{b\})$.

i) acl(acl(A)) $=$ acl$(A) \supseteq A$. (enlargement)

ii) If $A \subseteq B$, then acl$(A) \subseteq$ acl(B). (containment)

iii) If $a \in$ acl(A), then $a \in$ acl(A_0) for some finite $A_0 \subseteq A$. (finite character)

iv) If $a \in \text{acl}(A, b) \setminus \text{acl}(A)$, then $b \in \text{acl}(A, a)$. (exchange property)

Proof.

iv) So we assume that $a \in \text{acl}(Ab) \setminus \text{acl}(A)$, so this means that a is one of the finite many solutions to a formula that contains b as a parameter, $\phi(x, b)$. There is a sentence that expresses the number of solutions for ϕ. Let $\psi(b)$ be the following sentence

$$|\phi(x, b)| = n$$

We now consider $\psi(v)$, where v is a free variable (x is not free). So, the previous sentence ($\psi(b)$) asserts that the number of solutions for ϕ with b as a parameter is n. If there are finitely many solutions for $\psi(v)$, assume m, there would be a contradiction because $b \in \text{acl}(A)$, and then the following formula has a as a solution and only uses parameters from A:

$$\exists v \phi(x, v) \wedge \psi(v)$$

We will prove that $\phi(a, v) \wedge \psi(v)$ defines a finite subset of D and so $b \in \text{acl}(A)$. We assume it is not, so $G = \phi(a, \mathcal{M}) \wedge \psi(\mathcal{M})$ is cofinitely. We will call this set the set of proper generators of a that generate exactly n elements (one of them is a). So there are finitely many non-proper generators for a, either non-generators or not generating exactly n elements. Without loss of generality, assume the number of non-generators of a is $|\bar{G}| = l$. We can see $|\bar{G}| = l$ as a property of the element a,

$$|\neg\phi(a, \mathcal{M}) \vee \neg\psi(\mathcal{M})| = l.$$

If there are finitely many elements like a with this property, then $a \in \text{acl}(Aa)$. So there must be cofinitely many. Take $n + 1$ of them $a_1, \ldots, a_{n+1}$, each one of them has l non-proper generators denoted as $\bar{G}_{a_i}, 1 \leq i \leq n + 1$. So the $\bigcup \bar{G}_{a_i}$ is finite, take $g \notin \bigcup \bar{G}_{a_i}$ this is a proper generator for all $n + 1$ a_i's. This is a contradiction as it is non-proper because $\psi(g)$ is false. □

Because of the sentence $\mathcal{M} \models |\phi(\mathcal{M}) \wedge \psi(\mathcal{M}, \overline{a})| = n$, there are no new elements satisfying $\phi(v) \wedge \psi(v, \overline{a})$ in any elementary expansion over $A \supseteq a$. This has the following consequences.

Lemma 3.2.3. If p is algebraic over A, then it is isolated.

Proof. Let ψ be the algebraic formula and n be the number of its solutions. If we assume that there are $m > n$ different types that include ψ, then there is an elementary extension $\mathcal{N}$ realizing m solutions to ψ. Contradiction. So there are at most n different types that include ψ. For any of those types, if $q \neq p$, we have that there is a formula that $\psi_i \in p$ and $\psi_i \notin q$. Then the formula $\psi \wedge \bigwedge \psi_i$ isolates p. □

We will define the notion of independence, which generalizes the algebraic independence in algebraically closed fields.

Definition 3.2.4. We say that $A \subseteq D$ is **independent** if $a \notin \text{acl}(A \setminus \{a\})$ for all $a \in A$. If $C \subset D$, we say that A is **independent over a set of parameters** C if $a \notin \text{acl}(C \cup (A \setminus \{a\}))$ for all $a \in A$.

Definition 3.2.5. We say that A is a basis for $Y \subseteq D$ if $A \subseteq Y$ is independent and $\text{acl}(A) = \text{acl}(Y)$. Here, independence might also mean over a set of parameters.

