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Pitt's inequality with Stein-Weiss potentials

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Abstract

In this master thesis, we will study the Pitt's inequality and Pitt's inequality with gradient terms. For both of these inequalities we let $p = 2$, the dimension $n \geq 2$ and $0 < a < n$.

In Chapter 1, we define and study the spherical harmonic polynomials of degree k and the solid spherical harmonic polynomials of degree k. Then we define the space $\mathscr{H}^k$ and prove that $L^2(\mathbb{R}^n)$ can be decomposed as the direct sum of these spaces. In addition, we examine the action of the Fourier transform on these spaces. The last section is dedicated to the fractional Laplace operator.

In Chapter 2, we provide a detailed proof of Pitt's inequality and Pitt's inequality with gradient terms. In the proofs of both of these inequalities we will use the theory we developed in chapter 1 and the fact that the Fourier transform is unitary in $L^2(\mathbb{R}^n)$.

In Chapter 3, we compute the optimal constant for Pitt's inequality with gradient terms, considering different range of values for the parameter and the dimension.. We will use extensively the properties of the Gamma function, the digamma function and Stirling's formula for the Gamma function.

In Chapter 4, we will prove logarithmic uncertainty inequalities for both of the inequalities we have studied previously.

Introduction

In this thesis we will study the Pitt's inequality and its extension involving gradient terms. Pitt's inequality provides a pivotal insight into the relationship between the behavior of functions and their Fourier transforms in weighted L^p- spaces , particularly in the context of integrability and decay properties. His inequality forms a bridge between the classical inequalities of Hardy, Sobolev, and others, offering a unique perspective on the interplay between spatial and frequency domains. For the purpose of this thesis we assume that $p = 2$ a choice that simplifies the analysis while retaining significant generality.

Historical Context

In his seminal paper in 1939, Pitt established an inequality that bounded the weighted L^q-norm of a Fourier transform in terms of the weighted L^p -norm of the original function.This result was deeply rooted in the study of integral operators and the behavior of functions in $\mathbb{R}^n$.

Pitt's inequality can be viewed as a specific case of the broader class of fractional integration inequalities studied by Hardy, Littlewood, and Sobolev earlier in the 20th century. These results emphasized the role of weights and exponents in determining the boundedness of integral operators.

Over the decades, Pitt's inequality has been extended to encompass weighted settings involving higher-order differential operators. Among these, generalizations involving the Laplacian operator have received considerable attention, as they naturally arise in the study of Sobolev spaces and PDEs. These versions allow for more nuanced control over the decay and regularity properties of functions.

Scope of the Thesis

In addition to studying the classical Pitt's inequality, we will explore extensions involving gradient terms, which provide additional insight into the behavior of functions with respect to their spatial derivatives. This aspect connects the inequality to modern developments in Sobolev spaces, PDE theory, and functional analysis.

To study these inequalities, we will focus on spherical harmonic polynomials of degree k. Then we will prove that

$$L^2(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$$

where $\mathcal{H}^k$ is the space the closure of the subspace of all linear combinations of functions of the form $f(r)P(x)$, where f is radial and $P(x) = |x|^k Y_k$ where Y_k is spherical harmonic of degree k. Subsequently, we will examine the action of the Fourier transform on this spaces. We will also study the fractional Laplace operator.

Furthermore, we will focus on determining the sharp constant in Pitt's inequality, analyzing its dependence on different parameter values and dimensions. Special attention will be given to understanding how the dimension n and various choices of weights affect the constant, providing insights into its behavior across diverse settings.

Finally, we will prove a logarithmic uncertainty inequality for both the classical and gradient-augmented Pitt's inequalities, establishing a connection between these inequalities and the broader framework of uncertainty principles in analysis.

Chapter 1

Fourier Transform and Spherical Harmonic Functions

In this chapter, we will study the spherical harmonic functions as well as the action of the Fourier transform on them. Finally, we will provide the definition of the fractional Laplace operator and study its connection with the Fourier transform.

1.1 Fourier Transform

In this section, we will study the Fourier transform and some of its fundamental properties. The proofs are omitted as they are elementary.

Definition 1.1.1 *If $f \in L^1(\mathbb{R}^n)$, the Fourier transform of f is the function $\widehat{f}$ defined by letting*

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t) e^{-2\pi i \xi \cdot t} \, dt$$

for all $\xi \in \mathbb{R}^n$.

It is easy to establish the following result

Theorem 1.1.2 *(a) The mapping $f \longmapsto \widehat{f}$ is a bounded linear transformation from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$. In fact, $\|\widehat{f}\|_\infty \leq \|f\|_1$.*
(b) If $f \in L^1(\mathbb{R}^n)$, then $\widehat{f}$ is uniformly continuous.

In addition to the vector-space operations $L^1(\mathbb{R}^n)$ is endowed with a "multiplication" operation. This operation is called convolution.

Definition 1.1.3 *If $f, g \in L^1(\mathbb{R}^n)$, their convolution $h = f * g$ is the function defined by*

$$h(x) = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

We have the following results.

Theorem 1.1.4 *(a) If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$ and $g \in L^1(\mathbb{R}^n)$, then $h = f * g$ is well defined and belongs to $L^p(\mathbb{R}^n)$. Moreover,*

$$\|h\|_p \leq \|f\|_p \|g\|_1$$

(b) If $f, g \in L^1(\mathbb{R}^n)$ then

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

Following this, we shall examine the relationship between the Fourier transform and differentiation. We have the following results.

Theorem 1.1.5 *(a) Suppose $f \in L^1(\mathbb{R}^n)$ and $x_k f \in L^1(\mathbb{R}^n)$, where x_k is the k-th coordinate function. Then $\widehat{f}$ is differentiable with respect to ξ_k and*

$$\frac{\partial \widehat{f}(\xi)}{\partial \xi_k} = (-2\pi i \widehat{x_k f}(x))(\xi)$$

(b) If $f \in L^1(\mathbb{R}^n)$ and there exists the partial derivative of f with respect to x_k then

$$\widehat{f_{x_k}}(\xi) = 2\pi i \xi_k \widehat{f}(\xi)$$

We can extend these results to higher derivatives.
For an n-tuple $a = (a_1, a_2, \cdots, a_n)$ of non-negative integers we let

$$x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

We define

$$D^a = \frac{\partial^{a_1 + a_2 \cdots + a_n}}{\partial x_1^{a_1} \partial x_2^{a_2} \cdots \partial x_n^{a_n}} \tag{1.1}$$

If P is a polynomial in the n-variables $x_1, \cdots, x_n$, then $P(D)$ is the differential operator obtained by replacing x^a by D^a in $P(x)$. Considering the above, we have the following formulas:

$$\textit{(i)}\ (P(D)\widehat{f})(\xi) = (P(\widehat{-2\pi it)}f(t))(\xi)$$
$$\textit{(ii)}\ (\widehat{P(D)}f)(\xi) = P(2\pi i\xi)\widehat{f}(\xi)$$

Proposition 1.1.6 *If $a > 0$, $f \in L^1(\mathbb{R}^n)$ and δ_a denotes a dilation by a then we have*

$$(\delta_a f)(x) = f(ax)$$

The Fourier transform of the dilated function is

$$(\widehat{\delta_a f})(\xi) = a^{-n}\widehat{f}(a^{-1}\xi)$$

Up until this point, the Fourier transform is defined on $L^1(\mathbb{R}^n)$. In order to define it on $L^2(\mathbb{R}^n)$ we will use the following theorem:

Theorem 1.1.7 *If $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, then $\widehat{f} \in L^2(\mathbb{R}^n)$ and $\|\widehat{f}\|_2 = \|f\|_2$*

This theorem asserts that the Fourier transform is a bounded linear operator that maps the dense subset $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ of $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. Therefore, there exists a unique bounded extension, $\mathcal{F}$,of this operator to all of L^2. $\mathcal{F}$ will be called the Fourier Transform on L^2. We will often use the notation $\mathcal{F}f = \widehat{f}$.
In general, if $f \in L^2(\mathbb{R}^n)$, then $\widehat{f}$ is defined as the L^2 limit of the sequence $\{\widehat{h_k}\}_{k\in\mathbb{N}}$, where $\{h_k\}_{k\in\mathbb{N}}$ is any sequence in $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ converging to f in the L^2 norm. We select the sequence $\{h_k\}_{k\in\mathbb{N}}$ as follows: $h_k(t) = f(t)$ if $|t| \le k$ and $h_k(t) = 0$ otherwise. Thus $\widehat{f}$ is the L^2 limit of functions $\widehat{h_k}$ defined by

$$\widehat{h_k}(\xi) = \int_{\mathbb{R}^n} h_k(t)e^{-2\pi i\xi\cdot t}\,dt = \int_{|t|\le k} f(t)e^{-2\pi i\xi\cdot t}\,dt$$

Definition 1.1.8 *A linear operator on $L^2(\mathbb{R}^n)$ that is an isometry and maps onto $L^2(\mathbb{R}^n)$ is called unitary.*

From Theorem 1.1.7 we know that the Fourier transform is isometry. Furthermore, the following properties of the Fourier transform can be easily established.

Theorem 1.1.9 *(a) The Fourier transform is a unitary operator on $L^2(\mathbb{R}^n)$.*

(b) The inverse of the Fourier transform $\mathcal{F}^{-1}$ is given by

$$(\mathcal{F}^{-1}g)(x) = (\mathcal{F}g)(-x)$$

for all $f \in L^2(\mathbb{R}^n)$.

Theorem 1.1.9 is usually referred to as the Plancherel Theorem.
Using the above we can extend Theorem 1.1.4(b) for $L^2(\mathbb{R}^n)$ as follows:

Theorem 1.1.10 *If $f, g \in L^2(\mathbb{R}^n)$ then*

$$\widehat{(f * g)}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$$

for $\xi \in \mathbb{R}^n$.

Next, we provide the following definition:

Definition 1.1.11 *The Schwartz space on $\mathbb{R}^n$ is defined as follows:*

$$\mathcal{S}(\mathbb{R}^n) = \left\{ \phi \mid \phi \in C^\infty(\mathbb{R}^n) \text{ and } \forall\, a, b \in \mathbb{N}_0^n,\ \sup_{x \in \mathbb{R}^n} |x^a (D^b \phi)(x)| < \infty \right\}$$

where $C^\infty(\mathbb{R}^n)$ denotes the space of smooth functions on $\mathbb{R}^n$, and x^a the monomial as defined on (1.1).

The following results can be easily established

Theorem 1.1.12 *(a)If $\phi \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\phi} \in \mathcal{S}(\mathbb{R}^n)$.*
(b)The Fourier Transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself.
(c)If $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, then $\phi * \psi \in \mathcal{S}(\mathbb{R}^n)$.

1.2 Fourier Transform and Radial functions

Definition 1.2.1 *If f is a locally integrable function on $\mathbb{R}^n$, then its radial part is the function ϕ defined by:*

$$\phi(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(rx')\, dx'$$

where $r = |x|$ and $\omega_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ is the area of the surface of the unit sphere S^{n-1}.

It is clear that ϕ is radial.
Now we will focus on radial functions. We observe that a function f defined on $\mathbb{R}^n$ is radial if and only if for all $O \in \mathcal{O}(n)$, where $\mathcal{O}(n)$ denotes the group of orthogonal transformations on $\mathbb{R}^n$, and all $x \in \mathbb{R}^n$ it is true that $f(Ox) = f(x)$. Now we will establish the following result:

Theorem 1.2.2 *The Fourier transform, $\mathcal{F}$, commutes with the orthogonal transformations. This means that if $O \in \mathcal{O}(n)$, let $\mathcal{R}_O$ be the operator defined by*

$$(\mathcal{R}_O f)(x) = f(Ox)$$

for every $x \in \mathbb{R}^n$, then if $f \in L^1(\mathbb{R}^n)$,

$$(\mathcal{F}\mathcal{R}_O f)(\xi) = (\mathcal{R}_O \mathcal{F} f)(\xi)$$

Proof of Theorem 1.2.2. We let $g(x) = f(Ox)$. We have

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} f(Ox) e^{-2\pi i \xi \cdot x}\, dx$$

Because $O \in \mathcal{O}(n)$ and so its adjoint is its inverse and the Jacobian in the change of variables $w = Ox$ is one, we have

$$\widehat{g}(\xi) = \int_{\mathbb{R}^n} f(w) e^{-2\pi \xi \cdot O^{-1} w}\ dw = \int_{\mathbb{R}^n} f(w) e^{-2\pi O\xi \cdot w}\ dw = \widehat{f}(O\xi)$$

□
An immediate consequence of Theorem 1.2.2 is the following

Corollary 1.2.3 *If f is radial on $\mathcal{S}(\mathbb{R}^n)$ then $\widehat{f}$ is also radial.*

Definition 1.2.4 *We define*

$$\mathcal{H}^0 = \{ f \in L^2(\mathbb{R}^n) \mid f \text{ is radial } \}$$

The subspace $\mathcal{H}^0$ is a closed subspace of $L^2(\mathbb{R}^n)$. According to Corollary 1.2.3 $\mathcal{H}^0$ is left invariant by the Fourier Transform, $\mathcal{F}$, on $L^2(\mathbb{R}^n)$.

1.3 Spherical Harmonics

In this section, we will study spherical harmonic functions and their properties. Next, we will introduce a special class of spherical harmonics, known as zonal spherical harmonics. After defining the spaces $\mathcal{H}^k$, we will prove that $L^2(\mathbb{R}^n)$ can be expressed as a direct sum of the spaces $\mathcal{H}^k$. Finally, we will demonstrate that the Fourier transform maps each $\mathcal{H}^k$ into itself.

Definition and Elementary Properties of Spherical Harmonics

Definition 1.3.1 *The restriction to the surface of the unit sphere S^{n-1} of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k.*

We remind the reader that a function f defined on $\mathbb{R}^n$ is called homogeneous of degree k if $f(\lambda x) = \lambda^k f(x)$, for every $x \in \mathbb{R}^n$ and for every $\lambda > 0$.

Definition 1.3.2 *We let $\mathscr{P}_k$ be the set of all homogeneous polynomials of degree k on $\mathbb{R}^n$ with complex coefficients.*

Thus, if $P \in \mathscr{P}_k$ then

$$P(x) = \sum_{|a|=k} c_a x^a$$

where a denotes an n-tuple $a = (a_1, a_2, ..., a_n)$ of non-negative integers, $|a| = \sum_{i=1}^{n} a_i$ and $x^a = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. It is clear that the set of monomials x^a where $|a| = k$ is a basis for the space $\mathscr{P}_k$. The number of such monomials, d_k , and so the dimension of $\mathscr{P}_k$ is the number of ways an n-tuple $a = (a_1, a_2, ..., a_n)$ where $a_i \in \mathbb{N}_0$ for every $i \in \{1, ..., n\}$ can be chosen so that $k = \sum_{i=1}^{n} a_i$.

In order to compute d_k we select $n-1$ boxes out of linearly ordered array of $n+k-1$ boxes and let us fill each of the remaining k boxes with a ball. Then, there will be a_1 balls preceding the first box we chose, a_2 balls in the boxes between the first and the second one we chose and so on until we reach the a_n balls that follow the last box we chose. In this way, we obtain n non-negative integers $a_1, a_2, ..., a_n$ satisfying $k = \sum_{i=1}^{n} a_i$ and all such n-tuples are obtainable in this manner. Thus there are precisely as many such n-tuples as there are ways of selecting $n-1$ boxes out of a collection of $n+k-1$. Consequently, the dimension of $\mathscr{P}_k$ is

$$d_k = \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!k!}$$

Next we introduce an inner product on $\mathscr{P}_k$ by letting

$$\langle P, Q\rangle = P(D)\overline{Q} \tag{1.2}$$

for every $P, Q \in \mathscr{P}_k$, where $P(D)$ is the differential operator defined at Section 1 of this chapter.

Lemma 1.3.3 *The function $\langle\cdot,\cdot\rangle : \mathscr{P}_k \times \mathscr{P}_k \longrightarrow \mathbb{C}$ is an inner product on $\mathscr{P}_k$.*

Proof. Since P and Q are homogeneous polynomials of degree k then $\langle P, Q\rangle$ is scalar valued. In addition, it is clearly linear in the first variable and conjugate linear in the second variable. We need to prove that $\forall P \in \mathscr{P}_k$, $\langle P, P\rangle \geq 0$ and $\langle P, P\rangle = 0$ if and only if $P \equiv 0$.
However, if $(a_1, ..., a_n) \neq (b_1, ..., b_n)$ and $|a| = |b| = k$ then

$$\left(\frac{\partial^{a_1}}{\partial x_1^{a_1}}\frac{\partial^{a_2}}{\partial x_2^{a_2}}\cdots\frac{\partial^{a_n}}{\partial x_n^{a_n}}\right)x_1^{b_1}x_2^{b_2}\cdots x_n^{b_n} = 0$$

and if $a = b$, then this equals $a_1!\,a_2!\cdots a_n! = a!$. Consequently, if

$$P(x) = \sum_{|a|=k} c_a x^a$$

then

$$\langle P, P\rangle = \sum_{|a|=k} |c_a|^2 a! \geqslant 0$$

The last expression vanishes if and only if $c_a = 0$ for every c_a. Lastly, it is easy to establish that $\langle\cdot,\cdot\rangle$ is hermitian symmetric. Thus we have proven that $\langle\cdot,\cdot\rangle$ is an inner product on $\mathscr{P}_k$.
□
Now, we will establish the following theorem.

Theorem 1.3.4 *If $P \in \mathscr{P}_k$ then*

$$P(x) = P_0(x) + |x|^2 P_1(x) + ... + |x|^{2l} P_l(x)$$

where P_j is homogeneous harmonic polynomial of degree $k-2j$, where $j \in \{0,1,...,l\}$.

Proof. Any polynomial of degree less than 2 is harmonic. Thus we assume $k \geq 2$. We consider the mapping $\phi : \mathscr{P}_k \longrightarrow \mathscr{P}_{k-2}$ defined by $\phi(P) = \Delta P$ for $P \in \mathscr{P}_k$ where Δ is the Laplace operator. First, we will prove that ϕ is well defined. If $\phi(\mathscr{P}_k) \nsubseteq \mathscr{P}_{k-2}$ then $\exists Q \in \mathscr{P}_{k-2}$ where $Q \not\equiv 0$ such that Q is orthogonal to the range of ϕ, with respect to the inner product (1.2). That is

$$\overline{\langle \Delta P, Q\rangle} = \langle Q, \Delta P\rangle = 0$$

for every $P \in \mathscr{P}_k$. But if $Q \in \mathscr{P}_{k-2}$ then $P(x) = |x|^2 Q(x) \in \mathscr{P}_k$. Thus

$$0 = \langle Q, \Delta P\rangle = Q(D)\Delta\overline{P}$$

and because P, Q are polynomials, P, Q are smooth functions and so $Q(D), \Delta$ commute. Consequently,

$$0 = \Delta Q(D)\overline{P} = P(D)\overline{P} = \langle P, P\rangle$$

But this cannot be true because $Q \not\equiv 0$.
We let $\mathscr{A}_j \subset \mathscr{P}_j$, $j \geq 2$ to be the class of all polynomials in $\mathscr{P}_j$ that are harmonic. We claim that $\mathscr{P}_j$ is the orthogonal direct sum of $\mathscr{A}_j$ and

$$\mathscr{B}_j = |x|^2 \mathscr{A}_{j-2} = \{P \in \mathscr{P}_j \mid P(x) = |x|^2 Q(x) \textit{with } Q \in \mathscr{P}_{j-2}\}$$

If

$$R(x) = |x|^2 Q(x)$$

with $Q \in \mathscr{P}_{j-2}$, then

$$\langle R, P\rangle = 0$$

for all $Q \in \mathscr{P}_{j-2}$, if and only if

$$Q(D)\Delta\overline{P} = 0$$

for all $Q \in \mathscr{P}_{j-2}$, which is true if and only if

$$\langle Q, \Delta P\rangle = 0$$

for all $Q \in \mathscr{P}_{j-2}$, which in turn is true if and only if

$$\Delta P = 0$$

.

For $j = k$, and $P \in \mathscr{P}_k$, using the result above we have $P(x) = P_0(x)+|x|^2 Q(x)$ where P_0 harmonic and $Q \in \mathscr{P}_{k-2}$. Applying the same result for the space $\mathscr{P}_{k-2}$ we obtain $Q(x) = P_1(x)+|x|^2 Q_1(x)$ where $P_1(x)$ harmonic and $Q_1 \in \mathscr{P}_{k-4}$. Thus $P(x) = P_0(x) + |x|^2 P_1(x) + |x|^4 Q_1(x)$ and by the induction the result is established.

□

A consequence of the Theorem 1.3.4 is :

Corollary 1.3.5 *The restriction to the surface of the unit sphere S^{n-1} of any polynomial of n-variables is a sum of restrictions to S^{n-1} of harmonic polynomials.*

We let $\mathscr{H}_k$ be the space of spherical harmonics of degree k. It is clear that $\mathscr{H}_k$ coincides with the collection of all restrictions on S^{n-1} of the members of $\mathscr{A}_k$.

We consider the restriction Y of $P \in \mathscr{A}_k$. So if $x' \in S^{n-1}$ then $Y(x') = P(x')$. Because P is homogeneous, we have $P(x) = |x|^k Y\left(\frac{x}{|x|}\right)$ for every $x \neq 0$. Thus, the restriction map $P \longrightarrow Y$ has a trivial kernel and therefore must be an isomorphism between $\mathscr{A}_k$ and $\mathscr{H}_k$. Consequently,

$$\begin{aligned}\dim \mathscr{H}_k &= \dim \mathscr{A}_k = \dim \mathscr{P}_k - \dim \mathscr{P}_{k-2} \\ &= d_k - d_{k-2} = \binom{n+k-1}{k} - \binom{n+k-3}{k-2}\end{aligned}$$

for $k \geq 2$.

The space $\mathscr{A}_k$ is the space of solid spherical harmonics of degree k, whereas $\mathscr{H}_k$ is the space of surface spherical harmonics. Now we will begin to study some important properties of spherical harmonics.

Proposition 1.3.6 *The collection of all finite linear combination of elements of $\bigcup\limits_{k=0}^{\infty} \mathscr{H}_k$ is:*

(a) dense in the space of all continuous functions on S^{n-1} with respect to the L^∞ norm.

(b) dense on $L^2(S^{n-1})$.

Proof. Generally, it is well-known that the Stone-Weierstrass theorem can be extended to continuous real-valued functions over an arbitrary Hausdorff and compact topological space X. In particular, because S^{n-1} is compact subset of $\mathbb{R}^n$, this means that if g is continuous on S^{n-1}, we can approximate it uniformly by polynomials restricted on S^{n-1}. But by Corollary 1.3.5 these restricted polynomials are finite linear combination of elements of $\bigcup\limits_{k=0}^{\infty} \mathscr{H}_k$. This establishes (a).

For (b), we know that the space of continuous function is dense on $L^2(S^{n-1})$.Then given an $f \in L^2(S^{n-1})$ and for every $\epsilon > 0$ we can choose a continuous g such that $\|f-g\|_2 < \frac{\epsilon}{2}$. Because of (a) , we can find a finite linear combination of elements of $\bigcup\limits_{k=0}^{\infty} \mathscr{H}_k$, h , such that $\|g-h\|_\infty < \frac{\epsilon}{2\sqrt{\omega_{n-1}}}$ where ω_{n-1} is the area of the surface of the unit sphere. Then,

$$\|g-h\|_2 = \left(\int\limits_{S^{n-1}} |g(x') - h(x')|^2 \, dS(x') \right)^{\frac{1}{2}} \leq \|g-h\|_\infty \sqrt{\omega_{n-1}}$$

Thus by using the above and the triangular inequality we have

$$\|f-h\|_2 \leq \|f-g\|_2 + \|g-h\|_2 \leq \frac{\epsilon}{2} + \|g-h\|_\infty \sqrt{\omega_{n-1}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This establishes (b) and finishes the proof.