Lemma 3.2.6. If A and B are bases for $Y \subseteq D$, then $|A| = |B|$.

Proof. We will first prove the following claim.

Claim 3.2.7. (Swapping base elements) Suppose $A_0 \subseteq A$ and $B_0 \subseteq B$ are subsets such that $A_0 \cup B_0$ is a basis for D. Then if $a \in A \setminus A_0$, there is some $b \in B_0$ so that $A_0 \cup \{a\} \cup B_0 \setminus \{b\}$ is a basis for D.

Let $B_1 \subseteq B_0$ be of minimal cardinality such that $a \in \text{acl}(A_0 \cup B_1)$. Let $b \in B_1$, because of the minimality of B_1, $a \in \text{acl}(A_0 \cup B_1) \setminus \text{acl}(A_0 \cup B_1 \setminus \{b\})$, because b is essential in constructing the algebraic formula a satisfies. By the exchange principle, we have $b \in \text{acl}(A_0 \cup \{a\} \cup (B_1 \setminus \{b\}))$. We can increase $B_1 \setminus \{b\}$ back to $B_0 \setminus \{b\}$ and have $b \in \text{acl}(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}))$ because all the elements used to create the formula b is a solution are in $(A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$. Since $\text{acl}(Y) = \text{acl}(\text{acl}(Y))$ and $b \in \text{acl}(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}))$ it means that $\text{acl}(A_0 \cup \{a\} \cup (B_0 \setminus \{b\})) = \text{acl}(A_0 \cup B_0) = D$ because $A_0 \cup B_0$ is a basis for D.
We now need to check the independence of $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$. We only need to check the independence of a as all other elements are in the basis $A_0 \cup B_0$. For the sake of contradiction, suppose that $a \in \text{acl}(A_0 \cup (B_0 \setminus \{b\}))$ then $\text{acl}(A_0 \cup (B_0 \setminus \{b\})) = \text{acl}(A_0 \cup \{a\} \cup (B_0 \setminus \{b\}))$ and $b \in \text{acl}(A_0 \cup (B_0 \setminus \{b\}))$, because $A_0 \cup \{a\} \cup (B_0 \setminus \{b\})$ generates b. □

We distinguish two cases:

- if B is finite assuming the following, $|B| < |A|$, we will end in a contradiction. Let $|B| = n$ and $a_1, \ldots, a_{n+1}$ are all distinct elements of A. Let $A_0 = \emptyset$ and $B_0 = B$ we can apply the claim above n times to get that $\{a_1 \ldots, a_n\} \cup (B \setminus \{b_1 \ldots, b_n\}) = \{a_1 \ldots, a_n\}$ has the same span as B, so is a basis for D. But this is a contradiction as $a_{n+1} \in \text{acl(B)}$, so $a_{n+1} \in \text{acl}(\{\text{a}_1, \ldots, \text{a}_\text{n}\})$ contradiction because A is independent. Swap the roles of A and B to get $|A| = |B|$.
- if B is infinite then we can see B as the union of all of its finite subsets $B_0 \subset B$ which are $|B|$ is total. Notice that $\text{acl}(B_0)$ is finite and any $d \in \text{acl}(B)$ holds that $d \in \text{acl}(B_0)$ for some B_0, so $\bigcup_{B_0 \subset B finite} \text{acl}(B_0) = acl(B) = D$ and $A \subseteq D$.

$$|A| \leq |\bigcup_{B_0 \subset B finite} \text{acl}(B_0)|.$$

  This leads to $|A| \leq |B|$, because $|\bigcup_{B_0 \subset B finite}| = |B|$. We can then apply the same proof technique to A to get $|A| = |B|$.

Definition 3.2.8. If $Y \subseteq D$, then the dimension of Y is the cardinality of a basis for Y, denoted dim(Y).