□

Proposition 1.3.7 *If Y^k and Y^l are spherical harmonics of degree k and l respectively and $k \neq l$ then*

$$\int\limits_{S^{n-1}} Y^k(x') Y^l(x') \, dS(x') = 0$$

Proof. We recall that for $x \neq 0$ we used the following notations: $r = |x|$ and $x' = \frac{x}{r}$. We define $u(x) = r^k Y^k(x')$ and $v(x) = r^l Y^l(x')$ for $x \neq 0$. If $k, l \neq 0$ we put $u(0) = v(0) = 0$ and if either $k = 0$ or $l = 0$ we consider the corresponding function to be constant. The radial derivative of u and v at $x' \in S^{n-1}$ are

$$\frac{\partial (r^k Y^k(x'))}{\partial \overrightarrow{n}} = \frac{dr^k}{dr} Y^k(x') = k Y^k(x')$$

and

$$\frac{\partial(r^l Y^l(x'))}{\partial \overrightarrow{n}} = \frac{dr^l}{dr} Y^l(x') = l Y^l(x')$$

Furthermore, because Y^k and Y^l are surface spherical harmonics, we conclude that u and v are solid spherical harmonics. By using Green's Theorem we have

$$0 = \int\limits_{B(x,1)} (u\Delta v - v\Delta u)\, dx = \int\limits_{S^{n-1}} (u\frac{\partial v}{\partial \overrightarrow{n}} - v\frac{\partial u}{\partial \overrightarrow{n}})\, dS(x')$$
$$= \int\limits_{S^{n-1}} lY^k(x')Y^l(x') - kY^l(x')Y^k(x')\, dS(x') = (l-k) \int\limits_{S^{n-1}} Y^k(x')Y^l(x') dS(x')$$

and therefore we have proven that

$$\int\limits_{S^{n-1}} Y^k(x')Y^l(x') dS(x') = 0$$

for $l \neq k$.

□

Now, let us consider the space $\mathscr{H}_k$ as a subspace of $L^2(S^{n-1})$ with the usual inner product of L^2:

$$\langle f, g\rangle = \int\limits_{S^{n-1}} f(x')\overline{g(x')}\, dS(x')$$

If $\{Y_1^k, Y_2^k, \cdots, Y_{a_k}^k\}$ where $a_k = \dim \mathscr{H}_k$ is an orthonormal basis of $\mathscr{H}_k$ then from Proposition 1.3.7 follows that $\bigcup\limits_{k=0}^{\infty}\{Y_1^k, Y_2^k, \cdots, Y_{a_k}^k\}$ is an orthonormal basis of $L^2(S^{n-1})$. This follow if we use Proposition 1.3.6:
If $f \in L^2(S^{n-1})$ and $f \neq 0$, orthogonal to all these, then if h is a linear combination of $\bigcup\limits_{k=0}^{\infty}\{Y_1^k, Y_2^k, \cdots, Y_{a_k}^k\}$ we have

$$\|f-h\|_2^2 = \|f\|_2^2 - \langle f, h\rangle - \langle h, f\rangle + \|h\|_2^2 = \|f\|_2^2 + \|h\|_2^2 \geq \|f\|_2^2 > 0$$

which is impossible because by Proposition 1.3.6 we know that $\bigcup\limits_{k=0}^{\infty} \mathscr{H}_k$ is dense on $L^2(S^{n-1})$. Thus, we conclude that if $f \in L^2(S^{n-1})$ there exists a unique representation .

$$f = \sum_{k=0}^{\infty} Y^k \tag{1.3}$$

where the series converges to f in the L^2 norm and $Y^k \in \mathscr{H}_k$ and $Y^k = \sum_{i=0}^{a_k} b_i^k Y_i^k$ where $b_i^k = \langle f, Y_i^k \rangle$, $i \in \{1, \cdots, a_k\}$.

Zonal Harmonics

For a fixed point $x' \in S^{n-1}$, we consider the functional L on $\mathscr{H}_k$ that assigns each element $Y \in \mathscr{H}_k$ to the value $Y(x')$. From the self duality of the inner product space $\mathscr{H}_k$ follows that there exists a unique spherical harmonic $Z_{x'}^k \in \mathscr{H}_k$ such that

$$L(Y) = Y(x') = \int_{S^{n-1}} Y(t') Z_{x'}^k(t')\, dS(t')$$

for every $Y \in \mathscr{H}_k$.

Definition 1.3.8 *The function $Z_{x'}^k$, defined above, is called zonal harmonic of degree k and pole x'.*

In order to study zonal harmonics we need first to prove the following results

Proposition 1.3.9 *(a) If $\{Y_1^k, Y_2^k, \cdots, Y_{a_k}^k\}$ is an orthonormal basis of $\mathscr{H}_k$ then $Z_{x'}^k(t') = \sum_{m=1}^{a_k} \overline{Y_m^k(x')} Y_m^k(t')$.*
(b)$Z_{x'}^k$ is real valued and $Z_{x'}^k(t') = Z_{t'}^k(x')$.
(c) If O is a rotation [1] on $\mathbb{R}^n$ then $Z_{Ox'}^k(Ot') = Z_{x'}^k(t')$

Proof. Because $\mathscr{H}_k$ is a finite dimensional Hilbert Space and $Z_{x'}^k \in \mathscr{H}_k$, if $\{Y_1^k, \cdots, Y_{a_k}^k\}$ is an orthogonal basis, then

$$Z_{x'}^k = \sum_{m=1}^{a_k} \langle Z_{x'}^k, Y_m^k \rangle Y_m^k$$

But, by the definition of zonal harmonics we have

$$\langle Z_{x'}^k, Y_m^k \rangle = \int_{S^{n-1}} Z_{x'}^k(t') \overline{Y_m^k}(t')\, dS(t') = \overline{Y_m^k}(x')$$

[1] We recall that O is a rotation if O is an orthogonal transformation and if $\det O = 1$

and so by combining the above we have established (a).
When we derived the formula for the dimension of $\mathscr{H}_k$, we did not assume that $\mathscr{H}_k$ is complex valued or real valued. Consequently, we can choose an orthonormal basis in (a) that consists of real-valued functions. If we do this, we immediately conclude that $Z^k_{x'}$ is real valued and so (b) is proven.
For part (c), we will use the transformation $w' = Ot'$ on the following integral

$$\int\limits_{S^{n-1}} Z^k_{Ox'}(Ot')Y(t')\,dS(t') = \int\limits_{S^{n-1}} Z^k_{Ox'}(w')Y(O^{-1}w')\,dS(w') = Y(O^{-1}Ox') = Y(x')$$

for every $Y \in \mathscr{H}_k$. Thus by the uniqueness of the representation of linear functionals , we must have $Z^k_{Ox'}(Ot') = Z^k_{x'}(t')$.
□
Using Proposition 1.3.9 we can establish the following results.

Corollary 1.3.10 *(a)* $Z^k_{x'}(x') = a_k\omega^{-1}_{n-1}$*, where* $a_k = \dim \mathscr{H}_k$ *and* ω_{n-1} *is the surface area of* S^{n-1}*.*
(b) $\sum\limits_{m=1}^{a_k} |Y^k_m(x')|^2 = a_k\omega^{-1}_{n-1}$, $\forall x' \in S^{n-1}$*, independently of the orthonormal basis of* $\mathscr{H}_k$*.*
(c) $|Z^k_{x'}(t')| \leq a_k\omega^{-1}_{n-1}$ *for all* x' *and* t' *in* S^{n-1}*.*

Proof. Let $x'_1, x'_2 \in S^{n-1}$. Then we can find a rotation O such that $Ox'_1 = x'_2$. By part (c) of the previous proposition we have

$$Z^k_{x'_1}(x'_1) = Z^k_{x'_2}(x'_2)$$

So we conclude that $Z^k_{x'}(x')$ is a constant, lets say c, for every $x' \in S^{n-1}$. From (a) of the previous proposition we have that

$$c = \sum_{m=1}^{a_k} \overline{Y^k_m(x')}Y^k_m(x') = \sum_{m=1}^{a_k} |Y^k_m(x')|^2$$

where $\{Y^k_1, \cdots, Y^k_{a_k}\}$ is an orthonormal basis of $\mathscr{H}_k$. But then

$$a_k = \sum_{m=1}^{a_k} \int\limits_{S^{n-1}} |Y^k_m(x')|^2\,dS(x') = \int\limits_{S^{n-1}} \sum_{m=1}^{a_k} |Y^k_m(x')|^2\,dS(x') = \int\limits_{S^{n-1}} c\,dS(x') = c\omega_{n-1}$$

Thus,

$$c = a_k\omega^{-1}_{n-1}$$

which proves (a) and (b).
For (c), we observe that from the defining property of the zonal harmonics we have that for all $x', t' \in S^{n-1}$

$$Z^k_{t'}(x') = \int\limits_{S^{n-1}} Z^k_{x'}(w') Z^k_{t'}(w')\, dS(w') \tag{1.4}$$

On the other hand, if $\{Y^k_1, \cdots, Y^k_{a_k}\}$ is an orthonormal basis of $\mathscr{H}_k$, then from Proposition 1.3.9 (a),(b) and the previous result of this corollary we have that for all $u' \in S^{n-1}$:

$$\begin{aligned}\|Z^k_{u'}\|_2^2 &= \int\limits_{S^{n-1}} |Z^k_{u'}(w')|^2\, dS(w') = \int\limits_{S^{n-1}} Z^k_{u'}(w') Z^k_{u'}(w')\, dS(w') \\ &= \int\limits_{S^{n-1}} \left(\sum_{m=1}^{a_k} \overline{Y^k_m(u')} Y^k_m(w')\right)\left(\sum_{m=1}^{a_k} \overline{Y^k_m(w')} Y^k_m(u')\right) dS(w')\end{aligned}$$

where in the last equality we have used Proposition 1.3.9 (a) and (b). Because $\{Y^k_1, \cdots, Y^k_{a_k}\}$ is an orthonormal basis of $\mathscr{H}_k$ we have that if $i \neq j$ then

$$\int\limits_{S^{n-1}} \overline{Y^k_i(u')} Y^k_i(w') \overline{Y^k_j(w')} Y^k_j(u')\, dS(w') = \overline{Y^k_i(u')} Y^k_j(u') \int\limits_{S^{n-1}} Y^k_i(w') \overline{Y^k_j(w')}\, dS(w') = 0$$

and if $i = j$, then

$$\int\limits_{S^{n-1}} \overline{Y^k_i(u')} Y^k_i(w') \overline{Y^k_i(w')} Y^k_i(u')\, dS(w)' = |Y^k_i(u')|^2 \int\limits_{S^{n-1}} Y^k_i(w') \overline{Y^k_i(w')}\, dS(w') = |Y^k_i(u')|^2$$

So we conclude that

$$\|Z^k_{u'}\|_2^2 = \sum_{m=1}^{a_k} |Y^k_m(u')|^2 = a_k \omega_{n-1}$$

Using (1.4), the equality above and the Schwarz inequality we have that

$$|Z^k_{t'}(x')| \leq \|Z^k_{t'}\|_2 \|Z^k_{t'}\|_2 \leq \sqrt{a_k \omega_{n-1}} \sqrt{a_k \omega_{n-1}} = a_k \omega_{n-1}$$

and so (c) has been proven.
□

Definition 1.3.11 *We define a parallel of S^{n-1} orthogonal to a point e in S^{n-1} to be the intersection of the unit sphere with a hyperplane that is perpendicular to the line determined by the origin and e.*

If O is a rotation leaving e fixed then Proposition 1.3.9 gives

$$Z_e^k(x') = Z_e^k(Ox')$$

for all $x' \in S^{n-1}$. But such a rotation O must clearly map a parallel orthogonal to e into itself. Furthermore, given two points, x_1' and x_2' on this parallel there exists a rotation that leaves e fixed such that $Ox_1' = x_2'$. This can be achieved by choosing a rotation O that leaves the orthogonal complement of the plane spanned by x_1' and x_2' fixed. Consequently, the zonal harmonic Z_e^k is constant on the parallels of S^{n-1} orthogonal to e. We shall show that this property characterizes the zonal harmonics up to a constant multiplies. We observe that such functions are invariant under the rotations which leave e fixed.
In order to proceed to the proof of the property described above, we need first to prove the following lemma.

Lemma 1.3.12 *Suppose P is a polynomial on $\mathbb{R}^n$, $n \geq 2$, such that $P(Ox) = P(x)$ for all rotations O and all $x \in \mathbb{R}^n$. Then, there exist constants $c_0, c_1, \cdots, c_m$ such that*

$$P(x) = \sum_{k=0}^{m} c_k (x_1^2 + x_2^2 + \cdots + x_n^2)^k$$

Proof. We write

$$P(x) = \sum_{l=0}^{j} P_l(x)$$

where P_l is homogeneous of degree l. Then for all $\epsilon > 0$ and for all rotations O we have

$$\sum \epsilon^l P_l(x) = P(\epsilon x) = P(\epsilon O x) = \sum_{l=0}^{j} \epsilon^l P_l(Ox)$$

Consequently, we must have $P_l(Ox) = P_l(x)$ for all $l = 0, 1...., j$. If we let $F_l(x) = |x|^{-l} P_l(x)$, then F_l is homogeneous of degree 0, that is invariant under the rotation group. This implies that F_l is constant function : $F_l(x) = c_l$ for all $x \in \mathbb{R}^n$. Thus, $P_l(x) = c_l |x|^l$ for all $l = 0, 1...., j$. Since every P_l is polynomial,

l must be even when $c_l \neq 0$. By combining the above, we conclude that

$$P(x) = \sum_{k=0}^{m} c'_k |x|^{2k}$$

where $c'_k = c_{2k}$ for $k = 0, 1, ..., m$, and m is the largest integer that is less than or equal to $\frac{j}{2}$.

□

Theorem 1.3.13 *Suppose e is a point of S^{n-1}. Then $Y \in \mathscr{H}_k$ is constant on parallels of S^{n-1} orthogonal to e if and only if there exists a constant c such that $Y = cZ_e^k$*

Proof. We have already shown that zonal harmonics have this property. So we only need to show that if $Y \in \mathscr{H}_k$ is constant on parallels of S^{n-1} orthogonal to e then there exists a constant c such that $Y = cZ_e^k$.
Suppose that Y is constant on parallels of S^{n-1} orthogonal to e . If $e_1 = (1, 0, \cdots, 0) \in S^{n-1}$ and O is a rotation satisfying $e = Oe_1$, then the spherical harmonic W, where $W(x') = Y(Ox')$ is constant on parallels of S^{n-1} orthogonal to e_1. If we show that $W = cZ_{e_1}^k$ then Proposition 1.3.9 gives

$$Y(t') = W(O^{-1}t') = cZ_{e_1}^k(O^{-1}t') = cZ_{O^{-1}e}^k(O^{-1}t') = cZ_e^k(t')$$

for all $t' \in S^{n-1}$.
So it suffices to show that $W = cZ_{e_1}^k$. We define $P(x) = |x|^k W\left(\frac{x}{|x|}\right)$ for $x \neq 0$ and $P(0) = 0$ for $x = 0$. If O_1 is a rotation leaving e_1 fixed then $P(O_1 x) = P(x)$ for all $x \in \mathbb{R}^n$. Also all the polynomials of the form x_1^m where $m \in \mathbb{N}_0$ are invariant under the action of O_1. Consequently

$$P(x) = \sum_{j=0}^{k} x_1^{k-j} P_j(x_2, x_3, \cdots, x_n)$$

It follows that $P_0, P_1, ..., P_k$ are invariant under the action of O_1. We observe that $O_1(x_1, x_2, \cdots, x_n) = (x_1, x'_2, \cdots, x'_n)$ and the mapping $(x_2, \cdots, x_n) \to (x'_2, \cdots, x'_n)$ is a rotation on the $(n-1)$ dimensional space $\{x_1 = 0\}$. Moreover, by varying O_1 over all rotations that leave e_1 fixed we obtain all rotations of this $(n-1)$ dimensional space. Thus we can use Lemma 1.3.12 and therefore we conclude that P_j is zero if j is odd and $P_j = c_j(x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ if j is even.

By letting $R(x) = (x_2^2 + \cdots + x_n^2)^{\frac{1}{2}}$ we have shown that

$$P(x) = c_0 x_1^k + c_2 x_1^{k-2} R^2(x) + \cdots + c_{2l} x_1^{k-2l} R^{2l} \tag{1.5}$$

Because W is a surface spherical harmonic, P is a solid spherical harmonic. Thus

$$0 = \Delta P(x) = \sum_{j=0}^{l-1} [c_{2j} a_j + c_{2(j+1)} b_j] x_1^{k-2(j+1)} R^{2j}$$

where $a_j = (k-2j)(k-2j-1)$ and $b_j = 2(j+1)(n+2j-1)$. Hence $c_{2(j+1)} = (-\frac{a_j}{b_j}) c_{2j}$ for $j = 0, 1, \cdots, l-1$. This asserts that all the coefficients are determined by c_0. Consequently,any two zero non zero harmonic polynomials having the form (1.5) must be constant multiples of each other. But we have shown that any homogeneous polynomial of degree k whose restriction to S^{n-1} is constant on parallels of S^{n-1} orthogonal to e_1 must have the form (1.5). Since $|x|^k Z_{e_1}^k \left(\frac{x}{|x|}\right)$ has this property and it is harmonic it follow that

$$W(x') = P(x') = c Z_{e_1}^k (x')$$

for all $x' \in S^{n-1}$.

□

Corollary 1.3.14 *Suppose $F_{y'}(x')$ is defined for all $x', y' \in S^{n-1}$ and that $F_{y'}$ is a spherical harmonic of degree k for all $y' \in S^{n-1}$. Suppose also that for every rotation O we have that $F_{Oy'}(Ox') = F_{y'}(x')$. Then there exists a constant λ such that*

$$F_{y'}(x') = \lambda Z_{y'}^k (x')$$

for all $x', y' \in S^{n-1}$.

Proof. We fix a point $y' \in S^{n-1}$ and let O_1 be a rotation that leaves y' fixed. Then we have that

$$F_{y'}(x') = F_{O_1 y'}(O_1 x') = F_{y'}(O_1 x')$$

for all $x' \in S^{n-1}$. But then because $F_{y'}$ is spherical harmonic of degree k , which is constant on parallels of S^{n-1} orthogonal to y', by using Theorem

1.3.13, we have that there exists a constant $\lambda(y')$ such that

$$F_{y'} = \lambda(y')Z^k_{y'}$$

To complete the proof, we must show that for every $y'_1, y'_2 \in S^{n-1}$ we have $\lambda(y'_1) = \lambda(y'_2)$. To prove that we consider a rotation O_2 such that $O_2y'_1 = y'_2$. But then

$$\lambda(y'_2)Z^k_{y'_2}(O_2x') = F_{y'_2}(O_2x') = F_{O_2y'_1}(O_2x') = F_{y'_1}(x') = \lambda(y'_1)Z^k_{y'_1}(x')$$

On the other hand, by Proposition 1.3.9 (c)

$$Z^k_{y'_1}(x') = Z^k_{O_2y'_1}(O_2x') = Z^k_{y'_2}(O_2x')$$

and so $\lambda(y'_1) = \lambda(y'_2)$.
□

Direct Sum Decomposition

Definition 1.3.15 *We define the space $\mathcal{H}^k$ to be the closure of the subspace of all linear combinations of functions of the form $f(r)P(x)$, where f is radial and P a solid spherical harmonic of degree k, in such way that $f(r)P(x) \in L^2(\mathbb{R}^n)$.*

We observe that for $k = 0$, $\mathcal{H}^0$ is the space of radial function that we defined in a previous section.
Now we will proceed to the main result of this section.

Theorem 1.3.16 *The direct sum decomposition*

$$L^2(\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$$

holds in the sense that:
(a) each subspace $\mathcal{H}^k$ is closed.
(b) the spaces $\mathcal{H}^k$ are mutually orthogonal.
(c) Every element of $L^2(\mathbb{R}^n)$ is a limit of finite linear combinations of elements

belonging to $\mathcal{H}^k$.
In addition, the Fourier transform maps each $\mathcal{H}^k$ into itself.

Proof. For part (a), as we have proven previously on this section the space of solid spherical harmonics of degree k is isomorphic to $\mathscr{H}_k$ and thus has dimension a_k. Let $P_1, P_2, \cdots, P_{a_k}$ be an orthonormal basis of this space. We clarify that the inner product of this space is the inner product inherited by $L^2(S^{n-1})$. Clearly, every element of $\mathcal{H}^k$ can be written in the form $\sum\limits_{j=0}^{a_k} f_j(r)P_j(x)$. Moreover,

$$\int\limits_{\mathbb{R}^n} |f(x)|^2\,dx = \int\limits_0^\infty \int\limits_{S^{n-1}} |\sum_{j=1}^{a_k} f_j(r)P_j(y)|^2 r^{n-1}\,dS(y)\,dr$$

Using the facts that $\{P_1, P_2, \cdots, P_{a_k}\}$ is an orthonormal basis and that the sum is finite, gives

$$\int\limits_{\mathbb{R}^n} |f(x)|^2\,dx = \sum_{j=1}^{a_k} \int\limits_0^\infty |f_j(r)|^2 r^{n-1}\,dr$$

Using the formula above and considering a convergent sequence of elements of $\mathcal{H}^k$ we can easily establish (a).
For part (b), we have proven in Proposition 1.3.7 that two spherical harmonic functions of degree k and l, where $k \neq l$ are orthogonal. This result immediately proves (b): if $f \in \mathcal{H}^k$ and $g \in \mathcal{H}^l$ then

$$f(x) = \sum_{j=1}^{a_k} f_j(r)P_j(x)$$
$$g(x) = \sum_{i=1}^{a_l} g_i(r)P'_i(x)$$

where $\{P_1, P_2, \cdots, P_{a_k}\}$ is a basis of $\mathcal{H}^k$ and $\{P'_1, P'_2, \cdots, P'_{a_l}\}$ is a basis of $\mathcal{H}^l$, we have that

$$\langle f, g\rangle = \int\limits_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx = \int\limits_0^\infty \int\limits_{S^{n-1}} (\sum_{j=1}^{a_k} f_j(r)P_j(y))\overline{(\sum_{i=1}^{a_l} g_i(r)P'_i(y))} r^{n-1}\,dS(y)\,dx$$

and by using Proposition 1.3.7, (b) is established.

For part (c), it suffices to to prove that if a function is orthogonal to every $\mathcal{H}^k$ then it vanishes almost everywhere. But then by (1.3), this function must vanish a.e on almost every sphere centered at the origin and so we have (c). In order to prove that $\mathcal{H}^k$ is left invariant by the Fourier transform, it suffices to prove that if $f \in L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$ and f has the form $f(u) = f_0(t)P(u) = t^k f_0(t)Y(u')$ with $Y \in \mathscr{H}_k$, $t = |u|$ and $u' = \frac{u}{t}$, then $\widehat{f} \in \mathcal{H}^k$. If this is established, because finite linear combinations of functions of this form are dense in $\mathcal{H}^k$ we will have proven the result.
By letting $r = |\xi|$, $\xi = r\xi'$ and using polar coordinates we obtain

$$\widehat{f}(\xi) = \int\limits_{\mathbb{R}^n} e^{-2\pi i \xi \cdot u} f(u)du = \int\limits_0^\infty f_0(t)t^{k+n-1} \left[\int\limits_{S^{n-1}} e^{-2\pi i r t \xi' \cdot u'} Y(u')\, dS(u') \right] dt$$

If we can show that there exists a function ϕ, defined on $[0, \infty)$, such that

$$\int\limits_{S^{n-1}} e^{-2\pi i s \xi' \cdot u'} Y(u')\, dS(u') = \phi(s) Y(\xi') \tag{1.6}$$

then

$$\widehat{f}(\xi) = \left[\int\limits_0^\infty f_0(t)\phi(rt)t^{k+n-1}\, dt \right] Y(\xi')$$

By the defining property of the zonal harmonics $Z^k_{u'}$ and the property (Proposition 1.3.9 part (b)) $Z^k_{u'}(v') = Z^k_{v'}(u')$ we have

$$\begin{aligned} \int\limits_{S^{n-1}} e^{-2\pi i s \xi' \cdot u'} Y(u')\, dS(u') &= \int\limits_{S^{n-1}} e^{-2\pi i s \xi' \cdot u'} \left[\int\limits_{S^{n-1}} Y(v') Z^k_{u'}(v')\, dS(v') \right] dS(u') \\ &= \int\limits_{S^{n-1}} Y(v') \left[\int\limits_{S^{n-1}} e^{-2\pi i s \xi' \cdot u'} Z^k_{v'}(u')\, dS(u') \right] dS(v') \end{aligned}$$

We define

$$F_{\xi'}(v') = \int\limits_{S^{n-1}} e^{-2\pi i s \xi' \cdot u'} Z^k_{v'}(u')\, dS(u')$$

Then an immediate application of Fubini's Theorem and Proposition 1.3.7 give us the fact that $F_{\xi'}$ as a function of $v' \in S^{n-1}$ is orthogonal to all spaces

$\mathscr{H}_j$ where $j \neq k$. But then, (1.3) implies that $F_{\xi'} \in \mathscr{H}_k$. Because $Z^k_{\xi'}$ is invariant under the action of rotations (Proposition 1.3.9 part (c)), then for every rotation O and by the change of variables $u' = Ow'$, we obtain the property

$$F_{O\xi'}(Ov') = F_{\xi'}(v')$$

Thus by Corollary 1.3.14 there exists a constant $\lambda = \phi(s)$ such that

$$F_{\xi'}(v') = \lambda Z^k_{\xi'}(v')$$

for all $\xi', u' \in S^{n-1}$. Consequently,

$$\begin{aligned}\int\limits_{S^{n-1}} e^{-2\pi i s\xi'\cdot u'} Y(u')\,du' = \int\limits_{S^{n-1}} Y(v')F_{\xi'}(v')\,dv' = \int\limits_{S^{n-1}} Y(v')\phi(s)Z^k_{\xi'}(v')\,dv' \\ = \phi(s)Y(\xi')\end{aligned}$$

and thus (1.6) is established.
To conclude the proof, using the fact that $L^1(\mathbb{R}^n) \bigcap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, the result above is extended to all $L^2(\mathbb{R}^n)$.
□

1.4 The Action of the Fourier Transform on the spaces $\mathcal{H}^k$

So far, we have shown that the Fourier transform of a radial function, is a radial function and that every space $\mathcal{H}^k$ is mapped onto itself under the action of the Fourier transform. The main goal of this section is to derive a formula on the action of the Fourier transform on functions that belong on these spaces. In order to achieve that, we define certain Bessel functions.

Definition 1.4.1 *The function*

$$J_k(t) = \frac{1}{2\pi} \int\limits_0^{2\pi} e^{it\sin\theta} e^{-ik\theta}\,d\theta$$

where $k \in \mathbb{Z}$ is called Bessel function of degree k.

We will prove the following result

Lemma 1.4.2 (Poisson Representation of Bessel functions) *If $k \in \mathbb{N}_0$, then*

$$J_k(t) = \frac{(\frac{t}{2})^k}{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\frac{2k-1}{2}}\, ds$$

Proof. We define

$$J_k^*(t) = \frac{(\frac{t}{2})^k}{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int_{-1}^{1} e^{its}(1-s^2)^{\frac{2k-1}{2}}\, ds$$

For $k = 0$:

$$J_0(t) = \frac{1}{2\pi} \int_{0}^{2\pi} e^{it\sin\theta}\, d\theta$$

and

$$J_0^*(t) = \frac{1}{\pi} \int_{-1}^{1} e^{its}(1-s^2)^{-\frac{1}{2}}\, ds$$

In the last integral we change the variables $s = \sin\theta$, and so $ds = \cos\theta d\theta$ and thus

$$J_0^*(t) = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{it\sin\theta}(1-\sin^2\theta)^{\frac{-1}{2}}\cos\theta\, d\theta = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{it\sin\theta}\frac{\cos\theta}{\cos\theta}\, d\theta$$

because $\cos\theta$ is non negative for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. But the function $e^{itsin\theta}$ is periodic and so

$$J_0^*(t) = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{it\sin\theta}\, d\theta = \frac{1}{2\pi} \int_{0}^{2\pi} e^{it\sin\theta}\, d\theta = J_0(t)$$

Thus, to establish the lemma it suffices to prove the recursion relation

$$\frac{d(t^{-k}G_k(t))}{dt} = -t^{-k}G_{k+1}(t) \tag{1.7}$$

where $t \neq 0$ and $k \in \mathbb{N}_0$, for both the sequences $\{J_k\}_k$ and $\{J_k^*\}_k$. But

$$\begin{aligned}
\frac{d(t^{-k}J_k(t))}{dt} &= -t^{-k}\left[\frac{k}{t}J_k(t) - J_k'(t)\right] \\
&= -t^{-k}\left[\frac{k}{2\pi t}\int\limits_0^{2\pi} e^{it\sin\theta}e^{-ik\theta}\,d\theta - \frac{1}{2\pi}\int\limits_0^{2\pi} i\sin\theta e^{it\sin\theta}e^{-ik\theta}\,d\theta\right] \\
&= \frac{-t^{-k}}{2\pi}\left[\int\limits_0^{2\pi}\frac{i}{t}\left(\frac{d(e^{it\sin\theta}e^{-ik\theta})}{d\theta}\right)\cos\theta e^{it\sin\theta}e^{-ik\theta-i\sin\theta}e^{it\sin\theta}e^{-ik\theta}\,d\theta\right] \\
&= -\frac{t^{-k}}{2\pi}\int\limits_0^{2\pi} e^{it\sin\theta}e^{-i(k+1)\theta}\,d\theta = t^{-k}J_{k+1}(t)
\end{aligned}$$

and

$$\frac{d(t^{-k}J_k^*(t))}{dt} = \frac{2^{-k}}{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int\limits_{-1}^{1} ise^{its}(1-s^2)^{\frac{2k-1}{2}}\,ds$$

But integrating by parts gives

$$\int_{-1}^{1} ise^{its}(1-s^2)^{\frac{2k-1}{2}}\,ds = -\frac{1}{2\left(\frac{2k+1}{2}\right)}\left[ie^{its}(1-s^2)^{\frac{2k+1}{2}}\right]_{-1}^{1} + \frac{1}{2\left(\frac{2k+1}{2}\right)}\int_{-1}^{1} i^2te^{its}(1-s^2)^{\frac{2k+1}{2}}\,ds$$

In addition because $x\Gamma(x) = \Gamma(x+1)$ we have that

$$\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{2k+1}{2}\right) = \Gamma\left(\frac{2k+3}{2}\right)$$

So we conclude that

$$\begin{aligned}
\frac{d(t^{-k}J_k^*(t))}{dt} &= -\frac{2^{-k}}{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}\frac{1}{2\left(\frac{2k+1}{2}\right)}\int_{-1}^{1} ite^{its}(1-s^2)^{\frac{2k+1}{2}}\,ds \\
&= -t^{-k-1}\frac{t^{k+1}2^{-k-1}}{\Gamma\left(\frac{2k+3}{2}\right)\Gamma\left(\frac{1}{2}\right)}\int_{-1}^{1} e^{its}(1-s^2)^{\frac{2k+1}{2}}\,ds \\
&= -t^{-k-1}J_{k+1}^*(t)
\end{aligned}$$

and the lemma has been established.

□

Because the integral in the Lemma 1.4.2 is well defined for $k > -\frac{1}{2}$, we can extend the Bessel function J_k for every real number k where $k > -\frac{1}{2}$ by letting

$$J_k(t) = \frac{\left(\frac{t}{2}\right)^k}{\Gamma\left(\frac{2k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)} \int\limits_{-1}^{1} e^{its}(1-s^2)^{\frac{2k-1}{2}}\, ds$$

for $t > 0$. Now we are ready to proceed to the proof of the following Theorem.

Theorem 1.4.3 *Suppose f is a radial function in $\mathcal{S}(\mathbb{R}^n)$, where $n \geq 2$. Thus $f(x) = f_0(|x|)$ for almost every $x \in \mathbb{R}^n$. Then the Fourier transform $\widehat{f}$, is also radial and is given by*

$$\widehat{f}(\xi) = F_0(|\xi|) = F_0(r) = 2\pi r^{-\frac{n-2}{2}} \int\limits_0^\infty f_0(s) J_{\frac{n-2}{2}}(2\pi rs) s^{\frac{n}{2}}\, ds$$

Proof. As we have proven in Corollary 1.2.3 if $f \in \mathcal{S}(\mathbb{R}^n)$ radial then also $\widehat{f}$ is radial. Thus $f(x) = f_0(|x|)$ almost everywhere in $\mathbb{R}^n$ and so $\widehat{f}(\xi) = F_0(|\xi|)$ almost everywhere in $\mathbb{R}^n$. Thus if $r = |\xi|$, $\xi = r\xi'$, $s = |u|$ and $u = su'$ we have

$$\widehat{f}(\xi) = F_0(r) = \int\limits_{\mathbb{R}^n} f(u) e^{-2\pi i \xi \cdot u}\, du = \int_0^\infty f_0(s) \left(\int\limits_{S^{n-1}} e^{-2\pi i r s(\xi' \cdot u')}\, du' \right) s^{n-1}\, ds$$

To evaluate the inner integral we will firstly integrate over the parallel

$$L_\theta = \{u' \in S^{n-1} |\, \xi' \cdot u' = \cos\theta\}$$

which is orthogonal to ξ'. Thus we will obtain a function of θ, where $0 \leq \theta \leq \pi$, which will be integrated over that interval. But then $e^{-2\pi i r s(\xi' \cdot u')} = e^{-2\pi i r s \cos\theta}$ is constant on L_θ and the measure of L_θ is $\omega_{n-2} \sin^{n-2}\theta$. Thus by the change of variable $y = \cos\theta$

$$\begin{aligned}
\int\limits_{S^{n-1}} e^{-2\pi i r s(\xi' \cdot u')}\, du' &= \int_0^\pi e^{-2\pi i r s \cos\theta} \omega_{n-2} \sin^{n-2}\theta\, d\theta \\
&= \omega_{n-2} \int_{-1}^1 e^{-2\pi i r s y}(1-y^2)^{\frac{n-3}{2}}\, dy \\
&= \frac{2\pi^{\frac{n-1}{2}}\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)(\pi r s)^{\frac{n-2}{2}}} J_{\frac{n-2}{2}}(2\pi r s)
\end{aligned}$$

and so

$$\widehat{f}(\xi) = 2\pi r^{-\frac{n-2}{2}} \int\limits_0^\infty f_0(s) J_{\frac{n-2}{2}}(2\pi r s) s^{\frac{n}{2}}\, ds$$

and thus Theorem 1.4.3 is established.

□

Using Theorem 1.4.3 we can compute the action of the Fourier transform on a function $f \in \mathcal{H}^0$. We now turn our attention to the spaces $\mathcal{H}^k$ where $k \geq 1$. But first we will prove the following theorem

Theorem 1.4.4 *Suppose $f(u) = e^{-\pi|u|^2} P(u)$ for all $u \in \mathbb{R}^n$, where $P(u)$ is a solid spherical harmonic of degree k. Then*

$$\widehat{f}(v) = i^{-k} f(v)$$

for all $v \in \mathbb{R}^n$.

Proof. We fix a $t \in \mathbb{R}^n$ and we have

$$\int\limits_{\mathbb{R}^n} e^{-\pi|u|^2} P(u+t)\, du = \int\limits_0^\infty e^{-\pi r^2} r^{n-1} \left(\int\limits_{S^{n-1}} P(t + ru')\, dS(u') \right) dr$$

Since P is harmonic it satisfies the mean-value property and thus

$$\int\limits_{S^{n-1}} P(t + ru')\, du' = \omega_{n-1} P(t)$$

Consequently, using the fact $\int\limits_{S^{n-1}} 1\, du' = \omega_{n-1}$ we have that

$$\int\limits_{\mathbb{R}^n} e^{-\pi|u|^2} P(u+t)\, du = P(t) \int\limits_0^\infty e^{-\pi r^2} r^{n-1} \omega_{n-1}\, dr = P(t) \int\limits_{\mathbb{R}^n} e^{-\pi|u|^2}\, du$$
$$= P(t)$$

Since $P(t) = P(t_1, \cdots, t_n)$ is a polynomial, it has an obvious analytic continuation $P(z) = P(z_1, \cdots, z_n)$ to all of $\mathbb{C}^n$. It then follows by applying of the multidimensional Cauchy's Integral Theorem that

$$\int_{\mathbb{R}^n} e^{-\pi|u|^2} P(u+z)\, du = P(z)$$

for all $z = x + iy = (x_1 + iy_1, \cdots, x_n + iy_n) \in \mathbb{C}^n$. In particular for $z = -iv$ we have

$$\int_{\mathbb{R}^n} e^{-\pi|u|^2} P(u-iv)\, du = P(-iv) = (-i)^k P(v)$$

where last equality is a consequence of the homogeneity of P. But then by change of variables we have

$$\int_{\mathbb{R}^n} e^{-\pi(u+iv)\cdot(u+iv)} P(u)\, du = \int_{\mathbb{R}^n} e^{-\pi x\cdot x} P(x-iv)\, dx = (-i)^k P(v)$$

where

$$(u+iv)\cdot(u+iv) = \sum_{j=1}^{n} (u_j + iv_j)^2$$

Now we multiply the left and the right side by $e^{-\pi|v|^2}$ and we have that

$$\int_{\mathbb{R}^n} e^{-\pi(u+iv)\cdot(u+iv)} e^{-\pi|v|^2} P(u)\, du = (-i)^k f(v)$$

But

$$\begin{aligned}(u+iv)\cdot(u+iv) + |v|^2 &= \sum_{j=1}^{n} (u_j^2 + 2u_j v_j i - v_j^2 + v_j^2) \\ &= \sum_{j=1}^{n} (u_j^2 + 2u_j v_j i) = |u|^2 + 2iu\cdot v\end{aligned}$$

and thus

$$\begin{aligned}(-i)^k f(v) &= \int_{\mathbb{R}^n} e^{-\pi(u+iv)\cdot(u+iv)} e^{-\pi|v|^2} P(u)\, du = \int_{\mathbb{R}^n} e^{-2\pi i u\cdot v} e^{-\pi|u|^2} P(u)\, du \\ &= \int_{\mathbb{R}^n} e^{-2\pi i u\cdot v} f(u)\, du = \widehat{f}(v)\end{aligned}$$

which is the desired equality.

□

In order to complete this section, we will prove the following result. Observe that this Theorem can be extended to $L^2(\mathbb{R}^n)$ by using the density of $L^1(\mathbb{R}^n)\bigcap L^2(\mathbb{R}^n)$ as we discussed previously on this section.

Theorem 1.4.5 *Suppose $n \geq 2$ and $f \in \mathcal{S}(\mathbb{R}^n)$ has the form*

$$f(x) = f_0(|x|)P(x)$$

where P is a solid spherical harmonic of degree k. Then $\widehat{f}$ has the form $\widehat{f}(\xi) = F_0(|\xi|)P(\xi)$ where

$$F_0(r) = 2\pi i^{-\frac{n+2k-2}{2}} \int_0^\infty f_0(s) J_{\frac{n+2k-2}{2}}(2\pi rs) s^{\frac{n+2k}{2}}\, ds = i^{-k}(\mathcal{F}_{n+2k} f_0(r)) \tag{1.8}$$

Proof. To avoid confusion we will denote $\psi(u) = e^{-\pi|u|^2}P(u)$ the function that we have defined in Theorem 1.4.4.
We assume that $a > 0$ and we let

$$g(x) = e^{-\pi|ax|^2}P(x) = a^{-k}e^{-\pi|ax|^2}P(ax) = a^{-k}\psi(ax)$$

where in the second equality we have used the homogeneity of P. If we let δ_a denote the dilation by a it follows from Proposition 1.1.6, Theorem 1.4.4 and the homogeneity of P that

$$\widehat{g}(\xi) = a^{-k}(\widehat{\delta_a\psi})(\xi) = a^{-k}a^{-n}\widehat{\psi}(a^{-1}\xi) = i^{-k}a^{-n-2k}e^{-\pi\frac{|\xi|^2}{a^2}}P(\xi) \tag{1.9}$$

On the other hand, if h is the radial function

$$h(x) = e^{-\pi|ax|^2}$$

for $x \in \mathbb{R}^{n+2k}$ we can easily prove, as one can see in [7], that

$$\widehat{h}(\xi) = a^{-n-2k}e^{-\pi\frac{|\xi|^2}{a^2}} \tag{1.10}$$

In order to interpret (1.9) and (1.10) , let $\mathscr{D}_m$ be the Hilbert Space $L^2((0.\infty); r^{m-1})$, where the norm of an element $f \in \mathscr{D}_m$ is given by

$$\|f\|^2_{\mathscr{D}_m} = \int_0^\infty |f(r)|^2 r^{m-1}\, dr$$

Thus for the space $\mathscr{D}_{n+2k}$ and for $P(x)$ to be a non zero solid spherical harmonic of degree k we consider the function g with $g(x) = \phi(|x|)P(x)$ for $x \in \mathbb{R}^n$. Then, if

$$\|P\|_\Sigma = \left(\int_{S^{n-1}} |P(x')|^2\, dx' \right)^{\frac{1}{2}}$$

and $\phi \in \mathscr{D}_{n+2k}$ we have that

$$\|g\|_2^2 = \int_{\mathbb{R}^n} |\phi(|x|)P(x)|^2\, dx = \left(\int_0^\infty |\phi(r)|^2 r^{n+2k-1}\, dr \right) \|P\|^2_\Sigma < \infty$$

Thus, $g \in L^2(\mathbb{R}^n)$ and by the Plancherel Theorem, $\widehat{g} \in L^2(\mathbb{R}^n)$ with $\|\widehat{g}\|_2 = \|g\|_2$. Using Theorem 1.3.16 and the last equality we conclude that $\widehat{g}(\xi) = v(|\xi|)P(x)$ and thus $\|\phi\|_{\mathscr{D}_{n+2k}} = \|v\|_{\mathscr{D}_{n+2k}}$. Therefore we can define a bounded linear operator T^n_k, on $\mathscr{D}_{n+2k}$ by letting

$$T^n_k \phi = v$$

Because, $\|\phi\|_{\mathscr{D}_m} = \|v\|_{\mathscr{D}_m}$, we have that T^n_k is an isometry. Similarly, by considering a radial function h where $h = \phi(|x|)$ for all $x \in \mathbb{R}^{n+2k}$ we obtain the bounded linear operator T^{n+2k}_0, on $\mathscr{D}_{n+2k}$. This operator is well defined because , by Corollary 1.2.3 we know that the Fourier Transform of a radial function is a radial function or equivalently the Fourier Transform maps $\mathcal{H}^k$ onto itself. Thus, we observe that if h is radial then $\widehat{h}$ is also a radial function lets say $\widehat{h}(\xi) = \theta(|\xi|)$ for almost every $\xi \in \mathbb{R}^n$ and we put

$$T^{n+2k}_0 \phi = \theta$$

and by doing so we obtain the desired bounded linear operator on $\mathscr{D}_{n+2k}$. Then by the Plancherel Theorem we have that T^{n+2k}_0 is an isometry.
Equalities (1.9) and (1.10) show that $T^{n+2k}_0 \phi = i^k T^n_k \phi$, whenever $\phi(r) = e^{-\epsilon r^2}$,

where $\epsilon > 0$. Thus, T_0^{n+2k} and $i^k T_k^n$ agree when they are restricted to the space $\mathscr{W}$ of finite linear combinations of all such functions ϕ obtained by varying ϵ throughout the positive real numbers. By using this conclusion if we have that $\mathscr{W}$ is dense on the Hilbert Space $\mathscr{D}_{n+2k}$, we will conclude that $T_0^{n+2k}\phi = i^k T_k^n \phi$ for all $\phi \in \mathscr{D}_{n+2k}$. To prove this we assume that $\mathscr{W}$ is not dense on the $\mathscr{D}_{n+2k}$. Then by the characterization of the Hilbert Spaces there exists a $b \in \mathscr{D}_{n+2k}$ which is not equal almost everywhere to 0 such that

$$\int_0^\infty \phi(r) b(r) r^{n+2k-1}\, dr = 0$$

for all $\phi \in \mathscr{W}$. In fact, for all $\epsilon > 0$ we have that

$$\int_0^\infty e^{-\epsilon r^2} b(r) r^{n+2k-1}\, dr = 0 \tag{1.11}$$

We define

$$\Phi(s) = \int_0^s e^{-r^2} b(r) r^{n+2k-1}\, dr$$

for $s \geq 0$. Φ is obviously differentiable,with

$$\Phi'(r) = e^{-r^2} b(r) r^{n+2k-1}$$

and thus by letting $\epsilon = m+1$, where $m \in \mathbb{N}$ and integrating by parts (1.11) we have that

$$0 = \int_0^\infty e^{-mr^2} \Phi'(r)\, dr = [e^{-ms^2}\Phi(s)]_0^\infty - \int_0^\infty (-2mr) e^{-mr^2} \Phi(r)\, dr$$

$$= 2m \int_0^\infty e^{-mr^2} \Phi(r) r\, dr$$

because $[e^{-ms^2}\Phi(s)]_0^\infty = 0$. Thus by the change of variables $u = e^{-r^2}$ we obtain

$$0 = -m \int_0^1 u^{m-1} \Phi\left(\sqrt{\ln \frac{1}{u}}\right) du \text{ , and so}$$

$$0 = \int_0^1 u^{m-1} \Phi\left(\sqrt{\ln \frac{1}{u}}\right) du$$

and because $m \in \mathbb{N}$ was arbitrary, the equality above is true for every $m \in \mathbb{N}$. Since the polynomials are uniformly dense in the space of continuous functions on the closed interval $[0, 1]$, this can only be true if $\Phi\left(\sqrt{\ln \frac{1}{u}}\right) = 0$ and thus

$$\Phi'(r) = e^{-r^2} b(r) r^{n+2k-1} = 0$$

for almost every $r \in (0, \infty)$ and this is contradiction to the hypothesis that $b(r)$ is not almost everywhere 0.
Since the operators T_0^{n+2k} and $i^k T_k^n$ are bounded and agree on the dense subspace $\mathscr{W}$ they must be equal on $\mathscr{D}_{n+2k}$. Thus we have proven that

$$T_0^{n+2k}\phi = i^k T_k^n \phi \tag{1.12}$$

for all $\phi \in \mathscr{D}_{n+2k}$. In Theorem 1.4.3 we expressed the Fourier Transform of a radial function in terms of an integral involving a Bessel function. Equalities (1.6) (1.12) show that the Fourier Transform of a function in $\mathcal{H}^k$ of the form $f_0(|x|)P(x)$, where $P(x)$ is a solid harmonic of degree k on $\mathbb{R}^n$, can be expressed in terms of the Fourier transform of the radial function whose values are $h(y) = f_0(|y|)$ for $y \in \mathbb{R}^{n+2k}$. Thus we have proven that if $|\xi| = r$ we have that

$$\widehat{f}(\xi) = 2\pi i^{-\frac{n+2k-2}{2}} \int_0^\infty f_0(s) J_{\frac{n+2k-2}{2}(2\pi rs)} s^{\frac{n+2k}{2}} P(\xi)\, ds = i^{-k} (\mathcal{F}_{n+2k} f_0(r)) P(\xi)$$

and thus we obtain the desired result.
□
Theorem 1.4.5, with the fact that the spaces $\mathcal{H}^k$ are spanned by functions of the form $f_0(|x|)P(x)$ gives us the promised formula of the action of the Fourier Transform on $\mathcal{H}^k$.