Lemma 3.2.9. If D is uncountable and $\mathcal{L}$ is countable, then dim$(D) = |D|$.

Proof. First, note that dim(D) cannot be more than $|D|$ because any base is a subset of D. Let A be a basis for D and dim$(D) < |D|$ since the language is countable and the finite subsets of A are $|A|$ for each $\phi(\overline{v}, \overline{w})$ formula we can make $|A|$ formulas by inserting constants in the variables $\overline{w}$. So each of the $|A|$, $\mathcal{L}_A$-formulas contributes a finite number of elements to the algebraic closure alc(A)=$|A| \cdot n = |A| < |D|$, so it is not a basis of D. □

Lemma 3.2.10. Let $\mathcal{M}, \mathcal{N} \models T$, and $\phi(v)$ be a strongly minimal formula with parameters from A_0, where $A_0 \subseteq M_0$ where $\mathcal{M}_0 \models T, \mathcal{M}_0 \prec \mathcal{M}$, and $\mathcal{M}_0 \prec \mathcal{N}$. If $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in \phi(\mathcal{M})$ are independent over A_0 and $b_1, \ldots, b_n \in \phi(\mathcal{N})$ are independent over A_0, then tp$^{\mathcal{M}}(\overline{a}/A_0) =$ tp$^{\mathcal{N}}(\overline{b}/A_0)$.

Proof. We will use induction over n, which is the number of independent elements. For $n = 1$, $a_1, b_1 \notin$ acl(A_0), they both realize the same 1-type. Indeed, take $\mathcal{M} \models \psi(a_1) \wedge \phi(a_1)$, we know that $\psi \wedge \phi$ has infinite solutions, so $\neg\psi \wedge \phi$ has finite many. If $\mathcal{M} \models \neg\psi(b_1) \wedge \phi(b_1)$, then $b_1 \in$ acl(A_0), a contradiction.

Assume tp$^{\mathcal{M}}(a_1, \ldots, a_n/A_0) =$ tp$^{\mathcal{N}}(b_1, \ldots, b_n/A_0)$ is true for n, we will show that it is also true for $a_1, \ldots, a_n, a_{n+1}$ and $b_1, \ldots, b_n, b_{n+1}$. Let $\overline{a} = (a_1, \ldots, a_n)$ and take $\mathcal{M} \models \psi(\overline{a}, a_{n+1})$. We can view ψ as a formula with parameters from $\overline{a} \cup A_0$; we will denote it as $\psi_{\overline{a}}$. Because $a_{n+1} \notin$ acl$(A_0, \overline{a})$, $\psi_{\overline{a}}(v) \wedge \phi(v)$ has infinite many realizations hence $\neg\psi_{\overline{a}}(v) \wedge \phi(v)$ is finite. $\mathcal{M} \models |\neg\psi_{\overline{a}}(v) \wedge \phi(v)| = k$ for some $k \in \mathbb{N}$ is a property of the elements $\overline{a}$ over A_0, so we can use the inductive hypothesis that tp$^{\mathcal{M}}(\overline{a}/A_0) =$ tp$^{\mathcal{N}}(\overline{b}/A_0)$ and get $\mathcal{N} \models |\neg\psi_{\overline{b}}(v) \wedge \phi(v)| = k$ as a property of $\overline{b}$. Because ϕ is strongly minimal $\psi_{\overline{b}}(v) \wedge \phi(v)$ is infinite. If $\mathcal{N} \models \neg\psi_{\overline{b}}(b_{n+1}) \wedge \phi(b_{n+1})$ then $b_{n+1} \in$ acl$(A_0, \overline{b})$ a contradiction. So $\mathcal{N} \models \psi_{\overline{b}}(b_{n+1}) \wedge \phi(b_{n+1}) \Rightarrow \mathcal{N} \models \psi(\overline{b}, b_{n+1})$. □

3.3 Extending Partial Isomorphism of Strongly Minimal Sets

Corollary 3.3.1. Let B and C be independent subsets over A_0 of $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$, respectively, with the same cardinality. Any bijection $f : B \cup A_0 \to C \cup A_0$ that fixes A_0 is elementary.