1.5 Haar Measure

We consider the group $(\mathbb{R}_+, \cdot)$ where $\mathbb{R}_+ = (0, \infty)$. The group $(\mathbb{R}_+, \cdot)$ is a locally compact Hausdorff group if we restrict the topology of $\mathbb{R}$ to $\mathbb{R}_+$. The Haar measure on this group is

$$\mu(A) = \int\limits_A \frac{dt}{t}$$

for $A \subset \mathbb{R}_+$ where dt is the Lebesgue measure on $\mathbb{R}$. The proof of this assertion is elementary. For example we will prove that μ is left-translation invariant. Lets assume that $\kappa > 0$ is arbitrary and $A \subset \mathbb{R}_+$. Then for the subset κA, by the change of variables $t = \kappa\sigma$ we obtain

$$\mu(\kappa A) = \int\limits_{\kappa A} \frac{dt}{t} = \int\limits_{\kappa A} \frac{\kappa d\sigma}{\kappa\sigma} = \mu(A)$$

and thus we have the desired result. For every unimodular non-compact Lie group, and thus for the group $(\mathbb{R}_+, \cdot)$, the Young inequality holds :

$$\|g * f\|_{L^p(\mathbb{R}_+)} = \| \ g\|_{L^1(\mathbb{R}_+)} \|f\|_{L^p(\mathbb{R}_+)}$$

for $1 < p < \infty$. The spaces $L^1(R_+)$ and $L^p(\mathbb{R}_+)$ are with respect to the the Haar measure of $(\mathbb{R}_+, \cdot)$. Furthermore, if g is non-negative , this inequality is sharp with no extremal functions [3].

1.6 Some Useful Computations

We define the function

$$\Phi_a = |x|^{-a}$$

Using Theorem 1.3.4 and the properties of the Bessel functions we can easily prove that

$$\mathcal{F}[\Phi_a] = C_{n,a}\Phi_{n-a}$$

for some constant $C_{n,a}$. Indeed one can see at [2] that this equality holds and that

$$C_{n,a} = \frac{\pi^{-n/2+a}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}$$

Next we will prove the following formula :

$$\Phi_\beta * \Phi_\delta = \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{n-\delta}{2}\right)\Gamma\left(\frac{\beta+\delta-n}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{2n-\beta-\delta}{2}\right)} \right] \Phi_{\beta+\delta-n} \tag{1.13}$$

for $0 < \beta < n$, $0 < \delta < n$ and $n < \beta + \delta < 2n$.
In order to prove this we will use the Theorem 1.1.9(B) and the fact that Φ_a is radial And so we have :

$$\mathcal{F}^{-1}[\mathcal{F}[|\Phi_\beta * \Phi_\delta]] = \Phi_\beta * \Phi_\delta$$

First we compute :

$$\mathcal{F}[\Phi_\beta * \Phi_\delta] = \mathcal{F}[\Phi_\beta]\mathcal{F}[\Phi_\delta] = \pi^{-n+\beta+\delta} \left[\frac{\Gamma\left(\frac{n-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \frac{\Gamma\left(\frac{n-\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right] \Phi_{2n-\beta-\delta}$$

and by defining $C = \pi^{-n+\beta+\delta} \left[\frac{\Gamma\left(\frac{n-\beta}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)} \frac{\Gamma\left(\frac{n-\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right]$, we have:

$$\begin{aligned}
\mathcal{F}^{-1}\left[\mathcal{F}\left[\Phi_\beta * \Phi_\delta\right]\right] &= \mathcal{F}^{-1}\left[C\Phi_{\beta+\delta-n}\right] = C\mathcal{F}^{-1}\left[\Phi_{\beta+\delta-n}\right] \\
&= C\pi^{\frac{3}{2}-\beta-\delta} \left[\frac{\Gamma\left(\frac{\beta+\delta-n}{2}\right)}{\Gamma\left(\frac{2n-\beta-\delta}{2}\right)} \right] \Phi_{\beta+\delta-n} \\
&= \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{n-\beta}{2}\right)\Gamma\left(\frac{n-\delta}{2}\right)\Gamma\left(\frac{\beta+\delta-n}{2}\right)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\delta}{2}\right)\Gamma\left(\frac{2n-\beta-\delta}{2}\right)} \right] \Phi_{\beta+\delta-n}
\end{aligned}$$

which is the desired result.

1.7 Fractional Laplace Operator

In this section we will study the fractional Laplace operator

Definition 1.7.1 *For every $f \in \mathcal{S}(\mathbb{R}^n)$ and $s \in (0,1)$ we define*

$$(-\Delta)^s f(x) = C(n,s) PV \int\limits_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dy = C(n,s) \lim_{\epsilon \to 0^+} \int\limits_{\partial B(x,\epsilon)} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dS(y) \tag{1.14}$$

and

$$C(n,s) = \left(\int\limits_{\mathbb{R}^n} \frac{1-\cos\zeta_1}{|\zeta|^{n+2s}}\, d\zeta \right)^{-1} \tag{1.15}$$

where ζ_1 is the first component of ζ.

Remark 1.7.2 *Because of the singularity of the kernel, the right hand side of* (1.14) *is not well defined in general. On the other hand, if $s \in (0, \frac{1}{2})$ then the integral is not really singular near x*

Indeed for $f \in \mathcal{S}(\mathbb{R}^n)$ by the Taylor expansion we have

$$f(x) - f(y) = -\nabla f(x)\cdot(y-x) + O(|y-x|^2)$$

Therefore

$$\begin{aligned} PV \int\limits_{B(x,1)} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dy &= \lim_{\epsilon\to 0^+} \int\limits_{B(x,1)\setminus B(x,\epsilon)} \frac{f(x)-f(y)}{|x-y|^{n+2s}}\, dy \\ &= \lim_{\epsilon\to 0^+} \int\limits_{B(x,1)\setminus B(x,\epsilon)} \frac{-\nabla f(x)\cdot(y-x) + O(|y-x|^2)}{|x-y|^{n+2s}}\, dy \\ &= \lim_{\epsilon\to 0^+} \int\limits_{B(x,1)\setminus B(x,\epsilon)} \frac{O(|y-x|^2)}{|x-y|^{n+2s}}\, dy \\ &< C \lim_{\epsilon\to 0^+} \int\limits_{B(x,1)\setminus B(x,\epsilon)} \frac{1}{|x-y|^{n+2s-2}}\, dy < \infty \end{aligned}$$

where C is a positive constant. About the third equality, since $\nabla f(x)$ is fixed, for every $y_1 \in \partial B(x,r)$ there exists a corresponding $y_2 \in \partial B(x,r)$ such that

$$\nabla f(x)\cdot(y_1 - x) = -\nabla f(x)\cdot(y_2 - x)$$

Hence

$$\int\limits_{B(x,1)\setminus B(x,\epsilon)} -\frac{\nabla f(x)\cdot(y-x)}{|x-y|^{n+2s}}\, dy = 0$$

and thus the third equality holds.

Lemma 1.7.3 *Let $s \in (0,1)$ and $(-\Delta)^s$ be the fractional Laplacian operator defined by* (1.14). *Then for every $f \in \mathcal{S}(\mathbb{R}^n)$ we have*

$$(-\Delta)^s f(x) = -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{f(x+y)+f(x-y)-2f(x)}{|y|^{n+2s}}\,dy \tag{1.16}$$

for every $x \in \mathbb{R}^n$

Proof. We use the transformations $z = x - y$ and we have

$$(-\Delta)^s f(x) = -C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x+z)-f(x)}{|z|^{n+2s}}\,dz$$

We substitute $\overline{z} = -z$ and we have

$$(-\Delta)^s f(x) = -C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x-\overline{z})-f(x)}{|\overline{z}|^{n+2s}}\,d\overline{z}$$

and after we relabel $\overline{z}$ as z we get

$$(-\Delta)^s f(x) = -C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x-z)-f(x)}{|z|^{n+2s}}\,dz$$

Therefore

$$\begin{aligned}
2(-\Delta)^s f(x) &= -C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x+z)-f(x)}{|z|^{n+2s}}\,dz - C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x-z)-f(x)}{|z|^{n+2s}}\,dz \\
&= C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x+z)+f(x-z)-2f(x)}{|z|^{n+2s}}\,dz
\end{aligned}$$

Therefore

$$(-\Delta)^s f(x) = -\frac{1}{2}C(n,s)PV\int\limits_{\mathbb{R}^n} \frac{f(x+z)+f(x-z)-2f(x)}{|z|^{n+2s}}\,dz$$

We can use the above representation to remove the singularity of the integral at the origin. For any smooth function f , a second order Taylor expansion yields

$$\frac{f(x+y)+f(x-y)-2f(x)}{|y|^{n+2s}} \leq \frac{\|D^2 f\|_{L^\infty(\mathbb{R}^n)}}{|y|^{n+2s}}$$

which is integrable near 0 for any fixed $s \in (0,1)$.
Therefore, since $f \in \mathcal{S}(\mathbb{R}^n)$, we can remove the PV and write (1.16) . Thus the lemma is established.
□

Theorem 1.7.4 *Let $s \in (0,1)$ and $(-\Delta)^s : \mathcal{S}(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the fractional Laplacian operator as defined by* (1.14)*. Then for every $f \in \mathcal{S}(\mathbb{R}^n)$*

$$(-\Delta)^s f = (2\pi)^{2s} \mathcal{F}^{-1}(|\xi|^{2s}(\mathcal{F}f)) \tag{1.17}$$

for every $\xi \in \mathbb{R}^n$.

Proof. We define the integral operator

$$\mathcal{L}f(x) = -\frac{1}{2}C(n,s) \int\limits_{\mathbb{R}^n} \frac{f(x+y) + f(x-y) - 2f(x)}{|y|^{n+2s}}\, dy$$

where $C(n,s)$ is the constant we defined in (1.15). As one can easily see $\mathcal{L}$ is a linear operator. We are searching for a function $g : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathcal{L}f = \mathcal{F}^{-1}[\ g(\mathcal{F}f)] \tag{1.18}$$

We want to prove that

$$g(\xi) = |\xi|^{2s} \tag{1.19}$$

Similarly to the computations we have done above we obtain

$$\frac{|f(x+y) + f(x-y) - 2f(x)|}{|y|^{n+2s}} \le 4(\chi_{B(y,1)}|y|^{2-n-2s} \sup_{B(x,1)} |D^2 f|$$
$$+ \chi_{\mathbb{R}^n \setminus B(y,1)}|y|^{-n-2s}(|f(x+y) + f(x-y) - 2f(x)|)) \in L^1(\mathbb{R}^{2n})$$

Using this inequality the Fubini Theorem, the linearity of the Fourier transform and the Fourier transform of the translation of a function we have

$$\begin{aligned}\mathcal{F}[(\mathcal{L}f)] &= -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{\mathcal{F}[f(x+y)+f(x-y)-2f(x)]}{|y^{n+2s}|}\,dy \\ &= -\frac{1}{2}C(n,s)\int\limits_{\mathbb{R}^n} \frac{e^{2\pi i\xi\cdot y}+e^{-2\pi i\xi\cdot y}-2}{|y^{n+2s}|}\,dy\ \mathcal{F}[f](\xi) \\ &= C(n,s)\int\limits_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy\ \mathcal{F}[f](\xi)\end{aligned}$$

Hence, it suffices to prove that

$$\int\limits_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy = (2\pi)^{2s}C(n,s)^{-1}|\xi|^{2s}$$

We observe that for $\zeta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in \mathbb{R}^n$ and ζ near 0 we have

$$\frac{1-\cos\zeta_1}{|\zeta|^{n+2s}} \le \frac{|\zeta_1|^2}{|\zeta|^{n+2s}} \le \frac{1}{|\zeta|^{n+2s-2}}$$

Thus

$$0 < \int\limits_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy < \infty \tag{1.20}$$

We define the function $I : \mathbb{R}^n \to \mathbb{R}$ with

$$I(\xi) = \int\limits_{\mathbb{R}^n} \frac{1-\cos(2\pi\xi\cdot y)}{|y|^{n+2s}}\,dy$$

We have that I is invariant under the action of a rotation, that is

$$I(\xi) = I(|\xi|e_1) \tag{1.21}$$

where e_1 denotes the first directional vector in $\mathbb{R}^n$. Indeed for $n = 1$ we have

$$I(-\xi) = I(\xi)$$

For $n \geq 2$ we consider a rotation O for which

$$O|\xi|e_1 = \xi$$

We consider O^T to be the transpose of O. We let $z = O^T y$ and we obtain

$$\begin{aligned} I(\xi) &= \int\limits_{\mathbb{R}^n} \frac{1 - \cos(2\pi(O|\xi|e_1) \cdot y)}{|y|^{n+2s}}\, dy = \int\limits_{\mathbb{R}^n} \frac{1 - \cos(2\pi|\xi|e_1 \cdot O^T y)}{|y|^{n+2s}}\, dy \\ &= \int\limits_{\mathbb{R}^n} \frac{1 - \cos(2\pi|\xi|e_1 \cdot z)}{|z|^{n+2s}}\, dz = I(|\xi|e_1) \end{aligned}$$

Therefore, by combining (1.20) and (1.21), using the transformation $\zeta = 2\pi|\xi|y$ and the fact that $e_1 \cdot y = y_1$ we obtain

$$\begin{aligned} I(\xi) = I(|\xi|e_1) &= \int\limits_{\mathbb{R}^n} \frac{1 - \cos(2\pi|\xi|y_1)}{|y|^{n+2s}}\, dy = \frac{1}{(2\pi|\xi|)^n} \int\limits_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{\left|\frac{\zeta}{2\pi|\xi|}\right|^{n+2s}}\, d\zeta \\ &= (2\pi)^{2s} C(n,s)^{-1} |\xi|^{2s} \end{aligned}$$

And therefore the proof is complete.
□
Using the formula of the theorem above we have for $s = \frac{a}{4}$ that

$$(-\Delta)^{\frac{a}{4}} f = (2\pi)^{\frac{a}{2}} \mathcal{F}^{-1}(|\xi|^{\frac{a}{2}}(\mathcal{F}f))$$

Therefore, we obtain

$$\int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{a}{4}} f|^2\, dy = (2\pi)^a \int\limits_{\mathbb{R}^n} |\mathcal{F}^{-1}(|\xi|^{\frac{a}{2}}(\mathcal{F}f))|^2 dy$$

We use the Plancherel Theorem, Theorem 1.19 and the change of variables $\xi = -y$ in the right-hand side we obtain

$$\frac{1}{(2\pi)^a} \int\limits_{\mathbb{R}^n} |(-\Delta)^{\frac{a}{4}} f|^2\, dy = \int\limits_{\mathbb{R}^n} |\xi|^a |\widehat{f}(\xi)|^2\, d\xi \tag{1.22}$$

Chapter 2

Pitt's Inequalities

In this chapter we will study Pitt's Inequality :

$$\int_{\mathbb{R}^n} \Phi_a(x)|f(x)|^2\,dx \le C_a \int_{\mathbb{R}^n} \Phi_{-a}(\xi)|\widehat{f}(\xi)|^2\,d\xi \tag{2.1}$$

and Pitt's Inequality with gradient terms

$$\int_{\mathbb{R}^n} \Phi_a(x)|\nabla f(x)|^2\,dx \le 4\pi^2 D_a \int_{\mathbb{R}^n} \Phi_{-a-2}(\xi)|\xi|^2|\widehat{f}(y)|^2\,d\xi \tag{2.2}$$

For both of those inequalities we assume that $f \in \mathcal{S}(\mathbb{R}^n)$, $n \le 2$ and $0 < a < n$. The function Φ_a is the function we defined on Section 1.6

$$\Phi_a(x) = |x|^{-a}$$

Note that from now on, whenever we integrate over the unit sphere S^{n-1} we will use mean value integral. This means that

$$⨍_{S^{n-1}} dS'(\xi) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} dS(\xi)$$

where $dS(\xi)$ it the surface measure and ω_{n-1} the surface area of S^{n-1}. It is evident that this does not affect significantly the computations of this chapter which are based on the results of the Sections 1.2-1.4.

2.1 Pitt's Inequality

We will prove the following theorem.

Theorem 2.1.1 (Pitt's Inequality) *Let $n \geq 2$, $0 \leq a < n$ and $f \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\int\limits_{\mathbb{R}^n} |x|^{-a} |f(x)|^2 \, dx \leq C_a \int\limits_{\mathbb{R}^n} |y|^a |\widehat{f}(y)|^2 \, dy \tag{2.3}$$

where

$$C_a = \pi^a \left[\frac{\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n+a}{4}\right)} \right]^2$$

One proof can be found in the paper by Yafaev: "Sharp Constants in the Hardy-Rellich Inequalities" (see [9]). An alternative proof can be obtained by observing that the Pitt inequality is equivalent to the following inequality:

Theorem 2.1.2 *Let $n \geq 2$, $0 \leq a < n$ and $g \in \mathcal{S}(\mathbb{R}^n)$, then*

$$\left| \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \overline{g}(x) \frac{1}{|x|^{a/2}} \frac{1}{|x-y|^{n-a}} \frac{1}{|y|^{a/2}} g(y) \, dx \, dy \right| \leq B_a \int\limits_{\mathbb{R}^n} |g(x)|^2 \, dx \tag{2.4}$$

where

$$B_a = \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] \left[\frac{\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n+a}{4}\right)} \right]^2$$

Before we prove the Theorems 2.1.1 and 2.1.2, we will illustrate that (2.3) and (2.4) are equivalent.

Lemma 2.1.3 *Let $n \geq 2$, $0 \leq a < n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. Then for $g(\xi) = |\xi|^{a/2} \widehat{f}(\xi)$ we have*

(i) $$\int\limits_{\mathbb{R}^n} |x|^{-a} |f(x)|^2 \, dx = \left| \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \overline{g}(x) \frac{1}{|x|^{a/2}} \frac{1}{|x-y|^{n-a}} \frac{1}{|y|^{a/2}} g(y) \, dx \, dy \right|$$

and

(ii) $$\int\limits_{\mathbb{R}^n} |y|^a |\widehat{f}(y)|^2 \, dy = \int\limits_{\mathbb{R}^n} |g(x)|^2 \, dx$$

Proof. As we know from Section 1.6 we have that

$$\mathcal{F}[|x|^{-a}] = C_{n,a} |x|^{a-n}$$

where

$$C_{n,a} = \frac{\pi^{-n/2+a}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}$$

Using these and Theorem 1.1.9 we obtain

$$\int_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx = \int_{\mathbb{R}^n} |x|^{-a}f(x)\cdot f(x)\,dx = \int_{\mathbb{R}^n} (\widehat{\Phi_a f})(\xi)\,\overline{\widehat{f}}(\xi)\,d\xi$$

We know that

$$\widehat{(\Phi_a f)}(\xi) = \int_{\mathbb{R}^n} \widehat{\Phi_a}(\xi-\eta)\widehat{f}(\eta)\,d\eta$$

and thus we obtain:

$$\begin{aligned}\int_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \widehat{\Phi_a}(\xi-\eta)\widehat{f}(\eta)\overline{\widehat{f}}(\xi)\,d\eta\,d\xi \\ &= C_{n,a}\int_{\mathbb{R}^n\times\mathbb{R}^n} \widehat{f}(\eta)\,\frac{1}{|\xi-\eta|^{n-a}}\overline{\widehat{f}}(\xi)\,d\eta\,d\xi\end{aligned}$$

Using the fact that

$$\int_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx \geq 0$$

we have

$$\int_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx = C_{n,a}\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \widehat{f}(\eta)\,\frac{1}{|\xi-\eta|^{n-a}}\overline{\widehat{f}}(\xi)\,d\eta\,d\xi\right|$$

Because $f \in \mathcal{S}(\mathbb{R}^n)$, by Theorem 1.1.12 part (B) we have that for $g(\xi) = |\xi|^{a/2}\widehat{f}(\xi)$ it is true that $g \in \mathcal{S}(\mathbb{R}^n)$. Hence

$$\int_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx = C_{n,a}\left|\int_{\mathbb{R}^n\times\mathbb{R}^n} \overline{g}(\xi)\frac{1}{|\xi|^{a/2}}\frac{1}{|\xi-\eta|^{n-a}}\frac{1}{|\eta|^{a/2}}g(\eta)\,d\eta\,d\xi\right|$$

Similarly the right-hand side of (2.3) transforms into :

$$C_a\int_{\mathbb{R}^n} |y|^a|\widehat{f}(y)|^2\,dy = C_a\int_{\mathbb{R}^n} |g(y)|^2\,dy$$

Note that $B_a = C_a C_{n,a}^{-1}$ Therefore if we set $\xi = x$ and $\eta = y$ and multiply by $C_{n,a}^{-1}$, inequality (2.3) becomes:

$$\left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \overline{g}(x)\frac{1}{|x|^{a/2}}\frac{1}{|x-y|^{n-a}}\frac{1}{|y|^{a/2}}g(y)\,dx\,dy\right| \leq B_a \int\limits_{\mathbb{R}^n} |g(x)|^2\,dx$$

where

$$B_a = \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\right] \left[\frac{\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n+a}{4}\right)}\right]^2$$

Thus we conclude that (2.3) and (2.4) are equivalent.

□

Now we are ready to proceed to the proof of the theorems.

Proof of Theorems 2.1.1 and 2.1.2. The proof will be based on Young's inequality for the group $(R_+,\cdot)$.The spaces $L^1(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+)$ are considered with respect to the Haar measure of $(\mathbb{R}_+,\cdot)$, and therefore Young's inequality can be applied.

Step 1. We assume that f is radial. Thus by Corollary 1.2.3 $\widehat{f}$ is also radial. Therefore, $g(\xi) = |\xi|^{a/2}\widehat{f}(\xi)$ is radial. We set $t = |x|$, $|y| = s$ and $h(t) = t^{\frac{n}{2}}g(x)$. We use the fact that then the left- hand side of the inequality of (2.4) transforms into:

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \overline{g}(x)\frac{1}{|x|^{a/2}}\frac{1}{|x-y|^{n-a}}\frac{1}{|y|^{a/2}}g(y)\,dx\,dy = \int\limits_{\mathbb{R}_+\times\mathbb{R}_+} \overline{h}(t)\frac{1}{t^{\frac{n+a}{2}}}\phi\left(\frac{t}{s}\right)\frac{1}{(ts)^{\frac{n-a}{2}}}\frac{1}{s^{\frac{n+a}{2}}} \times h(s)s^{n-1}t^{n-1}dsdt$$

where

$$\phi(t) = \fint_{S^{n-1}} \left[t+\frac{1}{t}-2\xi_1\right]^{\frac{-(n-a)}{2}} dS(\xi)$$

where $dS(\xi)$ denotes the surface measure and ξ_1 is the first component of ξ. The function ϕ follows from the fact that:

$$|x-y|^{n-a} = (|x|^2+|y|^2-2x\cdot y)^{\frac{n-a}{2}} = |x|^{\frac{n-a}{2}}|y|^{\frac{n-a}{2}}\left(\frac{|x|}{|y|}+\frac{|y|}{|x|}-2\frac{x\cdot y}{|x||y|}\right)^{\frac{n-a}{2}}$$
$$= t^{\frac{n-a}{2}}s^{\frac{n-a}{2}}\left(\frac{t}{s}+\frac{s}{t}-2\xi_1\right)^{\frac{n-a}{2}}$$

We observe that ϕ is non-negative on $\mathbb{R}_+$ and that it is symmetric, $\phi(t) = \phi\left(\frac{1}{t}\right)$. Thus we have

$$\left| \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \overline{g}(x) \frac{1}{|x|^{a/2}} \frac{1}{|x-y|^{n-a}} \frac{1}{|y|^{a/2}} g(y)\, dx\, dy \right| = \left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} \overline{h}(t) \phi\left(\frac{t}{s}\right) h(s) \frac{ds}{s} \frac{dt}{t} \right|$$

Similarly on the right-hand side of (2.4) we have

$$B_a \int\limits_{\mathbb{R}^n} |g(x)|^2\, dx = C_a \left[\frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{a}{2}\right)}{2\pi^a \Gamma\left(\frac{n-a}{2}\right)} \right] \int\limits_{\mathbb{R}_+} |h(t)|^2 \frac{dt}{t}$$

Using the Cauchy Schwarz inequality and the fact that $\|\overline{h}\|_{L^2(\mathbb{R}^n)} = \|h\|_{L^2(\mathbb{R}^n)}$ we obtain

$$\left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} \overline{h}(t) \phi\left(\frac{t}{s}\right) h(s) \frac{ds}{s} \frac{dt}{t} \right| \leq \int\limits_{\mathbb{R}_+} |h(t)| |\phi * h| \frac{dt}{t} \leq \|h\|_{L^2(\mathbb{R}_+)} \|\phi * h\|_{L^2(\mathbb{R}_+)}$$

And therefore, using Young's inequality for the convolution, we have that

$$\left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} \overline{h}(t) \phi\left(\frac{t}{s}\right) h(s) \frac{ds}{s} \frac{dt}{t} \right| \leq \|h\|^2_{L^2(\mathbb{R}_+)} \|\phi\|_{L^1(\mathbb{R}_+)}$$

Therefore it suffices to show that

$$\|\phi\|_{L^1(\mathbb{R}_+)} = C_a \left[\frac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{a}{2}\right)}{2\pi^a \Gamma\left(\frac{n-a}{2}\right)} \right]$$

Using the co-area formula [6], and taking $x \in S^{n-1}$, it follows that:

$$\|\phi\|_{L^1(\mathbb{R}_+)} = \int\limits_0^\infty \left[\fint_{S^{n-1}} \left[t + \frac{1}{t} - 2\xi_1 \right]^{\frac{-(n-a)}{2}} d\xi \right] \frac{dt}{t} = \left[\frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \right] \int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^{n-a}} \frac{1}{|y|^{\frac{n+a}{2}}}\, dy$$

We recall that

$$\omega_{n-1} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

We recall that when we defined the function ϕ we had $dS(\xi)$ was the normalized the surface measure. In order to compute the integral above, we observe that

$$\int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^{n-a}} \frac{1}{|y|^{\frac{n+a}{2}}}\, dy = \Phi_{n-a} * \Phi_{\frac{n+a}{2}}$$

One can easily prove that $0 < n-a < n$, $0 < \frac{n+a}{2} < n$ and that $n < n-a+\frac{n+a}{2} < 2n$. Thus from (1.13) we obtain,

$$\Phi_{n-a} * \Phi_{\frac{n+a}{2}} = \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n+a}{4}\right)} \right] \Phi_{\frac{n-a}{2}}$$

Taking into account that $|x| = 1$ it follows that

$$\begin{aligned} \|\phi\|_{L^1(\mathbb{R}_+)} &= \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{n/2}} \pi^{\frac{n}{2}} \left[\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n+a}{4}\right)} \right] = \frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n-a}{2}\right)} \left[\frac{\Gamma\left(\frac{n-a}{4}\right)}{\Gamma\left(\frac{n+a}{4}\right)} \right]^2 \\ &= C_a \frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n}{2}\right)}{2\pi^a \Gamma\left(\frac{n-a}{2}\right)} \end{aligned}$$

Therefore, (2.4) holds for f radial and thus we obtain the desired result (2.3) in this case, due to the equivalence we demonstrated previously.

Step 2. For $f \in \mathcal{S}(\mathbb{R}^n)$ as we have shown in the previous chapter we can assume that

$$f(x) = \sum_{k=0}^{\infty} f_k(|x|) P_k(x)$$

where f_k is radial , P_k is a harmonic polynomial of degree k and thus:

$$P_k(x) = |x|^k Y_k(\xi), \quad \xi = \frac{x}{|x|}, \quad \fint_{S^{n-1}} |Y_k(\xi)|^2\, dS(\xi) = \frac{\omega_{n-1+2k}}{\omega_{n-1}}$$

where Y_k is spherical harmonic of degree k, ω_m the surface area of the unit sphere S^m and $dS(\xi)$ denotes the normalized surface measure on S^{n-1}. Then we shall prove

$$\int\limits_{\mathbb{R}^n} |f(x)|^2\, dx = \sum_{k=0}^{\infty} \int\limits_{\mathbb{R}^{n+2k}} |f_k(|x|)|^2 dx \tag{2.5}$$

Indeed, using Theorem 1.3.16 and the Beppo Levi Theorem we have

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |f(x)|^2\,dx &= \int\limits_{\mathbb{R}^n} \left|\sum_{k=0}^{\infty} f_k(|x|)P_k(x)\right|^2 dx = \int\limits_{\mathbb{R}^n} \sum_{k=0}^{\infty} |f_k(|x|)P_k(x)|^2\,dx \\ &= \omega_{n-1}\sum_{k=0}^{\infty}\int\limits_0^{\infty} \fint_{S^{n-1}} |f_k(r)P_k(r\xi)|^2 r^{n-1}\,dS(\xi)\,dr \\ &= \omega_{n-1}\sum_{k=0}^{\infty}\int\limits_0^{\infty} \fint_{S^{n-1}} |f_k(r)|^2 r^{n+2k-1}|Y_k(\xi)|^2\,dS(\xi)\,dr \\ &= \omega_{n-1}\sum_{k=0}^{\infty}\int\limits_0^{\infty} |f_k(r)|^2 r^{n+2k-1}\frac{\omega_{n-1+2k}}{\omega_{n-1}}\,dr \\ &= \omega_{n+2k-1}\sum_{k=0}^{\infty}\int\limits_0^{\infty} |f_k(r)|^2 r^{n+2k-1}\,dr = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^{n+2k}} |f_k(|x|)|^2\,dx\end{aligned}$$

We denote the Fourier Transform on $\mathbb{R}^n$ by $\mathcal{F}_n$. Thus by Theorem 1.4.5 we obtain

$$\mathcal{F}_n[f_k(|x|)P_k(x)] = i^{-k}\mathcal{F}_{n+2k}[f_k(|x|)]P_k(x) \tag{2.6}$$

Therefore the right-hand side of (2.3) changes into

$$\int\limits_{\mathbb{R}^n} |y|^a|\widehat{f}(y)|^2\,dy = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^{n+2k}} |y|^a|\widehat{f}_k(|y|)|^2\,dy \tag{2.7}$$

Indeed, again by using Theorem 1.3.16 and the Beppo Levi Theorem, we compute

$$\begin{aligned}\int\limits_{\mathbb{R}^n} |y|^a|\widehat{f}(y)|^2\,dy &= \int\limits_{\mathbb{R}^n} |y|^a\left|\mathcal{F}_n\left[\sum_{k=0}^{\infty} f_kP_k\right](y)\right|^2 dy = \int\limits_{\mathbb{R}^n} |y|^a\left|\sum_{k=0}^{\infty}\mathcal{F}_{n+2k}[f_k](|y|)P_k(y)\right|^2 dy \\ &= \int\limits_0^{\infty}\fint_{S^{n-1}} r^a\sum_{k=0}^{\infty}|\widehat{f}_k(r)P_k(r\xi)|^2 r^{n-1}dS(\xi)dr \\ &= \sum_{k=0}^{\infty}\int\limits_0^{\infty} r^a|\widehat{f}_k(r)|^2 r^{n+2k-1}\,dr = \sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^{n+2k}} |y|^a|\widehat{f}_k(|y|)|^2\,dy\end{aligned}$$

Similarly, the left-hand side of (2.3) transforms into

$$\int\limits_{\mathbb{R}^n} |f(x)|^2 |x|^{-a}\, dx = \sum_{k=0}^{\infty} \int\limits_{\mathbb{R}^{n+2k}} |f_k(|x|) P_k(x)|^2 |x|^{-a}\, dx = \sum_{k=0}^{\infty} \int\limits_{\mathbb{R}^{n+2k}} |f_k(|x|)|^2 |x|^{-a}\, dx$$

and using the result of Step 1 of the proof $\forall k \in \mathbb{N}$ we obtain :

$$\int\limits_{\mathbb{R}^{n+2k}} |f_k(|x|)|^2 |x|^{-a}\, dx \le C_a \int\limits_{\mathbb{R}^{n+2k}} |y|^a |\widehat{f_k}(|y|)|^2\, dy$$

Hence we have that

$$\int\limits_{\mathbb{R}^n} |f(x)|^2 |x|^{-a}\, dx \le C_a \sum_{k=0}^{\infty} \int\limits_{\mathbb{R}^{n+2k}} |y|^a |\widehat{f_k}(|y|)|^2\, dy = C_a \int\limits_{\mathbb{R}^n} |y|^a |\widehat{f}(y)|^2\, dy$$

which is the desired result.
□

2.2 Pitt's Inequality with gradient terms

The inequality that we will prove in this subsection is the following

Theorem 2.2.1 *Let $n \ge 2$, $0 < a < n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We have*

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx \le 4\pi^2 D_a \int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2 |y|^{a+2}\, dy \tag{2.8}$$

where

$$D_a = \pi^a \max_{k \in \mathbb{N}_0} \left\{ \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)} \right]^2 \left(1 + \frac{4ka}{(n+2k-a-2)^2} \right) \right\}$$

Before we prove Theorem 2.2.1 we will prove the following

Lemma 2.2.2 *Let $n \ge 2$, $0 < a < n$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We have*

(i) $$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = \left| \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{g(x)}{|x|^{\frac{a}{2}}} \frac{x \cdot y}{|x||y|} \frac{1}{|x-y|^{n-a}} \frac{\overline{g}(y)}{|y|^{\frac{a}{2}}}\, dy\, dx \right|$$

(ii) $$4\pi^2 \int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2 |y|^{a+2}\, dy = \left[\pi^{\frac{n}{2}-a} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\right] \int\limits_{\mathbb{R}^n} |g(x)|^2\, dx$$

where

$$g(\xi) = \widehat{f}(\xi)|\xi|^{\frac{a+2}{2}}$$

Proof. As we know from Subsection 1.6 we have that

$$\mathcal{F}[|x|^{-a}] = C_{n,a}|x|^{a-n}$$

where $C_{n,a} = \frac{\pi^{-n/2+a}\Gamma\left(\frac{n-a}{2}\right)}{\Gamma\left(\frac{a}{2}\right)}$.
Using these and Theorem 1.1.9 we obtain

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = \int\limits_{\mathbb{R}^n} \nabla f(x) \cdot \nabla f(x) |x|^{-a}\, dx = \int\limits_{\mathbb{R}^n} \widehat{\nabla f}(\xi) \overline{(\widehat{\Phi_a \nabla f})(\xi)}\, d\xi$$

We know that

$$(\widehat{\Phi_a \nabla f})(\xi) = \int\limits_{\mathbb{R}^n} \widehat{\Phi_a}(\xi - \eta) \widehat{\nabla f}(\eta)\, d\eta$$

and thus:

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{\nabla f}(\xi)\, \widehat{\Phi_a}(\xi - \eta)\, \overline{\widehat{\nabla f}}(\eta)\, d\eta\, d\xi$$

equivalently

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx &= \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{\nabla f}(x)\, \widehat{\Phi_a}(x - y)\, \overline{\widehat{\nabla f}}(y)\, dy\, dx \\ &= C_{n,a} \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \widehat{\nabla f}(x)\, \frac{1}{|x-y|^{n-a}}\, \overline{\widehat{\nabla f}}(y)\, dy\, dx \end{aligned}$$

Using the fact that

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx \geq 0$$

and so

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = \left| \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx \right|$$

and the formula from Chapter 1 $\widehat{\nabla f}(y) = (-2\pi i y)\widehat{f}(y)$ it follows:

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = C_{n,a} \left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \widehat{f}(x)\, 4\pi^2 x\cdot y\, \frac{1}{|x-y|^{n-a}}\, \overline{\widehat{f}}(y)\, dy\, dx \right|$$

We define

$$f(x) = \widehat{g}(x)|x|^{-\frac{a}{2}-1}$$

and thus

$$4\pi^2 C_{n,a} \left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \widehat{f}(x)\, x\cdot y\, \frac{1}{|x-y|^{n-a}}\, \widehat{f}(y)\, dy\, dx \right| =$$

$$= 4\pi^2 C_{n,a} \left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{g(x)}{|x|^{\frac{a}{2}}}\, \frac{x\cdot y}{|x||y|}\, \frac{1}{|x-y|^{n-a}}\, \frac{\overline{g}(y)}{|y|^{\frac{a}{2}}}\, dy\, dx \right|$$

and we conclude that:

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = 4\pi^2 C_{n,a} \left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{g(x)}{|x|^{\frac{a}{2}}}\, \frac{x\cdot y}{|x||y|}\, \frac{1}{|x-y|^{n-a}}\, \frac{\overline{g}(y)}{|y|^{\frac{a}{2}}}\, dy\, dx \right|$$

Similarly for the right-hand side of the inequality we obtain

$$4\pi^2 \int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2 |y|^{a+2}\, dy = \left[\pi^{\frac{n}{2}-a} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] \int\limits_{\mathbb{R}^n} |g(\xi)|^2\, d\xi$$

Therefore the lemma has been proven.

□

Therefore Theorem 2.2.1 is equivalent to the following inequality :

Theorem 2.2.3 *For* $g \in \mathcal{S}(\mathbb{R}^n)$ *and* $0 < a < n$ *where* $n > 1$

$$\left| \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} \frac{g(x)}{|x|^{\frac{a}{2}}}\, \frac{x\cdot y}{|x||y|}\, \frac{1}{|x-y|^{n-a}}\, \frac{\overline{g}(y)}{|y|^{\frac{a}{2}}}\, dy\, dx \right| \le \left[\pi^{\frac{n}{2}-a} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] D_a \int\limits_{\mathbb{R}^n} |g(x)|^2\, dx \tag{2.9}$$

Proof of Theorems 2.2.1 and 2.2.3. The proof of Theorems 2.2.1 and 2.2.3 is similar to the proof of Theorems 2.1.1 and 2.1.2. The proof will be based on Young's inequality for the group $(\mathbb{R}_+, \cdot)$.The spaces $L^1(\mathbb{R}_+)$ and $L^2(\mathbb{R}_+)$ are with respect to the Haar measure of $(\mathbb{R}_+, \cdot)$, and therefore Young's inequality

can be applied.
Specifically, we will follow the steps below.
Step 1: In this step we will assume that f is radial. Therefore, $g(\xi) = \widehat{f}(\xi)|\xi|^{\frac{a+2}{2}}$ is radial . We set $t = |x|$, $s = |y|$ and $h(t) = |x|^{\frac{n}{2}} g(x)$.

$$|x-y|^{n-a} = (|x|^2 + |y|^2 - 2x \cdot y)^{\frac{n-a}{2}} = |x|^{\frac{n-a}{2}} |y|^{\frac{n-a}{2}} \left(\frac{|x|}{|y|} + \frac{|y|}{|x|} - 2\frac{x \cdot y}{|x||y|} \right)^{\frac{n-a}{2}}$$
$$= t^{\frac{n-a}{2}} s^{\frac{n-a}{2}} \left(\frac{t}{s} + \frac{s}{t} - 2\xi_1 \right)^{\frac{n-a}{2}}$$

We define

$$\psi_a(t) = \fint_{S^{n-1}} \xi_1 \left(t + \frac{1}{t} - 2\xi_1 \right)^{\frac{-(n-a)}{2}} \; dS(\xi)$$

where $dS(\xi)$ denotes the surface measure , and $\xi_1$is the first component of ξ. Therefore, the left-hand side of (2.9) transforms into :

$$\left| \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{g(x)}{|x|^{\frac{a}{2}}} \frac{x \cdot y}{|x||y|} \frac{1}{|x-y|^{n-a}} \frac{\overline{g}(y)}{|y|^{\frac{a}{2}}} \, dy \, dx \right|$$
$$= \left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} \frac{h(t)}{t^{n/2}} \frac{1}{t^{a/2}} \frac{1}{t^{\frac{n-a}{2}}} \psi_a\left(\frac{t}{s}\right) \frac{\overline{h}(s)}{s^{n/2}} \frac{1}{s^{a/2}} \frac{1}{s^{\frac{n-a}{2}}} t^{n-1} s^{n-1} \, dt \, ds \right|$$
$$= \left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} h(t) \, \psi_a\left(\frac{t}{s}\right) \overline{h}(s) \, \frac{dt}{t} \frac{ds}{s} \right|$$

The right-hand side of (2.9) changes into:

$$\left[\pi^{\frac{n}{2}-a} \frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] D_a \int\limits_{\mathbb{R}^n} |g(x)|^2 \, dx = \frac{D_a}{2\pi^a} \left[\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] \int\limits_{\mathbb{R}_+} |h(t)|^2 \, \frac{dt}{t}$$

Hence, inequality (2.9) is restated as:

$$\left| \int\limits_{\mathbb{R}_+ \times \mathbb{R}_+} h(t) \, \psi_a\left(\frac{t}{s}\right) h(s) \, \frac{dt}{t} \frac{ds}{s} \right| \leq \frac{D_a}{2\pi^a} \left[\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)} \right] \int\limits_{\mathbb{R}_+} |h(t)|^2 \, \frac{dt}{t}$$

The function ψ_a is symmetrical:

$$\psi_a\left(\frac{t}{s}\right) = \fint_{S^{n-1}} \xi_1 \left(\frac{t}{s} + \frac{s}{t} - 2\xi_1\right)^{\frac{-(n-a)}{2}} d\xi = \psi_a\left(\frac{s}{t}\right)$$

and by monotonicity we deduce it is positive.
Using the Cauchy-Schwarz and Young inequalities for convolution, we have that:

$$\begin{aligned}\left| \int\limits_{\mathbb{R}_+\times\mathbb{R}_+} h(t)\,\psi_a\left(\frac{t}{s}\right)\overline{h}(s)\,\frac{dt}{t}\,\frac{ds}{s}\right| &\leq \int\limits_{\mathbb{R}_+\times\mathbb{R}_+} \left|\frac{h(t)}{t}\,\psi_a\left(\frac{t}{s}\right)\frac{\overline{h}(s)}{s}\right| dt\,ds \\ &\leq \|\overline{h} * \psi_a\|_{L^2(\mathbb{R}_+)}\,\|h\|_{L^2(\mathbb{R}_+)} \\ &\leq \|\psi_a\|_{L^1(\mathbb{R}_+)}\|h\|^2_{L^2(\mathbb{R}_+)}\end{aligned}$$

It suffices to show that :

$$\|\psi_a\|_{L^1(\mathbb{R}_+)} = \frac{D_a}{2\pi^a}\left[\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\right]$$

To compute the norm, we will perform exactly the reverse process as before: we will take an arbitrary $x \in S^{n-1}$ and we will set $\xi_1 = \frac{x\cdot y}{|x||y|} = \frac{x\cdot y}{|y|}$ and $t = |y|$. Indeed, by using the co-area formula from [6] we have

$$\begin{aligned}\|\psi_a\|_{L^1(\mathbb{R}_+)} &= \int\limits_0^\infty \fint_{S^{n-1}} \xi_1\left(t + \frac{1}{t} - 2\xi_1\right)^{\frac{-(n-a)}{2}} dS(\xi)\,\frac{dt}{t} \\ &= \omega_{n-1}^{-1}\int\limits_{\mathbb{R}^n} \frac{x\cdot y}{|y|}\,\frac{1}{|x-y|^{n-a}}\,\frac{1}{|y|^{(n+a)/2}}\,dy\end{aligned}$$

The second integral will be computed for $0 < a < n-2$. We observe that the first integral is an analytic function of the parameters $n \geq 2$ and $b = n - a$. Therefore, any computation for the parameters will determine the value of $\|\psi_a\|_{L^1(\mathbb{R}_+)}$ due to continuity, for the requested values of the parameters $0 < a < n$ and $n > 1$. Next, we use the fact that:

$$|x-y|^2 = |x|^2 + |y|^2 - 2x\cdot y$$

and so

$$\frac{2x\cdot y}{|x||y|}=\frac{|x|}{|y|}+\frac{|y|}{|x|}-\frac{|x-y|^2}{|x||y|}$$

and thus because $|x|=1$ we have that:

$$\int\limits_{\mathbb{R}^n}\frac{2x\cdot y}{|y|}\frac{1}{|x-y|^{n-a}}\frac{1}{|y|^{(n+a)/2}}\,dy=\int\limits_{\mathbb{R}^n}|x-y|^{-(n-a)}|y|^{-\frac{n+a}{2}-1}\,dy+$$

$$+\int\limits_{\mathbb{R}^n}|x-y|^{-(n-a)}|y|^{-\frac{n+a}{2}+1}\,dy-\int\limits_{\mathbb{R}^n}|x-y|^{-(n-a-2)}|y|^{-\frac{n+a}{2}-1}\,dy:=I_1+I_2-I_3$$