Proof. For the sake of contradiction, assume that $b_1, \ldots, b_n \in B \cup A_0$ and $c_1, \ldots, c_n \in C \cup A_0$ their respective image under f, such that

$$\mathcal{M} \models \psi(b_1, \ldots, b_n) \text{ and } \mathcal{N} \not\models \psi(c_1, \ldots, c_n).$$

Let $I \subseteq \{1, \ldots, n\}$ be such that for each $i \in I$, $b_i \in A_0$. But we can view b_i's as parameters and get

$$\mathcal{M} \models \psi_{A_0}(\overline{b}) \text{ and } \mathcal{N} \not\models \psi_{A_0}(\overline{c})$$

with ψ_{A_0} being a formula with parameters from A_0. This is a contradiction since $\bar{b}$ and $\bar{c}$ are indiscernible[1] over A_0 because of 3.2.10. □

The next theorem will extend an elementary function of the transcendental basis of strongly minimal sets to $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$ to only later be extended to $\hat{f} : \mathcal{M} \to \mathcal{N}$.

Theorem 3.3.2. Let $\mathcal{M}, \mathcal{N} \models T$, and $\phi(v)$ be a strongly minimal formula with parameters from A_0, where $A_0 \subseteq M_0$ where $\mathcal{M}_0 \models T, \mathcal{M}_0 \prec \mathcal{M}$, and $\mathcal{M}_0 \prec \mathcal{N}$. If dim$(\phi(M)) =$ dim$(\phi(N))$, then there is a bijective partial elementary map $g : \phi(\mathcal{M}) \to \phi(\mathcal{N})$.

Proof. Take a B basis for the transcendental subset of $\phi(\mathcal{M})$ and C as a basis for the transcendental subset of $\phi(\mathcal{N})$. These are also bases for $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$ together with A_0, i.e., acl$(B \cup A_0) = \phi(\mathcal{M})$ and the same for C. We can deduce that $|B| = |C|$ since the dimensions of $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$ are equal. Then take $f : B \cup \{A_0\} \to C \cup \{A_0\}$ be any bijection that fixes A_0. Because of 3.3.1, f is a partial elementary function. We will extend f from the bases to the whole strongly minimal sets.

Let

$$I = \{g : B' \to C' : \\ B \cup A_0 \subseteq B' \subseteq \phi(\mathcal{M}),\ C \cup A_0 \subseteq C' \subseteq \phi(\mathcal{N}),\ f \subseteq g \text{ partial elementary}\}.$$

By Zorn's Lemma, there is a maximal $g : B' \to C'$. We will show that $B' = \phi(\mathcal{M})$. Suppose there is $b \in \phi(\mathcal{M}) \setminus B'$, that b is algebraic over $B \cup A_0$, since it is a basis. Let $\psi(v, \overline{d})$ isolating tp$^{\mathcal{M}}(b/B')$ because of 3.2.3 we will find a way to extend the function g by find a pair for b. Notice, because g is elementary, that $\mathcal{M} \models \exists v \psi(v, \overline{d})$ and so $\mathcal{N} \models \exists v \psi(v, g(\overline{d}))$, so there exists an element $\in \mathcal{N}$ that satisfies ψ. Let $c \in \mathcal{N}$ denote that element. It is true that tp$^{\mathcal{M}}(b/B') =$ tp$^{\mathcal{N}}(c/g(B'))$ so $\psi(v, g(\bar{d}))$, is isolating tp$^{\mathcal{M}}(c/g(B'))$. Then $c \in \phi(\mathcal{N})$, as this is one of the properties of b. We can now extend g by sending $b \to c$. This is a contradiction because g is maximal. Thus, we are concluding that $B' = \phi(\mathcal{M})$. The same argument works for $C' = \phi(\mathcal{N})$, because g is one-to-one. So $g : \phi(\mathcal{M}) \to \phi(\mathcal{N})$ is a bijective partial elementary function. □

Now we are ready to prove the second direction of the 1.6.13. We will use the bijective partial elementary function from the previous theorem for any two models of the same cardinality. This function connects their strongly minimal sets; we extend this to a total bijective elementary function between the models using prime models and the lack of Vaughtian pairs.