To compute these integrals we will use (1.13). Indeed, as one can easily observe

$$I_1=\Phi_{n-a}*\Phi_{(n+a)/2-1}$$

$$I_2=\Phi_{n-a}*\Phi_{(n-a)/2+1}$$

$$I_3=\Phi_{n-a-2}*\Phi_{(n+a)/2+1}$$

Moreover all the convolutions are of the form: $0<\beta<n$, $0<\delta<n$ and $n<\beta+\delta<2n$ and thus by using the formula we obtain:

$$I_1=\pi^{\frac{n}{2}}\left[\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n-a-2}{4}\right)\Gamma\left(\frac{n-a+2}{4}\right)}{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{n+a+2}{4}\right)\Gamma\left(\frac{n+a-2}{4}\right)}\right]$$

$$I_2=\pi^{\frac{n}{2}}\left[\frac{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{n-a-2}{4}\right)\Gamma\left(\frac{n-a+2}{4}\right)}{\Gamma\left(\frac{n-a}{2}\right)\Gamma\left(\frac{n+a+2}{4}\right)\Gamma\left(\frac{n+a-2}{4}\right)}\right]$$

$$I_3=\pi^{\frac{n}{2}}\left[\frac{\Gamma\left(\frac{a+2}{2}\right)\Gamma\left(\frac{n-a-2}{4}\right)\Gamma\left(\frac{n-a-2}{4}\right)}{\Gamma\left(\frac{n-a-2}{2}\right)\Gamma\left(\frac{n+a+2}{4}\right)\Gamma\left(\frac{n+a+2}{4}\right)}\right]$$

By using the formula $\Gamma(x+1)=x\Gamma(x)$ we get

$$I_1=I_2=\pi^{n/2}\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a-2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}\frac{(n-a-2)}{4}\frac{(n+a-2)}{4}$$

$$I_3=\pi^{n/2}\,\frac{a}{2}\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a-2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}\frac{(n-a-2)}{2}$$

And so

$$
\begin{aligned}
I_1+I_2-I_3 &= \pi^{n/2}\,\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a-2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}\left[2\frac{(n-a-2)}{4}\frac{(n+a-2)}{4}-\frac{a}{2}\frac{(n-a-2)}{2}\right]\\
&= \pi^{n/2}\,\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a-2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}\left[2\left(\frac{n-a-2}{4}\right)^2\right]=2\pi^{n/2}\,\frac{\Gamma\left(\frac{a}{2}\right)}{\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a+2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}
\end{aligned}
$$

and so we have proven that:

$$\|\psi_a\|_{L^1(\mathbb{R}_+)}=\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{a}{2}\right)}{2\Gamma\left(\frac{n-a}{2}\right)}\frac{\Gamma^2\left(\frac{n-a+2}{4}\right)}{\Gamma^2\left(\frac{n+a+2}{4}\right)}$$

Consequently for radial functions the inequalities (2.8) and (2.9) hold for the constant

$$D_a=\pi^a\left[\frac{\Gamma\left(\frac{n-a+2}{4}\right)}{\Gamma\left(\frac{n+a+2}{4}\right)}\right]^2$$

Step 2: For $f\in\mathcal{S}(\mathbb{R}^n)$ as we have demonstrated in Chapter 1 we have

$$f(x)=\sum_{k=0}^{\infty}f_k(x)P_k(x)$$

where f_k is radial, P_k harmonic polynomial of degree k for which we have that:

$$P_k(x)=|x|^kY_k(\xi),\quad \xi=\frac{x}{|x|},\quad \fint_{S^{n-1}}|Y_k(\xi)|^2\,dS(\xi)=\frac{\omega_{n-1+2k}}{\omega_{n-1}}$$

where Y_k spherical harmonic of degree k, ω_m the surface area of the unit sphere S^m and $dS(\xi)$ the normalized surface measure of S^{n-1}. Hence we easily compute

$$\int\limits_{\mathbb{R}^n}|f(x)|^2\,dx=\sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^{n+2k}}|f_k(x)|^2dx$$

which can be proven similarly as in the (2.5). Similarly to the previous subsection we denote the Fourier Transform on $\mathbb{R}^n$ by $\mathcal{F}_n$. Thus by Theorem 1.4.5 we obtain

$$\mathcal{F}_n[f_k(x)P_k(x)]=i^{-k}\mathcal{F}_{n+2k}[f_k(x)]P_k$$

and thus similar to (2.7) the right-hand side of (2.8) takes the form

$$\int\limits_{\mathbb{R}^n}|\widehat{f}(y)|^2|y|^{a+2}\,dy=\sum_{k=0}^{\infty}\int\limits_{\mathbb{R}^{n+2k}}|\widehat{f_k}(y)|^2|y|^{a+2}\,dy$$

The left-hand side of (2.8) transforms

$$I = \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a}\, dx = \sum_{k=0}^{\infty} \int\limits_{\mathbb{R}^{n+2k}} |\nabla [f_k(x) P_k(x)]|^2 \frac{1}{|x|^a}\, dx$$

We have the following

$$\nabla [f_k(x) P_k(x)] = P_k(x) \nabla f_k(x) + f_k(x) \nabla P_k(x)$$

thus

$$\begin{aligned} |\nabla [f_k(x) P_k(x)]|^2 &= (\nabla [f_k(x) P_k(x)]) \cdot (\nabla [f_k(x) P_k(x)]) \\ &= |\nabla f_k(x)|^2 |P_k(x)|^2 + |f_k(x)|^2 |\nabla P_k(x)|^2 \\ &+ 2[\nabla f_k(x) P_k(x)] \cdot [f_k(x) \nabla P_k(x)] \end{aligned}$$

and

$$[P_k(x) \nabla f_k(x)] \cdot [f_k(x) \nabla P_k(x)] = [f_k(x) \nabla f_k(x)] \cdot [P_k(x) \nabla P_k(x)]$$

Therefore

$$\begin{aligned} I = \sum_{k=0}^{\infty} \Bigg\{ &\int\limits_{\mathbb{R}^{n+2k}} |\nabla f_k(x)|^2 |x|^{-a}\, dx + \int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 |\nabla P_k(x)|^2 |x|^{-a}\, dx \\ &+ 2 \int\limits_{\mathbb{R}^{n+2k}} [f_k(x)(\nabla f_k(x))] \cdot [P_k(x) \nabla P_k(x)] |x|^{-a}\, dx \Bigg\}. \end{aligned}$$

We define

$$b_k = \int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 |\nabla P_k(x)|^2 |x|^{-a}\, dx$$

and

$$\gamma_k = 2 \int\limits_{\mathbb{R}^{n+2k}} [f_k(x)(\nabla f_k(x))] \cdot [P_k(x)(\nabla P_k(x))] |x|^{-a}\, dx$$

In γ_k we integrate by parts and we obtain

$$\begin{aligned}\gamma_k &= -\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 \text{div}[P_k(x)\nabla P_k(x)|x|^{-a}]\,dx \\ &= \int\limits_{\mathbb{R}^{n+2k}} -|f_k(x)|^2|\nabla P_k(x)|^2|x|^{-a}\,dx - \int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 P_k(x)\Delta P_k(x)|x|^{-a}\,dx \\ &- \int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 P_k(x)\nabla P_k(x)\cdot\frac{(-a)}{|x|^{a+2}}x\,dx\end{aligned}$$

Therefore

$$\gamma_k = -b_k + a\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 P_k(x)|x|^{-(a+2)}\nabla P_k(x)\cdot x\,dx$$

Furthermore, one can easily see that

$$\nabla P_k(x) = k|x|^{k-2}(x_1,\cdots,x_{n+2k})\cdot(x_1,\cdots,x_{n+2k})Y_k(\xi) + |x|^{k-1}\nabla[Y_k(\xi)]\frac{x}{|x|}$$

and thus

$$x\cdot\nabla P_k(x) = k|x|^k Y_k(\xi) = kP_k(x)$$

Hence

$$\gamma_k = -b_k + a\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2 k|P_k(x)|^2|x|^{-a}\,dx = -b_k + ka\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2|x|^{-(a+2)}\,dx$$

We conclude that

$$I = \sum_{k=0}^{\infty}\left\{\int\limits_{\mathbb{R}^{n+2k}} |\nabla f_k(x)|^2|x|^{-a}\,dx + ka\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2|x|^{-(a+2)}\,dx\right\}$$

For every k the function f_k is radial. Thus by using Step 1 of the proof we obtain:

$$\int\limits_{\mathbb{R}^{n+2k}} |\nabla f_k(x)|^2|x|^{-a}\,dx \le 4\pi^{a+2}\left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right]^2\int\limits_{\mathbb{R}^{n+2k}} |\widehat{f}_k(y)|^2|y|^{a+2}\,dy$$

and by (2.3)

$$\int\limits_{\mathbb{R}^{n+2k}} |f_k(x)|^2|x|^{-(a+2)}\,dx \leq \pi^{a+2}\left[\frac{\Gamma\left(\frac{n+2k-a-2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right]^2 \int\limits_{\mathbb{R}^{n+2k}} |\widehat{f}_k(y)|^2|y|^{a+2}\,dy$$

As a result, we have the following:

$$\begin{aligned}
I &= \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2|x|^{-a}\,dx \\
&\leq 4\pi^{a+2}\sum_{k=0}^{\infty}\left[\left(\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right)^2 + \frac{ka}{4}\left(\frac{\Gamma\left(\frac{n+2k-a-2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right)^2\right]\int\limits_{\mathbb{R}^{n+2k}} |\widehat{f}_k(y)|^2|y|^{a+2}dy \\
&\leq 4\pi^{a+2}\max_{k\in\mathbb{N}_0}\left\{\left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right]^2\left(1+\frac{4ka}{(n+2k-a-2)^2}\right)\right\}\int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2|y|^{a+2}\,dy
\end{aligned}$$

because

$$\Gamma\left(\frac{n+2k-a-2}{4}\right) = \Gamma\left(\frac{n+2k-a+2}{4}\right)\frac{4}{(n+2k-a-2)}$$

and the proof is complete.

□

Chapter 3

Computation of Optimal Constants

In this section, we will determine the optimal constants for the different values of a and n, in Pitt's Inequality with gradient terms (Theorem 2.2.1).

Proposition 3.0.1 *For $a = 2$ we have:*

$$\frac{D_a}{\pi^2} = \begin{cases} \frac{144}{25}, & \text{για } n = 3 \\ \frac{4}{3}, & \text{για } n = 4 \\ \frac{16}{n^2}, & \text{για } n > 4 \end{cases}$$

Proof. We know that

$$D_a = \pi^a \max_{k \in \mathbb{N}_0} \left\{ \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)} \right]^2 \left(1 + \left(\frac{4ka}{(n+2k-a-2)^2} \right) \right) \right\}$$

For $a = 2$

$$\frac{D_2}{\pi^2} = \max_{k \in \mathbb{N}} \left\{ \left[\frac{\Gamma\left(\frac{n+2k}{4}\right)}{\Gamma\left(\frac{n+2k+4}{4}\right)} \right]^2 \left(1 + \left(\frac{8k}{(n+2k-4)^2} \right) \right) \right\}$$

and thus

$$\frac{D_2}{\pi^2} = \max_{k \in \mathbb{N}} \left\{ \left(\frac{4}{n+2k} \right)^2 \left(1 + \left(\frac{8k}{(n+2k-4)^2} \right) \right) \right\}$$

We easily see that the sequence

$$\left\{\left(\frac{4}{n+2k}\right)^2\left(1+\left(\frac{8k}{(n+2k-4)^2}\right)\right)\right\}$$

is decreasing with respect to k for $k \geq 1$. Hence we need to determine whether a_0 or a_1 is larger. We have that

$$a_1 = \frac{16}{(n+2)^2}\left(1+\frac{8}{(n-2)^2}\right)$$

and

$$a_0 = \frac{16}{n^2}.$$

Thus to prove that a_1 is larger than a_0 it suffices to show

$$\frac{a_1}{a_0} = \frac{\frac{16}{(n+2)^2}\left(1+\frac{8}{(n-2)^2}\right)}{\frac{16}{n^2}} > 1$$

By computations we obtain the equivalent inequality:

$$4n^3 - 20n^2 + 16 < 0$$

We consider the function

$$P(n) = 4n^3 - 20n^2 + 16$$

We compute that:

$$P'(n) = 12n^2 - 40n$$

Therefore if :

$$P'(n) = 0$$

we have that

$$n = 0 \text{ or } n = \frac{10}{3}$$

We observe that :

$$P''(n) = 24n - 40$$

and thus if :

$$P''(n) = 0$$

we have that

$$n = \frac{5}{3}.$$

Consequently if $n > \frac{5}{3}$, $P'' > 0$ and this means that P' is increasing for $n > \frac{5}{3}$. Thus:

$$P(3) = -196 < 0 \Leftrightarrow a_1 > a_0 \text{ for } n = 3,$$

$$P(4) = -48 < 0 \Leftrightarrow a_1 > a_0 \text{ for } n = 4.$$

On the other hand for $n \geq 5$ we have that $P(5) = 16$ and P is increasing in the interval $\left(\frac{10}{3}, \infty\right)$, and so it is true that $a_0 > a_1$. Thus we conclude that

- For $n = 3$, $\frac{D_a}{\pi^a} = \frac{144}{25}$
- For $n = 4$, $\frac{D_a}{\pi^a} = \frac{4}{3}$
- For $n \geq 5$, $\frac{D_a}{\pi^a} = \frac{16}{n^2}$

and so we have the desired result.
□

Definition 3.0.2 *For every $t > 0$ we define the digammma function*

$$\psi(t) = \frac{d}{dt} \ln \Gamma(t)$$

Lemma 3.0.3 *: The function*

$$F(b) = \ln \Gamma(x + b) - \ln \Gamma(y + b)$$

$b > 0$, is decreasing and

$$F'(b) = \psi(x + b) - \psi(y + b) = -\sum_{k=0}^{\infty} \frac{y - x}{(x + b + k)(y + k + b)}$$

Proof. It is known ([8]) that $\forall z \in \mathbb{C} - \{-2, -1, 0\}$ is true that

$$\Gamma(z) = \mathrm{e}^{-\gamma z} \frac{1}{z} \prod_{n=1}^{\infty} \frac{\mathrm{e}^{\frac{z}{n}}}{1 + \frac{z}{n}}$$

where γ is the Euler–Mascheroni constant. It follows that

$$\ln \Gamma(z) = -\gamma z - \ln z + \sum_{n=1}^{\infty} \left(\frac{z}{n} - \ln \left(1 + \frac{z}{n} \right) \right)$$

and so

$$\ln \Gamma(y+b) = -\gamma(y+b) - \ln(y+b) + \sum_{n=1}^{\infty}\left(\frac{y+b}{n} - \ln\left(1+\frac{y+b}{n}\right)\right)$$

and similarly for $\ln \Gamma(x+b)$ Thus

$$\psi(y+b) = \frac{d}{db}\ln \Gamma(y+b) = \frac{\Gamma'(y+b)}{\Gamma(y+b)} = -\gamma - \frac{1}{y+b} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+y+b}\right)$$
$$= -\gamma - \frac{1}{y+b} + \sum_{n=1}^{\infty}\left(\frac{y+b}{(n+y+b)n}\right)$$

Similarly we obtain

$$\psi(x+b) = \frac{d}{db}\ln \Gamma(x+b) = -\gamma - \frac{1}{x+b} + \sum_{n=1}^{\infty}\left(\frac{x+b}{(n+x+b)n}\right)$$

Therefore

$$F'(b) = \psi(x+b) - \psi(y+b) = -\frac{1}{x+b} + \frac{1}{y+b} + \sum_{n=1}^{\infty}\left(\frac{x+b}{(n+x+b)n} - \frac{y+b}{(n+y+b)n}\right)$$
$$= \frac{x+b-y-b}{(x+b)(y+b)} + \sum_{n=1}^{\infty}\left(\frac{(x+b)(y+b+n)-(y+b)(n+x+b)}{(n+y+b)(n+x+b)n}\right)$$

$$F'(b) = -\frac{y-x}{(x+b)(y+b)} + \sum_{n=1}^{\infty}\left(\frac{n(x-y)}{(n+y+b)(n+x+b)n}\right)$$
$$= -\frac{y-x}{(x+b)(y+b)} - \sum_{n=1}^{\infty}\left(\frac{y-x}{(n+y+b)(n+x+b)}\right)$$

Thus F is decreasing and

$$F'(b) = -\sum_{k=0}^{\infty}\frac{y-x}{(x+b+k)(y+k+b)}$$

□

Proposition 3.0.4 *For $n = 2$ and $a < 2$ we have:*

$$D_a = \pi^a \left[\frac{\Gamma(\frac{3}{2} - \frac{a}{4})}{\Gamma(\frac{3}{2} + \frac{a}{4})}\right]^2 \left(\frac{a^2 + 4}{(a-2)^2}\right)$$

Proof. We recall from Theorem 2.2.1 that

$$D_a = \pi^a \max_{k \in \mathbb{N}_0} \left\{ \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right]^2 \left(1 + \frac{4ka}{(n+2k-a-2)^2}\right) \right\}$$

Thus for $n = 2$ we have

$$D_a = \pi^a \max_{k \in \mathbb{N}_0} \left\{ \left[\frac{\Gamma\left(\frac{4+2k-a}{4}\right)}{\Gamma\left(\frac{4+2k+a}{4}\right)}\right]^2 \left(1 + \frac{4ka}{(2k-a)^2}\right) \right\}$$

We define

$$a_k = \left[\frac{\Gamma\left(\frac{4+2k-a}{4}\right)}{\Gamma\left(\frac{4+2k+a}{4}\right)}\right]^2 \left(1 + \frac{4ka}{(2k-a)^2}\right)$$

Similarly to the proof of Proposition 3.0.1 for $k \geq 1$ this sequence is decreasing and in order to determine the value of D_a it suffices to prove that $a_1 > a_0$. For $k = 1$ we have that:

$$a_1 = \left[\frac{\Gamma\left(\frac{6-a}{4}\right)}{\Gamma\left(\frac{6+a}{4}\right)}\right]^2 \left(1 + \frac{4a}{(2-a)^2}\right) = \left[\frac{\Gamma\left(\frac{3}{2} - \frac{a}{4}\right)}{\Gamma\left(\frac{3}{2} + \frac{a}{4}\right)}\right]^2 \left(1 + \frac{4a}{(2-a)^2}\right)$$

and by letting $b = \dfrac{a}{4}$ and using the property of the Gamma function $\Gamma(x+1) = x\Gamma(x)$ we have that

$$\Gamma\left(\frac{3}{2} - \frac{a}{4}\right) = \left(\frac{1}{2} - \frac{a}{4}\right) \Gamma\left(\frac{1}{2} - \frac{a}{4}\right) = \left(\frac{2-a}{4}\right) \Gamma\left(\frac{1}{2} - \frac{a}{4}\right)$$

and so

$$a_1 = \left[\frac{\Gamma(\frac{1}{2} - b)}{\Gamma(\frac{3}{2} + b)} \sqrt{b^2 + \frac{1}{4}}\right]^2$$

For $k = 0$ we obtain

$$a_0 = \left[\frac{\Gamma(1-b)}{\Gamma(1+b)}\right]^2$$

We define

$$F(b) = \ln\sqrt{a_1} - \ln\sqrt{a_0}$$

We suppose that $0 < a < n = 2$ and so $0 < b < \frac{1}{2}$. Thus F is well defined on the interval $(0, \frac{1}{2})$ and is differentiable with respect to the variable b. Consequently

$$F(b) = \ln\Gamma\left(\frac{1}{2} - b\right) - \ln\Gamma\left(\frac{3}{2} + b\right) + \frac{1}{2}\ln\left(b^2 + \frac{1}{4}\right) - \ln\Gamma(1-b) + \ln\Gamma(1+b)$$

It suffices to prove that $F(b) > 0$. As one can easily see

$$F(0) = \ln\Gamma\left(\frac{1}{2}\right) - \ln\Gamma\left(\frac{3}{2}\right) + \frac{1}{2}\ln\frac{1}{4} - \ln\Gamma(1) + \ln\Gamma(1)$$

$$= \ln\Gamma\left(\frac{1}{2}\right) - \ln\Gamma\left(\frac{1}{2}\right) + \frac{1}{2}\ln\frac{1}{4} - \frac{1}{2}\ln\frac{1}{4} = 0$$

Therefore, we need only to prove that $F(b) > 0$ and thus it suffices to prove $F'(b) > 0$ in that interval, because by the monotonicity of F we will have the desired result.

Using the Lemma 3.0.3 gives :

$$F'(b) = -\psi\left(\frac{1}{2} - b\right) - \psi\left(\frac{3}{2} + b\right) + \psi(1-b) + \psi(1+b) + \frac{4b}{4b^2+1}$$

$$= \left(\frac{1}{2} + 2b\right)\sum_{k=0}^{\infty}\left(\frac{1}{(\frac{1}{2} - b + k)(1 + b + k)} - \frac{1}{(1 - b + k)(\frac{3}{2} + b + k)}\right) + \frac{4b}{1 + 4b^2} > 0$$

Thus $a_1 > a_0$ and so :

$$D_a = \pi^a \left[\frac{\Gamma(\frac{3}{2} - \frac{a}{4})}{\Gamma(\frac{3}{2} + \frac{a}{4})}\right]^2 \frac{a^2 + 4}{(a-2)^2}$$

Therefore Proposition 3.0.4 is established .