Theorem 3.3.3. If T is a complete theory in a countable $\mathcal{L}$, which is ω-stable and has no Vaughtian pairs, then it is κ-categorical, for $\kappa \geq \aleph_1$.

[1]Here we use the more general notion of indiscernibles rather than order indiscernibles, we want $\mathcal{M} \models \phi(\overline{a}) \leftrightarrow \phi(\overline{b})$ for all $\overline{a}, \overline{b} \in X$

Proof. Let $\phi(v)$ be the strongly minimal formula with parameters A_0 from M_0, the prime model of T, as in 3.1.5. Let $\mathcal{M}$ and $\mathcal{N}$ be models of T of the same cardinality $\kappa \geq \aleph_1$, due to M_0 being prime $\mathcal{M}_0 \prec \mathcal{M}$ and $\mathcal{M}_0 \prec \mathcal{N}$. Assume that $|\phi(\mathcal{M})| < \kappa$ then we have a (κ, λ)-model and a Vaughtian pair. So $|\phi(\mathcal{M})| = \kappa$ and as explained in 3.2.9 $\dim(\phi(\mathcal{M})) = \dim(\phi(\mathcal{N})) = \kappa$. By 3.3.2, we can find a partial elementary map $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$. Our goal is to extend this to a total elementary map. If we take $X = \phi(\mathcal{M})$ as our parameters, then every model of $Th_X(\mathcal{M})$ contains X. From 1.8.3 let $\mathcal{M}'$ be prime over X. If $M' \subset M$, then because $X = \phi(\mathcal{M})$ is definable and contained in M' there is a Vaughtian pair $(\mathcal{M}', \mathcal{M})$. So $M' = M$, i.e., $\mathcal{M}$ is the prime model over X. Notice that $\mathcal{N} \models Th_X(\mathcal{M})$ as there is the elementary map f between the parameters $X = \phi(\mathcal{M})$ and $\phi(\mathcal{N})$ and we can extend to an elementary $f' : \mathcal{M} \to \mathcal{N}$ because $\mathcal{M}$ is prime. This embedding is surjective. Assume otherwise, then $f'(M) \subset N$ and $\phi(\mathcal{N})$ is contained in $f'(M)$, so $(\mathcal{N}, f'(M))$ is a Vaughtian pair, a contradiction. $\square$

This theorem marked an important milestone in the development of Model Theory and was the start of exciting new directions for the subject. Saharon Shelah built on Morley's work by developing Stability Theory to classify theories based on how tame they are. The reader is advised to look into [3] to expand their knowledge of the results that sprouted after Morley's Theorem. One of the most famous open problems in Model theory is Vaught's Conjecture, which states that any first-order theory in a countable language has finite, $\aleph_0$ or $2^{\aleph_0}$ countable models. Much of Shelah's work has revolved around counting the models of a theory per cardinality. The closest attempt to prove this conjecture in its full generality is actually a theorem of Morley proving that the number of countable models is finite, $\aleph_0$, $\aleph_1$, $2^{\aleph_0}$ which significantly narrows it down to excluding only $\aleph_1$ when the Continuum Hypothesis fails. However, narrowing it down to specific classes has proven true in many cases [4], [5]. In closing, to this day, we don't know the absoluteness of Vaught's Conjecture, meaning it could be independent of set theory, just like the Continuum Hypothesis was.

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