□

Proposition 3.0.5 *For $n \geq 3$ and $0 < a < n$ we have:*

$$D_a = \pi^a \max_{k\in\{0,1\}} \left\{ \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)} \right]^2 \left(1 + \frac{4ka}{(n+2k-a-2)^2}\right) \right\}$$

Proof. We define

$$G(2k) = \ln \left\{ \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)} \right] \sqrt{\left(1 + \frac{4ka}{(n+2k-a-2)^2}\right)} \right\}$$

We let $b = 2k$ and so

$$G(b) = \ln \left\{ \left[\frac{\Gamma\left(\frac{n+b-a+2}{4}\right)}{\Gamma\left(\frac{n+b+a+2}{4}\right)} \right] \sqrt{\left(1 + \frac{2ba}{(n+2k-a-2)^2}\right)} \right\}$$

It suffices to prove that $G'(b) < 0$ for $b \geq 2$. The function G is obviously differentiable with respect to b for $a \in (0, n)$. We have:

$$G'(b) = \frac{1}{4}\left[\psi\left(\frac{n+b-a+2}{4}\right) - \psi\left(\frac{n+b+a+2}{4}\right)\right] + a\left[1 - \frac{2b}{(n+b-a-2)}\right]$$
$$\times \left[\frac{1}{(n+b-a-2)^2 + 2ba}\right]$$

because

$$\frac{d}{db}\left(\ln\left(1 + \frac{2ba}{(n+b-a-2)^2}\right)\right) = 2a\left[1 - \frac{2b}{n+b-a-2}\right]\left(\frac{1}{(n+b-a-2)^2 + 2ba}\right)$$

By using Lemma 3.0.3 we get:

$$\begin{aligned}
&G'(b) \\
&= -\frac{2a}{16}\sum_{k=0}^{\infty}\left[\frac{n+b-a+2}{4} + k\right]^{-1}\left[\frac{n+b+a+2}{4} + k\right]^{-1} \\
&+ a\left[\frac{n-b-a-2}{n+b-a-2}\right]\left[(n+b-a-2)^2 + 2ba\right]^{-1} \\
&= a\left[-\frac{1}{8}\sum_{k=0}^{\infty}\left[\left(\frac{n+b+2}{4} + k\right)^2 - \frac{a^2}{16}\right]^{-1} + \left(\frac{n-b-a-2}{n+b-a-2}\right)\left((n+b-a-2)^2 + 2ba\right)^{-1}\right]
\end{aligned}$$

For $n = 3$ or $n = 4$ and for $b \geq 2$ and $a \in (0, n)$, we get $G'(b) < 0$ because $a > 0$ and so

$$-\frac{1}{8}\sum_{k=0}^{\infty}\left[\left(\frac{n+b+2}{4}+k\right)^2-\frac{a^2}{16}\right]^{-1} \leq 0$$

and

$$(n+b-a-2)^2+2ba \geq 0$$

and

$$\frac{n-b-a-2}{n+b-a-2} < 0$$

For $n \geq 5$ using Riemann sums we have :

$$\frac{1}{8}\sum_{k=0}^{\infty}\left(\left(\frac{n+b+2}{4}+k\right)^2-\frac{a^2}{16}\right)^{-1} > \frac{1}{8}\sum_{k=0}^{\infty}\left(\frac{n+b+2}{4}+k\right)^{-2}$$

$$> \frac{1}{8}\int_0^{\infty}\left(\frac{n+b+2}{4}+x\right)^{-2}dx = \frac{1}{2}\left(n+b+2\right)^{-1}$$

Thus:

$$G'(b) < a\left[-\frac{1}{2}(n+b+2)^{-1}+\frac{n-b-a-2}{n+b-a-2}\left[(n+b-2)^2+a^2-2a(n-2)\right]^{-1}\right]$$

The right side of the inequality is negative if :

$$-\frac{1}{2}(n+b+2)^{-1}+\frac{n-b-a-2}{n+b-a-2}\left[(n+b-2)^2+a^2-2a(n-2)\right]^{-1} < 0$$

and so

$$\frac{n-b-a-2}{n+b-a-2} < \frac{(n+b-2)^2+a^2-2a(n-2}{2(n+b+2)}$$

which is true of $n-b-a-2 < 0$.
We let $\delta = n-b-a-2$ and so

$$-\frac{n-b-a-2}{n+b-a-2}+\frac{(n+b-2)^2+a^2-2a(n-2)}{2(n+b+2)} = \frac{-\delta}{\delta+2b}+\frac{(2b+\delta+a)^2+a^2-2a(\delta+b+a)}{2(\delta+2b+a+4)}$$

Therefore it suffices to prove that this expression is positive for $\delta > 0$. Or equivalently:

$$H(\delta) = (\delta+2b)\left[(2b+\delta+a)^2+a^2-2a(\delta+b+a)\right]-2\delta(\delta+2b+a+4) > 0$$

We have that

$$H(\delta) = (\delta + 2b)\left[(2b+\delta+a)^2 + a^2 - 2a(\delta+b+a)\right] - 2\delta(\delta+2b+a+4)$$
$$= (\delta+2b)(\delta+2b+a)^2 - (\delta+2b)(a^2+2a\delta+2ab) - 2\delta(\delta+2b+a+4)$$
$$= (\delta+2b)[\delta^2+4b^2+2ba+4b-2\delta] - 2\delta(a+4)$$

By differentiation we have

$$H'(\delta) = \delta^2+4b^2+2ba+4b-2\delta+(\delta+2b)(2\delta-2)-2(a+4)$$
$$= \delta^2+4b^2+2ba+4b-2\delta+4\delta^2-2d+4b\delta-4b-2a-8$$
$$= 5\delta^2+4\delta(b-1)4b^2+2ba+4b-4b-2a-8 \geq 0$$

if $\delta > 0$ because $b \geq 2$ and so $4b^2 > 8$ and $b > 1$. In addition, $H(0) > 0$ and so for $\delta \geq 0$ we have $H(\delta) > 0$ which is the desired result . On the other hand for $\delta < 0$ by definition we also have the same result.
□

Theorem 3.0.6 (Hardy- Rellich Trace Inequality) *: For $f \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 2$ we have*

$$\int_{\mathbb{R}^n} |\nabla f|^2 |x|^{-1}\, dx \leq \frac{D_1}{2\pi}\int_{\mathbb{R}^n} |(-\Delta)^{3/4} f|^2\, dx$$

where

$$\frac{D_1}{2\pi} = \begin{cases} \frac{5}{2}\left[\frac{\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{7}{4}\right)}\right]^2, & \text{for } n=2 \\ \frac{\pi}{4}, & \text{for } n=3 \\ \frac{1}{2}\left[\frac{\Gamma\left(\frac{n+1}{4}\right)}{\Gamma\left(\frac{n+3}{4}\right)}\right]^2, & \text{for } n \geq 4 \end{cases}$$

Proof. From Theorem 2.2.1 for $a = 1$ and (1.22) we have

$$\int_{\mathbb{R}^n} |\nabla f|^2 |x|^{-1}\, dx \leq\ 4\pi^2 D_1 \int_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^3\, dy = \frac{4\pi^2}{8\pi^3} D_1 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{3}{4}} f\right|^2\, dy$$
$$= \frac{1}{2\pi} D_1 \int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{3}{4}} f\right|^2\, dy$$

Hence it suffices to compute the optimal constants.
We define

$$a_k = \left[\frac{\Gamma\left(\frac{n+2k-a+2}{4}\right)}{\Gamma\left(\frac{n+2k+a+2}{4}\right)}\right]^2 \left(1 + \frac{4ka}{(n+2k-a-2)^2}\right) \tag{3.1}$$

for $k \in \mathbb{N}_0$. For $a = 1$ we have

$$a_k = \left[\frac{\Gamma\left(\frac{n+2k+1}{4}\right)}{\Gamma\left(\frac{n+2k+3}{4}\right)}\right]^2 \left(1 + \frac{4k}{(n+2k-3)^2}\right)$$

For $n = 2$ by Proposition 3.0.4 we get that the term $k = 1$ is larger and by letting $a = 1$ we obtain the desired result.
By Proposition 3.0.5 we know that for $n \geq 3$ we have

$$D_1 = \pi \max_{k \in \{0,1\}} \{a_k\}$$

and therefore we need to compare a_1, a_0 , where

$$a_1 = \left[\frac{\Gamma\left(\frac{n+3}{4}\right)}{\Gamma\left(\frac{n+5}{4}\right)}\right]^2 \left(1 + \frac{4}{(n-1)^2}\right)$$
$$a_0 = \left[\frac{\Gamma\left(\frac{n+1}{4}\right)}{\Gamma\left(\frac{n+3}{4}\right)}\right]^2$$

For $n = 3$ we get

$$a_1 = \left[\frac{\Gamma\left(\frac{6}{4}\right)}{\Gamma\left(\frac{8}{4}\right)}\right]^2 \left(1 + \frac{4}{2^2}\right) = \frac{\pi}{2}$$

Similarly

$$a_0 = \left[\frac{\Gamma(1)}{\Gamma\left(1 + \frac{1}{2}\right)}\right]^2 = \frac{4}{\pi}$$

and so $a_1 > a_0$. We conclude that

$$\frac{D_1}{2\pi} = \pi \frac{\frac{\pi}{2}}{2\pi} = \frac{\pi}{4}$$

and so for $n = 3$ we get the desired result.
For $n \geq 4$ we define

$$\Lambda(w) = \ln\left(\frac{a_1}{a_0}\right)$$

where $w = \dfrac{n}{4}$. Hence

$$\Lambda(w) = \ln\left[\frac{\Gamma\left(w + \frac{3}{4}\right)^4}{\Gamma\left(w + \frac{1}{4}\right)^2 \Gamma\left(w + \frac{5}{4}\right)^2}\left(1 + \frac{4}{(4w-1)^2}\right)\right]$$

Using the formula $\Gamma(x+1) = x\Gamma(x)$ again we have that

$$\Gamma\left(w+\frac{5}{4}\right) = \left(w+\frac{1}{4}\right)\Gamma\left(w+\frac{1}{4}\right)$$

and thus

$$\Lambda(w) = \ln\left[\frac{\Gamma\left(w+\frac{3}{4}\right)^4}{\Gamma\left(w+\frac{1}{4}\right)^4\left(\frac{4w+1}{4}\right)^2}\left(1+\frac{4}{(4w-1)^2}\right)\right]$$

We shall use the Stirling formula to show that

$$\Lambda(w) \to 0$$

when $w \to \infty$.

Indeed, using the asymptotic approximation of Stirling for the Gamma function, we obtain that

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}}\left(\frac{t}{e}\right)^t (1+o(1)) \text{ as } t \to \infty$$

and so

$$\Gamma\left(w+\frac{3}{4}\right) = \sqrt{\frac{2\pi}{w+\frac{3}{4}}}\left(\frac{w+\frac{3}{4}}{e}\right)^{w+\frac{3}{4}}(1+o(1))$$

and

$$\Gamma\left(w+\frac{1}{4}\right) = \sqrt{\frac{2\pi}{w+\frac{1}{4}}}\left(\frac{w+\frac{1}{4}}{e}\right)^{w+\frac{1}{4}}(1+o(1))$$

Thus

$$\begin{aligned}\frac{\Gamma\left(w+\frac{3}{4}\right)}{\Gamma\left(w+\frac{1}{4}\right)} &= \sqrt{\frac{w+\frac{1}{4}}{w+\frac{3}{4}}}\frac{(w+\frac{3}{4})^w}{(w+\frac{1}{4})^w}\frac{(w+\frac{3}{4})^{\frac{3}{4}}}{(w+\frac{1}{4})^{\frac{1}{4}}}\frac{e^{\frac{1}{4}}}{e^{\frac{3}{4}}}(1+o(1)) = \frac{\left(1+\frac{3}{4w}\right)^w}{\left(1+\frac{1}{4w}\right)^w}\frac{(w+\frac{3}{4})^{\frac{3}{4}}}{(w+\frac{1}{4})^{\frac{1}{4}}}\frac{1}{e^{\frac{1}{2}}}(1+o(1)) \\ &= \frac{e^{\frac{3}{4}}}{e^{\frac{1}{4}}}\frac{w^{\frac{3}{4}}}{w^{\frac{1}{4}}}\frac{1}{e^{\frac{1}{2}}}(1+o(1)) = \sqrt{w}(1+o(1))\end{aligned}$$

Consequently

$$\frac{\Gamma^2\left(w+\frac{3}{4}\right)}{\Gamma^2\left(w+\frac{1}{4}\right)\left(\frac{4w+1}{4}\right)} = \frac{1}{w+1/4}\left[\sqrt{w}(1+o(1))\right]^2 = 1+o(1)$$

that is

$$\frac{\Gamma^2\left(w+\frac{3}{4}\right)}{\Gamma^2\left(w+\frac{1}{4}\right)\left(\frac{4w+1}{4}\right)} \to 1 \text{ when } w \to \infty$$

Furthermore it is true that

$$1+\frac{4w}{(4w-1)^2} \to 1 \text{ when } w \to \infty$$

Therefore

$$\frac{\Gamma^4\left(w+\frac{3}{4}\right)}{\Gamma^4\left(w+\frac{1}{4}\right)\left(\frac{4w+1}{4}\right)^2}\left(1+\frac{4}{(4w-1)^2}\right) \to 1 \text{ when } w \to \infty$$

and thus

$$\Lambda(w) \to 0 \text{ when } w \to \infty$$

Moreover

$$\Lambda(1) = \ln\left\{\frac{117}{25}\left[\frac{\Gamma\left(\frac{3}{4}\right)^4}{\Gamma\left(\frac{1}{4}\right)^4}\right]\right\} \simeq -2.797$$

Hence for $w = 1$ the term a_0 is greater than the term a_1. Thus we have proven the desired result for $n = 4$. For $n > 4$ we use the fact that Λ is continuously differentiable and we get:

$$\Lambda'(w) = 4\left[\psi\left(w+\frac{3}{4}\right)-\psi\left(w+\frac{1}{4}\right)\right]+8\frac{(4w-1)}{(4w-1)^2+4}-\frac{32w}{16w^2-1}$$

And by using the Lemma 3.0.3 we obtain:

$$\Lambda'(w) = 2\sum_{k=0}^{\infty}\left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1}+8\frac{(4w-1)}{(4w-1)^2+4}-\frac{32w}{16w^2-1}$$

We observe that the function $f(k) = \left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1}$ is positive and decreasing. Therefore :

$$\sum_{k=0}^{\infty}\left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1} > \int_0^{\infty}\frac{1}{(x+w+\frac{1}{2})^2-\frac{1}{16}}\,dx$$

We let $u = x+w+\dfrac{1}{2}$ and we have:

$$\sum_{k=0}^{\infty}\left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1} > \int_{w+\frac{1}{2}}^{\infty}\frac{1}{u^2-\frac{1}{16}}\,du = 2\left[\ln\left(\frac{u-\frac{1}{4}}{u+\frac{1}{4}}\right)\right]_{w+\frac{1}{2}}^{\infty}$$

But $\lim_{u\to\infty} \ln\left(\frac{u-\frac{1}{4}}{u+\frac{1}{4}}\right) = 0$ and therefore:

$$\sum_{k=0}^{\infty}\left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1} > 2\ln\left(\frac{w+\frac{3}{4}}{w+\frac{1}{4}}\right) = h(w)$$

The function h is increasing and the function $g(w) = \frac{2}{2w+1}$ is decreasing , for $w \geq 1$ and $h(1) > g(1)$. Hence

$$\sum_{k=0}^{\infty}\left[\left(k+w+\frac{1}{2}\right)^2-\frac{1}{16}\right]^{-1} > \frac{2}{2w+1}$$

As a result we have :

$$\Lambda'(w) > \frac{4}{2w+1} + 8\frac{(4w-1)}{(4w-1)^2+4} - \frac{32w}{16w^2-1} > \frac{4}{2w+1} + \frac{8}{4w+1} - \frac{8}{4w-1}$$

Because for $w \geq 1$ we get :

$$w > \frac{3}{4} \Rightarrow 8w > 6$$

or equivalently :

$$16w^2 - 1 > 16w^2 - 8w + 1 + 4$$

and so

$$\frac{8(4w-1)}{(4w-1)^2+4} > \frac{8}{4w+1}$$

In addition for $w \geq 1$:

$$w > \frac{1}{4}$$

Thus

$$16w^2 - 4w < 16w^2 - 1$$

Consequently:

$$\frac{32w}{16w^2-1} < \frac{8}{4w-1}$$

Therefore:

$$-\frac{32w}{16w^2-1} > -\frac{8}{4w-1}$$

But on the other hand :

$$\frac{4}{2w+1} + \frac{8}{4w+1} - \frac{8}{4w-1} > 0$$

because for $w > \dfrac{1+\sqrt{6}}{4} \approx 0.862 < 1$ (we remind the reader that $w \geq 1$) it is true:

$$16w^2 - 8w - 5 > 0$$

and so:

$$16w^2 - 1 + 16w^2 + 4w - 2 - 16w^2 - 12w - 2 > 0$$

which in turn gives:

$$\frac{4}{2w+1} + \frac{8}{4w+1} - \frac{8}{4w-1} > 0$$

Therefore $\Lambda(w)$ is increasing for $w \geq 1$ that is $n \geq 4$ and because $\Lambda(w) \to 0$ when $w \to \infty$, $\Lambda(w)$ is negative for $w \geq 1$ and thus the term a_0 is greater than the term a_1 for $n \geq 4$. Therefore, we obtain the desired result by letting $k = 0$ and $a = 1$.
□

Theorem 3.0.7 *(A) For $n \geq 2$ and $n - 2 \leq a < n$*

$$D_a = \pi^a \left[\frac{\Gamma\left(\frac{n-a}{4} + 1\right)}{\Gamma\left(\frac{n+a}{4} + 1\right)} \right]^2 \left(1 + \frac{4a}{(n-a)^2}\right)$$

(B) For $n \geq 3$ and for a sufficiently near 0:

$$D_a = \pi^a \left[\frac{\Gamma\left(\frac{n-a}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4} + \frac{1}{2}\right)} \right]^2$$

Moreover, if $n \geq 4$ this estimate is true for $0 < a \leq n - 3$
(C) For fixed a , $D_a \simeq \left(\frac{4\pi}{n}\right)^a$ as $n \to \infty$.

Proof. By Proposition 3.0.4 we have the desired result for $n = 2$. By Proposition 3.0.5 for $n \geq 3$ It suffices to choose the larger of the terms a_0 and a_1 as previously defined.
The first step of the proof of part (A) is to show the result for $a = n - 2$. By letting $a = n - 2$ and for $k = 0$, $k = 1$ we obtain:

$$a_0 = \left[\frac{\Gamma(1)}{\Gamma\left(\frac{n}{2}\right)} \right]^2$$

$$a_1 = \left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \right]^2 (n-1)$$

As we have done in previous proofs of this section we define:

$$\Lambda(n) = \ln\left(\frac{a_1}{a_0}\right) = \ln\left\{ \left[\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)\,\Gamma\left(\frac{n+1}{2}\right)} \right]^2 (n-1) \right\} = \ln\left\{ \left[\frac{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi}}{\Gamma\left(\frac{n+1}{2}\right)2} \right]^2 (n-1) \right\}$$

And so

$$\Lambda(n) = \ln\left(\frac{\pi}{4}\right) + 2\ln\Gamma\left(\frac{n}{2}\right) - 2\ln\Gamma\left(\frac{n+1}{2}\right) + \ln(n-1)$$

We set $n = w$ and assume that w is a continuous variable for Λ. The function Λ is continuously differentiable, and I have that:

$$\Lambda'(w) = \psi\left(\frac{w}{2}\right) - \psi\left(\frac{w+1}{2}\right) + \frac{1}{w-1}$$

Using the formula for the digamma function found at [8] we obtain :

$$\psi(z) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-zt}}{1-e^{-t}} \right) dt$$

Therefore we have the following:

$$\psi\left(\frac{w}{2}\right) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{wt}{2}}}{1-e^{-t}} \right) dt$$

and

$$\psi\left(\frac{w+1}{2}\right) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-\frac{(w+1)t}{2}}}{1-e^{-t}} \right) dt$$

Thus

$$\psi\left(\frac{w}{2}\right)-\psi\left(\frac{w+1}{2}\right)=\int_0^\infty\left(-\frac{e^{-\frac{wt}{2}}e^{-\frac{t}{2}}}{1-e^{-t}}+\frac{e^{-\frac{wt}{2}}}{1-e^{-t}}\right)dt=-\int_0^\infty\frac{e^{-\frac{wt}{2}}\left(1-e^{-\frac{t}{2}}\right)}{1-e^{-t}}\,dt$$

$$=-\int_0^\infty\frac{e^{-\frac{wt}{2}}\left(1-e^{-\frac{t}{2}}\right)}{\left(1-e^{-\frac{t}{2}}\right)\left(1+e^{-\frac{t}{2}}\right)}\,dt=-\int_0^\infty\frac{e^{-\frac{wt}{2}}}{\left(1+e^{-\frac{t}{2}}\right)}\,dt$$

$$=-2\int_0^\infty\frac{e^{-wu}}{(1+e^{-u})}\,du$$

We set $\delta=w-1$ where $\delta>1$ and we have

$$\Lambda'(\delta+1)=\frac{1}{\delta}-2\int_0^\infty\frac{e^{-\delta u}e^{-u}}{(1+e^{-u})}\,du=\frac{1}{\delta}\left[1-2\delta\int_0^\infty\frac{e^{-\delta u}}{(1+e^{u})}\,du\right]$$

By computation we have :

$$-\delta\int_0^\infty\frac{e^{-\delta t}}{(1+e^t)}\,dt=\int_0^\infty\frac{\left(e^{-\delta t}\right)'}{1+e^t}\,dt=\left[\frac{e^{-\delta t}}{1+e^t}\right]_0^\infty+\int_0^\infty\frac{e^{-\delta t}}{(1+e^t)^2}e^t\,dt=-\frac{1}{2}+\int_0^\infty\frac{e^{-\delta t}}{(1+e^t)^2}e^t\,dt$$

and so

$$\Lambda'(\delta+1)=\frac{2}{\delta}\int_0^\infty\frac{e^{-\delta t}}{(1+e^t)^2}e^t\,dt>0$$

Therefore $\Lambda'(w)>0$ and $\Lambda(2)=0$, hence $\Lambda(n)>0$ for $n\geq 3$. We conclude that the result of part(A) is true for $a=n-2$. We will use the following Lemma:

Lemma 3.0.8 *If $w>1$, then:*

$$\frac{2}{2w+1}<\psi\left(\frac{w+1}{2}\right)-\psi\left(\frac{w}{2}\right)<\begin{cases}\dfrac{1}{w}+\dfrac{2}{w^2+w} & 1<w\leq 3,\\[2mm] \dfrac{1}{w-1} & w\geq 3\end{cases}$$

We will prove Lemma 3.0.8 after we have completed the proof of the theorem To complete the proof of part (A) we need to prove the equality for $n-2<a<n$. Therefore we must prove that $a_1>a_0$ for $a=n-2+2\delta$ and $0<\delta<1$. As we have done before we let $a=n-2+2\delta$ and for $k=0$, $k=1$

we obtain the following results:

$$a_0 = \left[\frac{\Gamma\left(1-\frac{\delta}{2}\right)}{\Gamma\left(\frac{n+\delta}{2}\right)}\right]^2$$

$$a_1 = \left[\frac{\Gamma\left(\frac{3-\delta}{2}\right)}{\Gamma\left(\frac{n+\delta+1}{2}\right)}\right]^2 \left(\frac{n+\delta^2-1}{(1-\delta)^2}\right)$$

We define the function:

$$\Lambda(\delta) = \ln\left\{\left[\frac{\Gamma\left(\frac{n+\delta}{2}\right)\Gamma\left(\frac{3-\delta}{2}\right)}{\Gamma\left(\frac{n+\delta+1}{2}\right)\Gamma\left(1-\frac{\delta}{2}\right)}\right]^2 \left(\frac{n+\delta^2-1}{(1-\delta)^2}\right)\right\}$$

The function Λ is continuously differentiable, and therefore we have that:

$$\Lambda'(\delta) = \psi\left(\frac{n+\delta}{2}\right) - \psi\left(\frac{n+\delta+1}{2}\right) + \psi\left(1-\frac{\delta}{2}\right) - \psi\left(\frac{3}{2}-\frac{\delta}{2}\right) + \frac{2\delta}{n+\delta^2-1} + \frac{2}{1-\delta}$$

Using Lemma 3.0.8, we have that:

$$\Lambda'(\delta) > \frac{2\delta}{n+\delta^2-1} + \frac{2}{1-\delta} - \frac{1}{n+\delta-1} - \frac{1}{2-\delta} - \frac{2}{(2-\delta)(3-\delta)}$$

But for $\delta \in (0,1)$ as one can easily see

$$\frac{1}{1-\delta} > \frac{1}{2-\delta} + \frac{2}{(2-\delta)(3-\delta)}$$

Therefore

$$\Lambda'(\delta) > \frac{2\delta}{n+\delta^2-1} + \frac{1}{1-\delta} - \frac{1}{n+\delta-1}$$

Because for those values of δ and n it is true that

$$\frac{1}{1-\delta} > \frac{1}{n+\delta-1}$$

we obtain

$$\Lambda'(\delta) > 0$$

so $\Lambda(\delta)$ is increasing and because $\Lambda(0) > 0$ we have that $\Lambda(\delta) > 0$ for all $\delta \in (0,1)$ and so we get the desired result. Therefore we have proven part (A). We now prove part (B).

For $n = 2$ by Proposition 3.0.4 we have that $a_1 > a_0$ for every $a \in (0,n)$ and therefore we have the desired result.

For $n \geq 3$ we define :

$$\Lambda(a) = \ln\left(\frac{a_1}{a_0}\right) = \ln\left\{\left[\frac{\Gamma\left(\frac{n-a}{4}+1\right)\Gamma\left(\frac{n+a}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4}+1\right)\Gamma\left(\frac{n-a}{4}+\frac{1}{2}\right)}\right]^2\left(1+\frac{4a}{(n-a)^2}\right)\right\}$$

And for $n = 3$ we obtain :

$$\Lambda(a) = \ln\left\{\left[\frac{\Gamma\left(\frac{7-a}{4}\right)\Gamma\left(\frac{5+a}{4}\right)}{\Gamma\left(\frac{7+a}{4}\right)\Gamma\left(\frac{5-a}{4}\right)}\right]^2\left(\frac{a^2-2a+9}{(3-a)^2}\right)\right\}$$

For small values of a. Λ is positive which means that $a_1 > a_0$. For example if $a = 0.2$, then $\Lambda \simeq 0.002 > 0$. But for $a = 0.1$ we have $\Lambda \simeq -0.0010 < 0$. Note that:

$$\Lambda'(0) = -\psi\left(\frac{7}{4}\right) + \psi\left(\frac{5}{4}\right) + \frac{4}{9} < 0$$

which, since $\Lambda(0) = 0$, implies that there exists a small neighbourhood around 0 where $\Lambda < 0$, and thus $a_0 > a_1$.

For $n \geq 4$, we set $n = a + 2\delta$ where $\delta \geq \frac{3}{2}$ and then we have

$$\Lambda(a) = \ln\left(\frac{a_1}{a_0}\right) = \ln\left\{\left[\frac{\Gamma\left(\frac{\delta}{2}+1\right)\Gamma\left(\frac{\delta+a+1}{2}\right)}{\Gamma\left(\frac{\delta+1}{2}\right)\Gamma\left(\frac{\delta+a+2}{2}\right)}\right]^2\left(1+\frac{a}{\delta^2}\right)\right\}$$

where $0 < a \leq n - 2\delta$ and $\delta > \frac{3}{2}$. By differentiation, we obtain

$$\Lambda'(a) = \psi\left(\frac{a+\delta+1}{2}\right) - \psi\left(\frac{a+\delta+2}{2}\right) + \frac{1}{a+\delta^2}$$

Using Lemma 3.0.8 for $w = a + \delta + 1$ we obtain following:

$$\Lambda'(a) < -\frac{2}{a+\delta+3} + \frac{1}{a+\delta^2}$$

We will prove that

$$-\frac{2}{a+\delta+3} + \frac{1}{a+\delta^2} < 0$$

Hence

$$a + \delta + 3 < 2\delta^2 + 2a$$

and so

$$0 < 2\delta^2 - \delta + a - 2$$

We define the function

$$f(\delta) = 2\delta^2 - \delta + a - 2$$

which is obviously differentiable and thus

$$f'(\delta) = 4\delta - 1$$

Thus f is increasing for $\delta > \frac{1}{4}$.
We have

$$f\left(\frac{3}{2}\right) = a > 0$$

Therefore for $\delta \geq \frac{3}{2}$ and $0 < a < n - 2\delta$ we have that

$$-\frac{2}{a+\delta+3} + \frac{1}{a+\delta^2} < 0$$

Therefore $\Lambda'(a) < 0$ and because $\Lambda(0) = 0$, Λ is negative for $a = n - 2\delta$ and so $a_0 > a_1$ for $0 < a \leq n - 3$ and $n \geq 4$. This concludes the proof of part(B).
For part (C). Let us assume that we have fixed a constant $a > 0$. Then, for sufficiently large n, we have that $0 < a < n - 3$, and thus from (B) we obtain that $a_0 > a_1$, that is:

$$D_a = \pi^a \left[\frac{\Gamma\left(\frac{n-a}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4} + \frac{1}{2}\right)}\right]^2$$

By the Stirling's formula for the Gamma function we get:

$$\Gamma(t) = \sqrt{\frac{2\pi}{t}}\left(\frac{t}{e}\right)^t (1 + o(1)) \text{ as } t \to \infty$$

Thus by applying the formula on the terms $\Gamma\left(\frac{n-a}{4} + \frac{1}{2}\right)$ and $\Gamma\left(\frac{n+a}{4} + \frac{1}{2}\right)$ we obtain:

$$\Gamma\left(\frac{n-a}{4} + \frac{1}{2}\right) = \sqrt{\frac{2\pi}{\frac{n-a}{4} + \frac{1}{2}}}\left(\frac{\frac{n-a}{4} + \frac{1}{2}}{e}\right)^{\frac{n-a}{4} + \frac{1}{2}} (1 + o(1))$$

and

$$\Gamma\left(\frac{n+a}{4} + \frac{1}{2}\right) = \sqrt{\frac{2\pi}{\frac{n+a}{4} + \frac{1}{2}}}\left(\frac{\frac{n+a}{4} + \frac{1}{2}}{e}\right)^{\frac{n+a}{4} + \frac{1}{2}} (1 + o(1))$$

We observe that:

$$\sqrt{\frac{\frac{n+a}{4} + \frac{1}{2}}{\frac{n-a}{4} + \frac{1}{2}}} = 1 + o(1)$$

and

$$\frac{e^{\frac{n+a}{4}+\frac{1}{2}}}{e^{\frac{n-a}{4}+\frac{1}{2}}} = e^{\frac{a}{2}}$$

and

$$\frac{\left(\frac{n}{4}+\frac{1}{2}-\frac{a}{4}\right)^{\frac{n}{4}}}{\left(\frac{n}{4}+\frac{1}{2}+\frac{a}{4}\right)^{\frac{n}{4}}} = \frac{\left(1+\frac{\frac{1}{2}-\frac{a}{4}}{\frac{n}{4}}\right)^{\frac{n}{4}}}{\left(1+\frac{\frac{1}{2}+\frac{a}{4}}{\frac{n}{4}}\right)^{\frac{n}{4}}} = \frac{e^{-\frac{a}{4}+\frac{1}{2}}}{e^{\frac{a}{4}+\frac{1}{2}}}(1+o(1)) = e^{-\frac{a}{2}}(1+o(1))$$

and

$$\frac{\left(\frac{n}{4}+\frac{1}{2}-\frac{a}{4}\right)^{\frac{1}{2}-\frac{a}{4}}}{\left(\frac{n}{4}+\frac{1}{2}+\frac{a}{4}\right)^{\frac{1}{2}+\frac{a}{4}}} = \left(\frac{n}{4}\right)^{-\frac{a}{2}}(1+o(1))$$

Combining the above we have that

$$\left[\frac{\Gamma\left(\frac{n-a}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4}+\frac{1}{2}\right)}\right]^2 = \left[e^{\frac{a}{2}}e^{-\frac{a}{2}}\left(\frac{n}{4}\right)^{-\frac{a}{2}}(1+o(1))\right]^2 = \left(\frac{4}{n}\right)^a(1+o(1))$$

And thus

$$D_a = \pi^a\left[\frac{\Gamma\left(\frac{n-a}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4}+\frac{1}{2}\right)}\right]^2 = \left(\frac{4\pi}{n}\right)^a(1+o(1))$$

And so we have the desired result.
In addition, from the Corollary 3.01 for $a = 2$ we obtain :

$$D_a = \left(\frac{4\pi}{n}\right)^2$$

which means that it becomes an equality for $a = 2$. This completes the proof of part (C) and of the theorem as well.

□

Proof of Lemma 3.0.8. As in the proof of Lemma 3.0.3 we have that

$$\ln\Gamma(z) = -\gamma z - \ln z + \sum_{n=1}^{\infty}\left(\frac{z}{n} - \ln\left(1+\frac{z}{n}\right)\right)$$

thus

$$\psi(y) = \frac{d}{dy}\ln\Gamma(y) = -\gamma - \frac{1}{y} + \sum_{n=1}^{\infty}\left(\frac{y}{n(n+y)}\right)$$

For $y = \frac{w}{2}$, $w > 1$ we obtain

$$\psi\left(\frac{w}{2}\right) = -\gamma - \frac{2}{w} + \sum_{n=1}^{\infty}\left(\frac{w}{n(2n+w)}\right)$$

and similarly for $y = \frac{w+1}{2}$, $w > 1$ we have

$$\psi\left(\frac{w+1}{2}\right) = -\gamma - \frac{2}{w+1} + \sum_{n=1}^{\infty}\left(\frac{w+1}{n(2n+w+1)}\right)$$

Therefore

$$\begin{aligned}\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) &= \frac{2}{w(w+1)} + \sum_{n=1}^{\infty}\frac{2}{(2n+w+1)(2n+w)} \\ &= 2\sum_{n=0}^{\infty}\frac{1}{(2n+w+1)(2n+w)}\end{aligned}$$

The function

$$f(n) = \frac{1}{(2n+w+1)(2n+w)}$$

is positive and decreasing. Therefore

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) = 2\sum_{n=0}^{\infty}\frac{1}{(2n+w+1)(2n+w)} > 2\int_0^{\infty}\frac{1}{(2x+w+1)(2x+w)}\,dx$$

By the change of variables $2x + w = u$ we obtain

$$\begin{aligned}\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) &> \int_w^{\infty}\frac{1}{u^2+u}\,du = \left[-\ln\left(\frac{1}{u}+1\right)\right]_w^{\infty} \\ &= \ln(w+1) - \ln w = h(w)\end{aligned}$$

One can easily see that

$$h(w) - g(w) > 0$$

for $w > 1$ where g is the function $g(w) = \frac{2}{2w+1}$ and $h(1) > g(1)$. Therefore for every $w > 1$ we obtain

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) > g(w) = \frac{2}{2w+1}$$

for $w > 1$.
We know from the proof of the Theorem 3.0.7 that

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) = 2\int_0^\infty \frac{e^{-wu}}{(1+e^{-u})}\,du$$

Because $w > 1$ and for $u \geq 0$ we have that

$$\frac{1}{1+e^{-u}} \leq \frac{1}{1+e^{-wu}}$$

Thus

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) \leq 2\int_0^\infty \frac{e^{-wu}}{(1+e^{-wu})}\,du$$

By the change of variables $x = e^{-wu}$ we have that

$$\begin{aligned}\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) &\leq 2\int_0^\infty \frac{e^{-wu}}{(1+e^{-wu})}\,du = -\frac{2}{w}\int_1^0 \frac{1}{1+x}\,dx \\ &= \frac{2}{w}[\ln(1+x)]_0^1 = \frac{2}{w}\ln 2\end{aligned}$$

The function $h_1(w) = \frac{2}{w+1}$ is decreasing for $1 < w \leq 3$. For $w = 3$ we have that $h_1(3) = \frac{1}{2} > 2\ln 2 - 1$. Thus for every $w \in (1, 3]$ we have that

$$2\ln 2 - 1 < h_1(w)$$

hence

$$\frac{2\ln 2}{w} - \frac{1}{w} < \frac{2}{w(w+1)}$$

consequently

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) \leq \frac{1}{w} + \frac{2}{w(w+1)}$$

For $w \geq 3$ we define the function $h_2(w) = \frac{1}{w-1}$ and for $w = 3$ we have that

$$h_2(3) = \frac{1}{2} > \frac{2\ln 2}{3}$$

For every $w \geq 3$ we define the function

$$h_3(w) = h_2(w) - \frac{2\ln 2}{w}$$

which is decreasing . We have $h_3(3) > 0$ and $h_3(w) \to 0$ when $w \to \infty$. Thus $h_3(w) > 0$ for $w \geq 3$. Therefore

$$\psi\left(\frac{w+1}{2}\right) - \psi\left(\frac{w}{2}\right) < \frac{2\ln 2}{w} < \frac{1}{w-1}$$

Therefore we have obtained the desired result.

□

Chapter 4

Logarithmic Uncertainty Inequality

4.1 Logarithmic Uncertainty Inequality for Pitt Inequality

In this section we will prove a logarithmic uncertainty inequality based on the result of the Theorem 2.1.1.

Theorem 4.1.1 *For* $f \in \mathcal{S}(\mathbb{R}^n)$

$$\int\limits_{\mathbb{R}^n} \ln|x||f(x)|^2\,dx + \int\limits_{\mathbb{R}^n} \ln|y||\widehat{f}(y)|^2\,dy \geq D \int\limits_{\mathbb{R}^n} |f(x)|^2\,dx$$

where $D = \psi\left(\frac{n}{4}\right) - \ln\pi$

Proof. From Theorem 2.1.1 we have that for $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\int\limits_{\mathbb{R}^n} |x|^{-a}|f(x)|^2\,dx \leq C_a \int\limits_{\mathbb{R}^n} |y|^a|\widehat{f}(y)|^2\,dy$$

where

$$C_a = \pi^a\left[\Gamma\left(\frac{n-a}{4}\right)/\Gamma\left(\frac{n+a}{4}\right)\right]^2$$

For $a = 0$ we have that $C_0 = 1$ and thus the inequality (2.3) for $a = 0$ is

$$\int\limits_{\mathbb{R}^n} |f(x)|^2\,dx \leq \int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2\,dy$$

But, by the Plancerel Theorem (Theorem 1.1.9) it is true that :

$$\int\limits_{\mathbb{R}^n} |f(x)|^2 \, dx = \int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2 \, dy$$

and thus for $a = 0$ the inequality (2.3) becomes an equality. Moreover, it is elementary to prove that C_a is differentiable with respect to a.
In order to prove the Theorem 4.1.1, we define the function:

$$h(a) = C_a \int\limits_{\mathbb{R}^n} |y|^a |\widehat{f}(y)|^2 \, dy - \int\limits_{\mathbb{R}^n} |x|^{-a} |f(x)|^2 \, dx$$

for $a \geq 0$.
Using (2.3) we conclude that

$$h(a) \geq 0$$

for $a \geq 0$, while $h(0) = 0$. Thus, we conclude that for a small neighbourhood of 0 the function h is increasing and using the fact that h is differentiable in this neighbourhood, and so it must be true that $g'(a) \geq 0$ and consequently $g'(0) \geq 0$. We have that

$$\frac{dC_a}{da} = \ln \pi \pi^a \left[\Gamma\left(\frac{n-a}{4}\right) / \Gamma\left(\frac{n+a}{4}\right) \right]^2 + 2\pi^a \left[\Gamma\left(\frac{n-a}{4}\right) / \Gamma\left(\frac{n+a}{4}\right) \right] \times \frac{d\left[\Gamma\left(\frac{n-a}{4}\right) / \Gamma\left(\frac{n+a}{4}\right)\right]}{a}$$

But

$$\frac{d\left[\Gamma\left(\frac{n-a}{4}\right) / \Gamma\left(\frac{n+a}{4}\right)\right]}{da} = \frac{1}{\Gamma^2\left(\frac{n+a}{4}\right)} \left[-\frac{1}{4}\Gamma'\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n+a}{4}\right) - \frac{1}{4}\Gamma'\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n-a}{4}\right) \right] = -\frac{1}{4\Gamma^2\left(\frac{n+a}{4}\right)} \left[\psi\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n-a}{4}\right)\Gamma\left(\frac{n+a}{4}\right) + \psi\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n+a}{4}\right)\Gamma\left(\frac{n-a}{4}\right) \right]$$

Thus

$$\left. \frac{d\left[\Gamma\left(\frac{n-a}{4}\right) / \Gamma\left(\frac{n+a}{4}\right)\right]}{da} \right|_{a=0} = -\frac{1}{4\Gamma^2\left(\frac{n}{4}\right)} 2\psi\left(\frac{n}{4}\right)\Gamma^2\left(\frac{n}{4}\right) = -\frac{\psi\left(\frac{n}{4}\right)}{2}$$

and so

$$\left.\frac{dC_a}{da}\right|_{a=0} = \ln \pi - \psi\left(\frac{n}{4}\right) = -D$$

Therefore

$$h'(a) = \frac{dC_a}{da}\int\limits_{\mathbb{R}^n} |y|^a |\widehat{f}(y)|^2\, dy + C_a \int\limits_{\mathbb{R}^n} |y|^a \ln|y| |\widehat{f}(y)|^2\, dy + \int\limits_{\mathbb{R}^n} |x|^{-a} \ln|x| |f(x)|^2\, dx$$

and hence for $a = 0$, because $h'a(0) \geq 0$ we have that

$$-D\int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2\, dy + \int\limits_{\mathbb{R}^n} \ln|y| |\widehat{f}(y)|^2\, dy + \int\limits_{\mathbb{R}^n} \ln|x| |f(x)|^2\, dx \geq 0$$

and thus

$$\int\limits_{\mathbb{R}^n} \ln|y| |\widehat{f}(y)|^2\, dy + \int\limits_{\mathbb{R}^n} \ln|x| |f(x)|^2\, dx \geq D\int\limits_{\mathbb{R}^n} |\widehat{f}(y)|^2\, dy$$

and by using the Theorem 1.1.9 we conclude that

$$\int\limits_{\mathbb{R}^n} \ln|y|\, |\widehat{f}(y)|^2\, dy + \int\limits_{\mathbb{R}^n} \ln|x|\, |f(x)|^2\, dx \geq D\int\limits_{\mathbb{R}^n} |f(x)|^2\, dx$$

which is the desired result.

□

4.2 Logarithmic Uncertainty Inequality for Pitt's Inequality with gradient terms

In Theorem 2.2.1 we have proven that for $f \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 2$ it is true that:

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2\, |x|^{-a}\, dx \leq 4\pi^2 D_a \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^{a+2}\, dy$$

For $a = 0$ one can see that $D_a = 1$ and thus the inequality is:

$$\int\limits_{\mathbb{R}^n} |\nabla f(x)|^2\; dx \leq 4\pi^2 \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2\, dy$$

As we have shown in Chapter 1 it is true that

$$\widehat{\nabla f}(y) = (-2\pi i y)\widehat{f}(y)$$

and thus by using Theorem 1.1.9 we have the following

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx = \int_{\mathbb{R}^n} \left|\widehat{\nabla f}(y)\right|^2 \, dy = 4\pi^2 \int_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \, dy \tag{4.1}$$

Consequently, the above inequality becomes an equality

Theorem 4.2.1 *For $f \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 2$, it is true that*

$$\int_{\mathbb{R}^n} |\nabla f(x)|^2 \ln|x| \, dx + 4\pi^2 \int_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \ln|y| \, dy \geq E \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx$$

where

$$E = \begin{cases} \psi\left(\dfrac{3}{2}\right) - \ln\pi - 1, & n = 2 \\ \psi\left(\dfrac{n}{4} + \dfrac{1}{2}\right) - \ln\pi, & n \geq 3 \end{cases}$$

Proof. From the Proposition 3.0.4 we have that for $n = 2$:

$$D_a = \pi^a \left[\frac{\Gamma(\frac{3}{2} - \frac{a}{4})}{\Gamma(\frac{3}{2} + \frac{a}{4})}\right]^2 \left(\frac{a^2 + 4}{(a-2)^2}\right)$$

On the other hand for $n \geq 3$ from the Theorem 3.0.7 (B) for a in a neighbourhood 0 it is true that :

$$D_a = \pi^a \left[\frac{\Gamma\left(\frac{n-a}{4} + \frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4} + \dfrac{1}{2}\right)}\right]^2$$

For every possible value of D_a it can be easily proven that D_a is differentiable with respect to the variable a.

To prove the Theorem 4.2.1 we will define the function

$$g(a) = 4\pi^2 D_a \int_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^{a+2} \, dy - \int_{\mathbb{R}^n} |\nabla f(x)|^2 |x|^{-a} \, dx$$

for $a \geq 0$

According to the Theorem 2.1.1 it is true $g(a) \geq 0$ for every $a \geq 0$, while

$g(0) = 0$. We conclude that in a neighbourhood of 0 the function g is increasing and also g is differentiable and so in that neighbourhood it must be true that $g'(a) \geq 0$ and so $g'(0) \geq 0$.
Firstly we will consider that $n = 2$.

$$\begin{aligned}\frac{dD_a}{da} =&(\ln \pi)\pi^a \left[\frac{\Gamma\left(\frac{3}{2}-\frac{a}{4}\right)}{\Gamma\left(\frac{3}{2}+\frac{a}{4}\right)}\right]^2 \left(\frac{a^2+4}{(2-a)^2}\right) \\ &+ 2\pi^a \left[\frac{\Gamma\left(\frac{3}{2}-\frac{a}{4}\right)}{\Gamma\left(\frac{3}{2}+\frac{a}{4}\right)}\right]\left[-\frac{1}{4}\frac{\Gamma'\left(\frac{3}{2}-\frac{a}{4}\right)}{\Gamma\left(\frac{3}{2}+\frac{a}{4}\right)} - \frac{1}{4}\frac{\Gamma\left(\frac{3}{2}-\frac{a}{4}\right)\Gamma'\left(\frac{3}{2}+\frac{a}{4}\right)}{\left(\Gamma\left(\frac{3}{2}+\frac{a}{4}\right)\right)^2}\right]\left(\frac{a^2+4}{(2-a)^2}\right) \\ &+ \pi^a \left[\frac{\Gamma\left(\frac{3}{2}-\frac{a}{4}\right)}{\Gamma\left(\frac{3}{2}+\frac{a}{4}\right)}\right]^2 \left[\frac{2a}{(2-a)^2} + \frac{2(a^2+4)}{(2-a)^3}\right]\end{aligned}$$

and for $a = 0$ we have the following

$$\left.\frac{dD_a}{da}\right|_{a=0} = \ln \pi - \psi\left(\frac{3}{2}\right) + 1$$

because

$$\left[\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}\right]\left[-\frac{1}{4}\frac{\Gamma'\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} - \frac{1}{4}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma'\left(\frac{3}{2}\right)}{\left(\Gamma\left(\frac{3}{2}\right)\right)^2}\right] = -\frac{1}{4}\left[\psi\left(\frac{3}{2}\right) + \psi\left(\frac{3}{2}\right)\right] = -\frac{1}{2}\psi\left(\frac{3}{2}\right)$$

Similarly for $n \geq 3$ we have that :

$$\begin{aligned}\left.\frac{dD_a}{da}\right|_{a=0} =& \left.\frac{d}{da}\right|_{a=0}[\pi^a]\left[\frac{\Gamma\left(\frac{n-a}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4}+\frac{1}{2}\right)}\right]^2 + \pi^a \left.\frac{d}{da}\right|_{a=0}\left[\left[\frac{\Gamma\left(\frac{n-a}{4}+\frac{1}{2}\right)}{\Gamma\left(\frac{n+a}{4}+\frac{1}{2}\right)}\right]^2\right] \\ =& \ln \pi - \psi\left(\frac{n}{4}+\frac{1}{2}\right)\end{aligned}$$

We define

$$E = \begin{cases} \psi\left(\frac{3}{2}\right) - \ln \pi - 1, & n = 2 \\ \psi\left(\frac{n}{4}+\frac{1}{2}\right) - \ln \pi, & n \geq 3 \end{cases}$$

As mentioned above :

$$g'(0) \geq 0$$

and by differentiating under the integrals we have that:

$$4\pi^2 \left.\frac{dD_a}{da}\right|_{a=0} \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \, dy + 4\pi^2 D_0 \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \ln|y| \, dy + \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \ln|x| \, dx \geq 0$$

Hence

$$4\pi^2 D_0 \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \ln|y| \, dy + \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \ln|x| \, dx \geq -4\pi^2 \left.\frac{dD_a}{da}\right|_{a=0} \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \, dy$$

Using (4.1) and the facts that for every possible n it is true that $-\left.\frac{dD_a}{da}\right|_{a=0} = E$ and that $D_0 = 1$ we have the following :

$$4\pi^2 \int\limits_{\mathbb{R}^n} \left|\widehat{f}(y)\right|^2 |y|^2 \ln|y| \, dy + \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \ln|x| \, dx \geq E \int\limits_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx$$

which is the desired result.

□

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